OSCILLATION OF THIRD-ORDER NONLINEAR NEUTRAL DIFFERENTIAL EQUATIONS WITH DISTRIBUTED TIME DELAY

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Abstract. With the development of modern society, research on properties of ordinary differential equation is becoming one of the hotspots in mathematical field. Neutral differential equation which is usually generated in natural science and engineering field is always extensively concerned by many scientific researchers for it can effectively describe multiple complex phenomena in natural world.

In recent years, differential equation and non-linear differential equation with time delay have attracted more and more attention. However, few researches concern about the properties of neutral differential equations with time delay. On account of this, we explored the oscillation of third-order nonlinear neural delay differential equations. Based on operators and integration techniques and with the help of proper comparison theorem, we established some oscillation sufficient conditions for several new solutions to such kind of equation. The obtained theorem which popularizes and improves results of the existing literature is applicable to neural differential equations. These results fully reflect the effects of time delay in oscillation of equation.

Keywords: third-order differential equation; neural; non-linear; oscillation.

1. Introduction

Owing to the demand on development of productivity and mathematicization of science in 17 and 18 century, calculus gained a further development and was gradually closely integrated with a series of subjects such as mechanics [1], physics, geometry [2] and acoustics [3]. Under such a background, differential equation theory emerged in response to the needs of times. Neutral differential equation which has been researched for a long time can be used for describing many natural phenomena. Hence studying neural differential equation is of great scientific and academic values theoretically and practically. Many interesting and important research topics deriving from study of neural differential equation have attracted much attention.

Due to abstraction and complexity [4], systems such as circuit signals, infection of epidemic disease [5] and automatic control usually have the defect of time delay. For instance, lossless transport network of switching circuit of high-speed computer can be affected by time delay feedback [6]. Time delay of dynamic equation can reflect effects of previous, current and future states of matters on change rules of matters [7] and deeply reveal the nature of matters [8]. Therefore, neural differential equation with time delay becomes the research object of scholars. Oscillation is a quite important research field in neural differential equation. With the development of science and technology, oscillation is a universal motion mode for substances [9].

In recent years, many scholars in this field have made several researches on oscillation of first-order and high-order, linear and non-linear neutral differential equation [10, 11] and obtained considerable academically valuable achievements. In 1836, Sturm proposed and deeply studied the problem of oscillation of second-order linear differential equation for the first time through studying heat conduction [12, 13], which laid a good research basis for the oscillation theory of differential equation [14]. Based on previous studies, this study explored the oscillation of third-order differential equation.

2. Theoretical basis of third-order nonlinear neutral differential equation with distributed time delay

In recent years, oscillation and asymptotic property of delay differential equation have been extensively concerned. Compared to oscillation of second-order delay differential equation, few results concern third-order delay differential equation, especially third-order or higher neural differential equation with time delay [15]. Previous researches considered asymptotic property of non-oscillatory solution of third-order nonlinear neutral differential equation with time delay [16]

$$[a(t)(b(t)(x(t)+p(t)x(\sigma(t)))')']'+q(t)x(\tau(t))=0,\ t\geq t_0 \tag{2.1}$$

as well as oscillation of solution of third-order neutral differential equation with distributed time delay [17]

$$(r(t)([x(t)+p(t)x(\sigma(t))]'')^{\alpha})'+\int_a^b q(t,\xi)f(x(g(t,\xi)))d\xi=0,\ t\geq t_0. \tag{2.2}$$

This study mainly considers oscillation of third-order neutral differential equation with more extensively distributed time delays:

$$\{r_1(t)(r_2(t)(x(t)+p(t)x(\sigma(t)))')')^{\gamma}\}'+\int_a^b q(t,\xi)f(x(\tau(t,\xi)))d\xi=0,\ t\geq t_0, \tag{2.3}$$

that is,

$$\{r_1(t)(r_2(t)(x(t)+p(t)x(\sigma(t)))')')^{\gamma}\}'+q(t)f(x(t),x(\tau(t)))=0,\ t\geq t_0. \tag{2.4}$$

Several sufficient conditions for equation oscillation can be obtained if we apply Riccati transformation and Hardy inequality method. t_0 is supposed as a real number. Besides, $I = [t_0, \infty)$ and $R^+ = (0, +\infty)$. Then, we establish the following conditions.

(H1) For all $t \in [t_0, +\infty)$, $r_1, r_2 \in C^1[t_0, +\infty)$ ($r_1(t), r_2(t) > 0$, $r_1'(t), r_2'(t) > 0$ and $\int_{t_0}^{+\infty} \frac{1}{r_2(s)} ds = +\infty$, $\int_{t_0}^{+\infty} r_1^{-\frac{1}{\gamma}}(s) ds = +\infty$);

(H2) For all $t \in [t_0, +\infty)$, $p, q \in C[t_0, +\infty)$ ($0 \le p(t) \le b < 1$, $q(t) > 0$, b refers to constant);

(H3) $\tau, \sigma \in C[t_0, +\infty)$, $\sigma(t) \le t$, $\tau(t) \le t$, $\lim\limits_{t\to+\infty} \sigma(t) = +\infty$, $\lim\limits_{t\to+\infty} \tau(t) = +\infty$;

(H4) $f \in C(R^2, R)$. There is a positive constant k that can make $\frac{f(x,y)}{y^\gamma} \ge k$ when $y \ne 0$; $q, \tau \in C(I \times [a, b], R^+), \tau(t, \xi) \le t, \tau(t, \xi)$ is non-decreasing with regard to t and ξ and moreover $\lim\limits_{t\to+\infty} \min\limits_{\xi\in[a,b]} \tau(t, \xi) = \infty$;

(H5) $\gamma = \frac{m}{n}$ (m and n are positive odd numbers); $f(u) \in C(R, R)$, $\frac{f(u)}{u^\gamma} \ge c > 0$, $u \ne 0$.

We suppose $z(t) = x(t) + p(t)x(\sigma(t))$. It can be known from $\lim\limits_{t\to+\infty} \sigma(t) = +\infty$ and $\lim\limits_{t\to+\infty} \tau(t) = +\infty$ that, there is T that can make $\sigma(t) > t_0$ when $t > T$; besides, there is a minimum value as $\sigma(t)$ is continuous in the interval of $[t_0, +\infty)$ and it is recorded as t_1. Similarly, for $\tau(t)$, there is also a minimum value t_2 in the interval of $[t_0, +\infty)$. Hence we take the value of t_* as $\min\{t_1, t_2\}$; then we get $\tau(t) \ge t_*$ and $\sigma(t) \ge t_*$. Therefore we consider the solution of equation (2.4) has definition in the interval of $[t_*, +\infty)$ and there is also T_x that can make $x(t)$ second order and continuously differentiable and make $r_1(t)((r_2(t)(x(t) + p(t)x(\tau(t)))')')^\gamma$ first order and continuously differentiable and meet equation (2.4) when $t \in [T_x, +\infty)$. If $x(t)$ is unbounded in zero locus set of $[T_x, +\infty)$, then solution of equation (2.4), i.e., $x(t)$, is considered as oscillatory, that is, the final result is neither positive nor negative. In contrast, if $x(t)$ is bounded in zero locus set, then solution of equation (2.4), i.e., $x(t)$, is non-oscillatory, that is, solution of equation (2.4) is positive or negative. Before proof of theorem, necessary prerequisite knowledge and lemma are given in the following.

3. Prerequisite knowledge

Lemma 3.1 *Suppose* (H1)$\sim$(H5) *are tenable and* $x(t)$ *is the final positive solution of equation* (2.3)*, then function* $z(t)$ *defined by* $z(t) = x(t) + p(t)x(\sigma(t))$ *only has one of the following properties.*

(1) $z(t) > 0$, $r_1(t)z'(t) > 0$, $(r_1(t)z'(t))' > 0$;

(2) $z(t) > 0$, $r_1(t)z'(t) < 0$, $(r_1(t)z'(t))' > 0$.

Proof. As $x(t)$ is the final positive solution of equation (2.3), $t \in [t_1, +\infty)$, we have:

$$x(t) > 0, x(\sigma(t)) > 0, x(\tau(t,\xi)) > 0.$$

It can be known from (H2) and $z(t) = x(t) + p(t)x(\sigma(t))$ that $z(t) \geq x(t) > 0$ when $t \in [t_1, +\infty)$.

It can be known from (H5) that,

$$\frac{f(x(\tau(t,\xi)))}{x\gamma(\tau(t,\xi))} \geq c > 0.$$

Hence, for all $t \in [t_1, +\infty)$, there is $\int_a^b q(t,\xi)f(x(\tau(t,\xi)))d\xi > 0$, then

$$(r_1(t)((r_2)z'(t))')^{\gamma})' = -\int_a^b q(t,\xi)f(x(\tau(t,\xi)))d\xi < 0,\ t \geq t_1 \tag{3.1}$$

We can know that $r_1(t)((r_2(t)(x(t)+p(t)x(\tau(t)))')')^{\gamma}$ is a decreasing function on $[t_1, +\infty)$ and it is positive or negative eventually. Thus we have verified the establishment of the first condition.

Otherwise, there is constant $G > 0$ and $t_2 \geq t_1$ that can make:

$$r_1(t)((r_2(t)z'(t))')^{\gamma} \leq -G < 0,\ t \geq t_2,$$

is the nth-autocommutator subgroup of G.

Now, we obtain the following series of characteristic subgroups. Integrating the above equation from t_2 to t and combining H1, we have:

$$r_2(t)z'(t) \leq r_2(t_2)z'(t_2) - G^{\frac{1}{\gamma}} \int_{t_2}^{t} \frac{1}{r_1^{\frac{1}{\gamma}}(s)} ds \to -\infty (t \to +\infty). \tag{3.2}$$

Hence, $r_2(t)z'(t)$ is negative eventually. Besides, it can be known from $(r_2(t)z'(t))' < 0$ and $r_2(t)z'(t) < 0$ that, $z(t)$ is negative eventually, which is contradictory with $z(t) > 0$. Therefore, $((r_2(t)z'(t))')^{\gamma}$ is positive, i.e., $(r_2(t)z'(t))' > 0$ holds.

It can be known from $(r_2(t)z'(t))' > 0$ that, $r_2(t)z'(t)$ is monotonically increasing in the interval of $[t_1, +\infty)$; therefore $r_2(t)z'(t)$ is either negative or positive. The proof of Lemma 3.1 is over. ∎

It can be known from the above requirement of the question and monotonic boundedness theory that, there is constant $d \geq 0$ that can make $\lim_{t\to\infty} z(t) = d$. Besides, due to limit existence of $z(t)$, $d = 0$. Otherwise, if $d > 0$, then for any $\varepsilon > 0$, there is

$$d < z(t) < d+\varepsilon, d < z(\sigma(t)) < d+\varepsilon, d < z(\tau(t,\xi)) < d+\varepsilon, \tag{3.3}$$

when $t_1 > t_0$

Therein, $L = \dfrac{d - p(d+\varepsilon)}{d+\varepsilon} > 0.$

It can be known from (H5) that, when $t \geq t_2$, there is $t_2 \in [t_1, +\infty)$ that can make $f(x(\tau(t,\xi))) \geq cx^\gamma(\tau(t,\xi)) \geq cL^\gamma z^\gamma(\tau(t,\xi))$.

Combining equation (3.3), we have:

$$(r_1(t)((r_2(t)z'(t))')^\gamma)' \leq -cL^\gamma d^\gamma \int_a^b q(t,\xi)d\xi \tag{3.4}$$

Hence, we can conclude that function $r_1(t)((r_2(t)z'(t))')^\gamma$ is monotonically decreasing in the interval of $[t_2, +\infty)$. It can be known from the monotonic boundedness theory that $r_1(t)((r_2(t)z'(t))')^\gamma > 0$; there is a finite limit, i.e., $\lim_{t\to\infty} r_1(t)((r_2(t)z'(t))')^\gamma \geq 0$. Integrating equation (3.4) in the interval of $[t, +\infty)$, we have:

$$r_1(t)((r_2(t)z'(t))')^\gamma - \lim_{t\to\infty} r_1(t)((r_2(t)z'(t))')^\gamma \geq -cL^\gamma d^\gamma \int_t^{+\infty} \int_a^b q(s,\xi)d\xi ds.$$

Hence, $r_1(t)((r_2(t)z'(t))')^\gamma \geq cL^\gamma d^\gamma \int_t^{+\infty} \int_z^b q(s,\xi)d\xi ds$, that is,

$$(r_2(t)z'(t))' \geq c^{\frac{1}{\gamma}} Ld \left(\frac{1}{r_1(t)} \int_t^{+\infty} \int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}}. \tag{3.5}$$

It can be known from $(r_2(t)z'(t))' > 0$ that $r_2(t)z'(t)$ is monotone increasing in the interval of $[t_2, +\infty)$. Besides, $r_2(t)z'(t) < 0$. It can be known from the monotonic boundedness theory that there is a finite limit, i.e., $\lim_{t\to\infty} r_2(t)z'(t) \leq 0$. Integrating equation (3.5) in the interval of $[t, +\infty)$, we have:

$$\lim_{t\to\infty} r_2(t)z'(t) - r_2(t)z'(t) \geq -c^{\frac{1}{\gamma}} Ld \int_t^{+\infty} \left(\frac{1}{r_1(u)} \int_u^{+\infty} \int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}} du.$$

Then, we get

$$-r_2(t)z'(t) \geq c^{\frac{1}{\gamma}} Ld \int_t^{+\infty} \left(\frac{1}{r_1(u)} \int_u^{+\infty} \int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}} du,$$

that is,

$$-z'(t) \geq \frac{c^{\frac{1}{\gamma}} Ld}{r_2(t)} \int_t^{+\infty} \left(\frac{1}{r_1(u)} \int_u^{+\infty} \int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}} du. \tag{3.6}$$

As $z'(t)$ is monotonically decreasing in the interval of $[t_2, +\infty)$ ($z'(t) < 0$ and $z(t) > 0$), we can know that there is a finite limit, i.e., $\lim_{t\to\infty} z(t) \geq 0$. Integrating equation (3.6) in the interval of $[t, +\infty)$, we have:

$$z(t_2) - \lim_{t\to\infty} z(t) \geq c^{\frac{1}{\gamma}} Ld \int_{t2}^{+\infty} \frac{1}{r_2(t)} \int_v^{+\infty} \left(\frac{1}{r_1(u)} \int_u^{+\infty} \int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}} dudv,$$

Hence,

$$z(t_2) \geq c^{\frac{1}{\gamma}} Ld \int_{t2}^{+\infty} \frac{1}{r_2(t)} \int_v^{+\infty} \left(\frac{1}{r_1(u)} \int_u^{+\infty} \int_a^b q(s,\xi)d\xi ds\right)^{\frac{1}{\gamma}} dudv.$$

The result obtained is contradictory with equation (3.2).

Because of $0 < x(t) < z(t)$, we know that $\lim_{t\to\infty} x(t) = \lim_{t\to\infty} z(t) = 0$. The proof of Lemma 3.1 is over. ■

Lemma 3.2 *Suppose there is $w(t) > 0, w'(t) > 0, w''(t) < 0, g(t) \leq t$ in the interval of $t \in [t_0, +\infty)$. Then, for any $\theta \in (0,1)$, there is $T_\theta \geq t_0$ that can make:*

$$w(g(t)) \geq \theta \frac{g(t)}{t} w(t),\ t \geq T_\theta.$$

Proof. Applying Lagrange's mean value theorem and considering that fact that $w'(t)$ is monotonically decreasing, we have:

$$w(t) - w(g(t)) \leq w'(g(t))(t - g(t))$$

Multiplying both sides of the above inequation with $\frac{1}{w(g(t))}$, we have:

$$\frac{w(t)}{w(g(t))} \leq 1 + \frac{w'(g(t))(t - g(t))}{w(g(t))}. \tag{3.7}$$

Using Lagrange's mean value theorem again, we have:

$$w(g(t)) \geq w(g(t)) - w(t_0) \geq w'(g(t))(g(t) - t_0).$$

Therefore, for any $\theta \in (0,1)$, there is $T_\theta \geq t_0$ that can make:

$$\frac{w(g(t))}{w'(g(t))} \geq g(t)\left(1 - \frac{t_0}{g(t)}\right) \geq \theta g(t). \tag{3.8}$$

Combining equation (3.7) and (3.8), we can get:

$$\frac{w(t)}{w(g(t))} \leq 1 + \frac{1}{\theta g(t)}(t - g(t)) \leq \frac{t}{\theta g(t)},$$

i.e., $w(g(t)) \geq \theta \frac{g(t)}{t} w(t), t \geq T_\theta$. The proof of Lemma 3.2 is over. ■

Lemma 3.3 *If $z(t) > 0, r_2(t)z'(t) > 0$ and $(r_2(t)z'(t))' > 0$ hold in the interval of $t \in [T_\theta, +\infty)$, then there is $\beta \in (0,1)$ and $T_\beta \geq T_\theta$ that can make:*

$$z(t) \geq \beta t z'(t),\ t \geq T_\beta.$$

Proof. Suppose $y(t) = \int_{T_\theta}^t T_2(s)z'(s)ds, t \in [T_\theta, +\infty)$. Then, the following conditions with regard to $y(t)$ hold.

$$y(t) > 0, y'(t) > 0, y''(t) > 0, y'''(t) \leq 0.$$

Besides, there is $\beta \in (0,1)$ and $T_\beta \geq T_\theta$ that can make $y(t) \geq \beta t y'(t), t \geq T_\beta$, that is,

$$\frac{\int_{T_\theta}^t r_2(s)z'(s)ds}{r_2(t)z'(t)} \geq \beta t,\ t \geq T_\beta. \tag{3.9}$$

Using the distribution integral method and taking (H1) into account, we have:

$$\int_{T_\theta}^{t} r_2(s)z'(s)ds = r_2(t)z'(t)\,\big|_{T_\theta}^{t} - \int_{T_\theta}^{t} r_2'(s)z(s)ds$$
$$= r_2(t)z(t) - r_2(T_\theta)z(T_\theta) - \int_{T_\theta}^{t} r_2'(s)z(s)ds < r_2(t)z(t).$$

Combining this with equation (3.9) yields:

$$\frac{r_2(t)z(t)}{r_2(t)z'(t)} \geq \beta t,\ t \geq T_\beta,$$

that is, $z(t) \geq \beta t z'(t), t \geq T_\beta$. The proof of Lemma 3.3 is over. ∎

Lemma 3.4 *Suppose A and B are non-negative constants. Then we have:*

$$A^\lambda - \lambda A B^{\lambda-1} + (\mu - 1)B^\lambda \geq 0,\ \lambda > 1.$$

The equation holds if and only if $A = B$.

4. Main results and proof

Theorem 4.1 *Suppose* (H1)∼(H5) *and equation* (3.2) *are tenable, and moreover*

$$\limsup_{t\to\infty} \int^{t} \left[\left(\frac{\beta \tau_1(s) R(\tau_1(s))}{r_2(\tau_1(s)} \right)^{\gamma} q_1(s) - \left(\frac{\gamma}{\gamma+1} \right)^{\gamma+1} \frac{R'(\tau 1(s))}{R(\tau 1(s))} \right] ds = +\infty \quad (4.1)$$

where β is given by Lemma 3.3. *For convenience, we write:*

$$\tau_1(s) = \tau(s,a)$$
$$R(\tau_1(s)) \int_{t0}^{\tau_1(s)} r_1^{-\frac{1}{\gamma}}(s)ds, q_1(s) = c(1-p)^\gamma \int_a^b q(s,\xi)d\xi.$$

Then, every solution of equation (2.3), *i.e.,* $x(t)$, *either oscillate or satisfies* $\lim\limits_{t\to+\infty} x(t) = 0$.

Proof. Suppose $x(t)$ is a non-oscillatory solution of equation (2.3) and it is positive. Then, we can know from Lemma 3.1 that, $z(t)$ can only satisfy property (1) or (2). On that condition that $z(t)$ satisfying property (1), there exists $t_1 \geq t_0$ such that

$$z(t) > 0, z(\sigma(t)) > 0, z'(t) > 0$$

when $t \geq t_1$. Then, $z(t)$ is considered monotonically increasing in the interval of $[t_1, +\infty)$. And then

$$x(t) = z(t) - p(t)x(\sigma(t)) \geq z(t) - pz(\sigma(t)) \geq (1-p)z(t).$$

Combining this with (H5), we have:

$$\begin{aligned}(r_1(t)((r_2(t)z(t))')^\gamma)' &= -\int_a^b q(t,\xi)f(x(\tau(t,\xi)))d\xi\\ &\le -c(1-p)^\gamma \int_a^b q(t,\xi)z^\gamma(\tau(t,\xi))d\xi\\ &\le -c(1-p)^\gamma(\tau(t,a))\int_a^b q(t,\xi)d\xi\\ &\le -q_1(t)z^\gamma(\tau_1(t))<0.\end{aligned}$$

Then, we get

$$(r_1(t)((r_2(t)z(t))')^\gamma)' = -q_1(t)z^\gamma(\tau_1(t))<0, \tag{4.2}$$

where $q_1(t) = c(1-p)^\gamma \int_a^b q(t,\xi)d\xi$.

$$W(t) = R^\gamma(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma$$

is defined.

It can be known from equation (3.1) that,

$$W(t) = R^\gamma(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma.$$

Thus, when $t \ge t_2$, there is $t_2 \ge t_1$ such that

$$z(t)>0, (r_2(t)z(t))'>0, r_2(\tau_1(t)z'(\tau_1(t))>0, (r_2(t)z'(t))''<0.$$

Then, $W(t)>0$ $(t\ge t_2)$. Considering Lemma 3.3 and equation (4.2), we have:

$$\begin{aligned}&W'(t)\\ &= \gamma R^{\gamma-1}(\tau_1(t))R'(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma\\ &\quad + R^\gamma(\tau_1(t))\left(\frac{r_1(t)((r_2(t)z'(t))')^\gamma}{(r_2(\tau_1(t))z'(\tau_1(t)))^\gamma}\right)\\ &\le \gamma R^{\gamma-1}(\tau_1(t))R'(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma - \frac{R^\gamma(\tau_1(t))q_1(t)z'(\tau_1(t))}{(r_2(\tau_1(t))z'(\tau_1(t)))^\gamma}\\ &\quad - R^\gamma(\tau_1(t))\frac{r_1(t)((r_2(t)z'(t))')^\gamma(r_2(\tau_1(t))z'(\tau_1(t)))'\tau_1(t))}{(r_2(\tau_1(t))z'(\tau_1(t)))^{\gamma+1}}\\ &\le \gamma R^{\gamma-1}(\tau_1(t))R'(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma - \left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))}\right)^\gamma q_1(t)\\ &\quad - R^\gamma(\tau_1(t))\frac{r_1(t)z_1'(t)((a(t)z'(t))')^{\gamma+1}}{(a(\tau_1(t))z'(\tau_1(t)))^{\gamma+1}}\\ &= \frac{\gamma R'(\tau_1(t))}{R(\tau_1(t))}R(\tau_1(t))r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(\tau_1(t))z'(\tau_1(t))}\right)^\gamma - \left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))}\right)^\gamma q_1(t)\end{aligned}$$

$$- \frac{\gamma\tau'(t)}{R(\tau_1(t))r_1^{\frac{1}{\gamma}}(\tau_1(t))} R^{\gamma+1}(\tau_1(t)) r_1^{\frac{\gamma+1}{\gamma}}(t) \left(\frac{(r_2(t)z_1'(t))'}{(r_2(\tau_1(t))z'(\tau_1(t))} \right)^{\gamma+1}$$

$$= - \left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))} \right)^{\gamma} q_1(t) + \left(W(t) - W^{\frac{\gamma+1}{\gamma}}(t) \right) \frac{\gamma R'(\tau_1(t))}{R(\tau_1(t))}.$$

It can be known from Lemma 3.4 that,

$$W'(t) \le - \left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))} \right)^{\gamma} q_1(t) + \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} \frac{R'(\tau_1(t))}{R(\tau_1(t))}.$$

Integrating both sides of the above equation, we have:

$$W(t) \le W(t_2) - \int_{t_2}^{t} \left[\left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s))} \right)^{\gamma} q_1(s) + \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} \frac{R'(\tau_1(s))}{R(\tau_1(s))} \right] ds.$$

Assume $t \to +\infty$. Considering equation (4.1), we know that, $W(t) \to -\infty$, which is contradictory with $W(t) > 0$. Therefore, $x(t)$ oscillates.

Then, if $z(t)$ satisfy property (2), we can know from equation (3.2) and Lemma 3.1 that $\lim\limits_{t\to+\infty} x(t) = 0$. The proof of the theorem is over. ∎

Deduction 4.2 *If equation* (4.1) *in Theorem* 4.1 *is substituted with*

$$\liminf_{t\to+\infty} \frac{1}{\ln R(\tau_1(t))} \int_{t_1}^{t} \left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s))} \right)^{\gamma} q_1(s)ds > \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1},$$

then the conclusion of Theorem 4.1 *still holds.*

Proof. If we only consider $z(t)$ satisfying property (1), then for all sufficiently large t, there exists $\varepsilon > 0$ and $t_2 \ge t_1$ that can make:

$$\frac{1}{\ln R(\tau_1(t))} \int_{t_2}^{t} \left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s))} \right)^{\gamma} q_1(s)ds \ge \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} + \varepsilon$$

that is,

$$\int_{t_2}^{t} \left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s))} \right)^{\gamma} q_1(s)ds - \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} \ln R(\tau_1(t)) \ge \varepsilon \ln R(\tau_1(t)).$$

Therefore,

$$\begin{aligned} \int_{t_2}^{t} \left[\left(\frac{\beta\tau_1(s)R(\tau_1(s))}{r_2(\tau_1(s))} \right)^{\gamma} q_1(s)ds - \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} \frac{R'(\tau_1(s))}{R(\tau_1(s))} \right] ds \\ \le \varepsilon \ln R(\tau_1(t)) + \left(\frac{\gamma+1}{\gamma} \right)^{\gamma+1} \ln R(\tau_1(t_2)) \end{aligned} \tag{4.3}$$

It can be known from (H1) that $\lim\limits_{t\to+\infty} R(\tau_1(t)) = \infty$. Hence, equation (4.3) ensures the holding of equation (4.1). ∎

Deduction 4.3 *If equation* (4.1) *in Theorem* 4.1 *is substituted with*

$$\liminf_{t\to+\infty}\left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))}\right)^{\gamma}\tau_1(t)\frac{R(\tau_1(t))}{R'(\tau_1(t))}>\left(\frac{\gamma+1}{\gamma}\right)^{\gamma+1}, \tag{4.4}$$

then the conclusion of Theorem 4.1 *holds.*

Proof. If equation (4.4) holds, then for all sufficiently large t, there exists $\varepsilon>0$ and $t_2\ge t_1$ that can make

$$\left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))}\right)^{\gamma}\tau_1(t)\frac{R(\tau_1(t))}{R'(\tau_1(t))}\ge\left(\frac{\gamma+1}{\gamma}\right)^{\gamma+1}+\varepsilon.$$

Multiplying both sides of the above equation with $\frac{R'(\tau_1(t))}{R(\tau_1(t))}$, we have:

$$\left(\frac{\beta\tau_1(t)R(\tau_1(t))}{r_2(\tau_1(t))}\right)^{\gamma}\tau_1(t)-\left(\frac{\gamma+1}{\gamma}\right)^{\gamma+1}\frac{R(\tau_1(t))}{R'(\tau_1(t))}\ge\varepsilon\frac{R(\tau_1(t))}{R'(\tau_1(t))}.$$

Therefore, the above equation ensures the holding of equation (4.4).

Next, we will prove the Philos oscillation criteria of equation (2.3). Considering set $D_0=\{(t,s):t>s>T_0\}$ and $D=\{(t,s):t\ge s>T_0\}$, we call the function H possesses property P if the function $H\in C(D,R)$ and satisfies the following conditions:

(I) $H(t,t)=0, t\ge t_0, H(t,s)>0, (t,s)\in D_0$;

(II) For all $(t,s)\in D_0$, $\frac{\partial H(t,s)}{\partial s}$ is continuous and non-positive.

Theorem 4.4 *Suppose* (H1)$\sim$(H5) *and equation* (3.2) *hold. Then there is the* P *function* $H(t,s), h\in C(D_0,R)$ *and* $\rho\in C'(I,R^+)$ $(I=[t_0,\infty),\ R^+=(0,+\infty))$ *that can make*

$$-\frac{\partial H(t,s)}{\partial s}=h(t,s)H^{\frac{\gamma}{\gamma+1}}(t,s)+\frac{\rho'(s)}{\rho(s)}H(t,s) \tag{4.5}$$

and

$$\limsup_{t\to+\infty}\frac{1}{H(t,t_0)}\int_{t_0}^{t}\left(H(t,s)Q(s)-\frac{\rho(s)r_1(s)}{(\gamma+1)^{\gamma+1}}\left|h(t,s)\right|^{\gamma+1}\right)ds=+\infty, \tag{4.6}$$

where

$$Q(s)=\frac{\rho(s)q_1(s)}{r_2^{\gamma}(s)}\left(\frac{\beta\theta\tau'(s)}{s}\right)^{\gamma},$$

$\beta,\theta,g_1(s)$ *and* $q_1(s)$ *are defined according to Lemmas* 3.2 *and* 3.3. *Then, every solution of equation* (2.3), *i.e., oscillates or satisfies* $\lim_{t\to+\infty}x(t)=0$.

Proof. If $x(t)$ is a non-oscillatory solution of equation (2.3) and does not lose generality, then we can suppose $x(t) > 0$. Similar to the proof of Theorem 4.1, we consider $z(t)$ satisfies property (1) or (2). If $z(t)$ satisfies property (1), i.e., there exists $t_1 \geq t_0$ that can make

$$z(t) > 0, r_2(t)z'(t) > 0, (r_2(t)z'(t))' > 0,$$

when $t \geq t_1$.

Moreover, equation (4.2) holds. The function $W(t)$ is defined as follows.

$$W(t) = \rho(t)r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(t)z'(t)}\right)^{\gamma},\ t \geq t_1.$$

Then $W(t) > 0$ $(t \geq t_1)$. Combining equation (2.3) and (4.2), we have:

$$\begin{aligned} W(t) &= \rho'(t)r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(t)z'(t)}\right)^{\gamma} + \rho(t)\left(\frac{r_1(t)((r_2(t)z'(t))')^{\gamma}}{(r_2(t)z'(t))^{\gamma}}\right)' \\ &= \rho'(t)r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(t)z'(t)}\right)^{\gamma} + \rho(t)\left(\frac{r_1(t)((r_2(t)z'(t))')^{\gamma}}{(r_2(t)z'(t))^{\gamma}}\right)' \\ &\quad - \rho(t)\frac{\gamma r_1(t)((r_2(t)z'(t))')^{\gamma+1}}{(r_2(t)z'(t))^{\gamma+1}} \\ &\leq \frac{\rho'(t)}{\rho(t)}\rho(t)r_1(t)\left(\frac{(r_2(t)z'(t))'}{r_2(t)z'(t)}\right)^{\gamma} - \frac{\rho'(t)q_1(t)}{r_2^{\gamma}(t)}\left(\frac{z(\tau_1(t))}{z'(t)}\right)^{\gamma} \\ &\quad - \frac{\gamma}{(\rho(t)r_1(t))^{\frac{1}{\gamma}}}\rho^{1+\frac{1}{\gamma}}(t)r_1^{1+\frac{1}{\gamma}}(t)\left(\frac{(r_2(t)z_1'(t))'}{r_2(t)z_1'(t)}\right)^{\gamma+1} \\ &= \frac{\rho'(t)}{\rho(t)}W(t) - \frac{\rho'(t)q_1(t)}{r_2^{\gamma}(t)}\left(\frac{z(\tau_1(t))}{z'(t)}\right)^{\gamma} - \frac{\gamma}{(\rho(t)r_1(t))^{\frac{1}{\gamma}}}W^{\frac{1+\gamma}{\gamma}}(t), \end{aligned}$$

that is,

$$W'(t) \leq \frac{\rho'(t)}{\rho(t)}W(t) - \frac{\rho'(t)q_1(t)}{r_2^{\gamma}(t)}\left(\frac{z(\tau_1(t))}{z'(t)}\right)^{\gamma} - \frac{\gamma}{(\rho(t)r_1(t))^{\frac{1}{\gamma}}}W^{\frac{1+\gamma}{\gamma}}(t). \tag{4.7}$$

In Theorem 4.4, suppose $w(t) = z'(t)$. Then, we get $\frac{1}{z'(t)} \geq \frac{\theta\tau_1(t)}{tz'(\tau_1(t))}$, $t \geq T_\theta \geq t_1$. Combining this with Theorem 4, we have:

$$\frac{1}{z'(\tau_1(t))} \geq \frac{\beta\tau_1(t)}{z(\tau_1(t))},\ t \geq T_\beta \geq T_\theta,$$

then $\frac{1}{z'(t)} \geq \frac{\theta\beta\tau_1^2(t)}{tz(\tau_1(t))}, t \geq T_\beta$. Combining this with equation (4.7), we have:

$$W'(t) \leq \frac{\rho'(t)}{\rho(t)}W(t) - \frac{\rho(t)q_1(t)}{r_2^{\gamma}(t)}\left(\frac{\beta\theta\tau_1'(t)}{t}\right)^{\gamma} - \frac{\gamma}{(\rho(t)r_1(t))^{\frac{1}{\gamma}}}W^{\frac{1+\gamma}{\gamma}}(t),\ t \geq T_\beta. \tag{4.8}$$

Both sides of equation (4.8) multiply with $H(t,s)$ and then it is integrated from $T(\geq T_\beta)$ to t. Combining the above result with equation (4.5), we get the following equation:

$$\begin{aligned}
&\int_T^t H(t,s)Q(s)ds \\
&\leq \int_T^t H(t,s)\left[-W'(s)+\frac{\rho'(s)}{\rho(s)}W(s)-\gamma(\rho(s)r_1(s))^{-\frac{1}{\gamma}}W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \\
&\leq -H(t,s)W(s)\left.\right|_T^t + \\
&\quad \int_T^t\left[\frac{\partial H(t,s)}{\partial s}W(s)+\frac{\rho'(s)}{\rho(s)}H(t,s)W(s)-\gamma H(t,s)(\rho(s)r_1(s))^{-\frac{1}{\gamma}}W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \\
&= H(t,T)W(T)+ \\
&\quad \int_T^t\left[-h(t,s)H^{\frac{\gamma+1}{\gamma}}(t,s)W(s)-\gamma H(t,s)(\rho(s)r_1(s))^{-\frac{1}{\gamma}}W^{\frac{\gamma+1}{\gamma}}(s)\right]ds \\
&\leq H(t,T)W(T)+ \\
&\quad \int_T^t\left[|h(t,s)|\,H^{\frac{\gamma+1}{\gamma}}(t,s)W(s)-\gamma H(t,s)(\rho(s)r_1(s))^{-\frac{1}{\gamma}}W^{\frac{\gamma+1}{\gamma}}(s)\right]ds
\end{aligned} \tag{4.9}$$

Combining the above result with Theorem 5, we suppose

$$A=W(s),\ B=\frac{\rho(s)r_1(s)\,|h(t,s)|^{\gamma}}{(\gamma+1)^{\gamma}H^{\frac{\gamma}{\gamma+1}}(t,s)},\ \lambda=1+\frac{1}{\gamma}.$$

Then, we have:

$$\left(1+\frac{1}{\gamma}\right)W(s)\frac{(\rho(s)r_1(s))^{\frac{1}{\gamma}}\,|h(t,s)|^{\gamma}}{(\gamma+1)H^{\frac{\gamma}{\gamma+1}}(t,s)}-W^{1+\frac{1}{\gamma}}(s)\leq\frac{1}{\gamma}\frac{(\rho(s)r_1(s))^{\frac{\gamma+1}{\gamma}}\,|h(t,s)|^{\gamma+1}}{(\gamma+1)^{\gamma+1}H(t,s)}.$$

Multiplying both ends of the above equation with $\gamma H(t,s)(\rho(s)r_1(s))^{-\frac{1}{\gamma}}$, we have:

$$|h(t,s)|\,H^{1+\frac{1}{\gamma}}(t,s)W(s)-\gamma H(t,s)(\rho(s)r_1(s))^{-\frac{1}{\gamma}}W^{1+\frac{1}{\gamma}}(s)\leq\frac{\rho(s)r_1(s)\,|h(t,s)|^{\gamma+1}}{(\gamma+1)^{\gamma}}.$$

Then, equation (4.9) transforms into:

$$\int_T^t H(t,s)Q(s)ds\leq H(t,T)W(T)+\int_T^t\frac{\rho(s)r_1(s)\,|h(t,s)|^{\gamma+1}}{(\gamma+1)^{1+\gamma}}ds.$$

Therefore,

$$\frac{1}{H(t,T_\beta)}\int_{T_\beta}^t\left(H(t,s)Q(s)-\frac{\rho(s)r_1(s)\,|h(t,s)|^{\gamma+1}}{(\gamma+1)^{1+\gamma}}\right)ds\leq W(T_\beta),$$

which is contradictory with equation (4.6). It follows that $x(t)$ is the oscillation solution of equation (2.3).

Then, if $x(t)$ satisfies property (2), we can get $\lim_{t\to+\infty} x(t) = 0$ from equation (3.2) and the theorem. ■

5. Some properties of autonilpotent groups

Note that when $r_2(t) = 1$, equation (2.3) transforms into:

$$\{r(t)((x(t) + p(t)x(\sigma(t)))')')^{\gamma}\}' + \int_a^b q(t,\xi)f(x(\tau(t,\xi)))d\xi = 0, \ t \geq t_0.$$

Corresponding theorem of the above equation can be obtained from the theorem of this study. Therefore, we can say, theorem of this study generalizes corresponding theorem of the above equation.

6. Conclusion

In conclusion, this study established oscillatory criteria for equation (2.3). The novelty of this study lies in the innovation of the differential equations we studied, understandable inference process and convenient use of the conclusion obtained. However, we found that, the restrictive conditions given by the equation were too strict. Whether restrictive conditions can be reduced or relaxed and whether such kind of method can be used for exploring oscillation of higher-order functional differential equation are expected to be further explored in the future.

Exploration of boundary value problem of differential equation is an important part of differential equation, which is of great significance to many problems theoretically. Due to practical condition and demand of practical engineering, more and more scholars will deeply discuss over boundary value problem of differential equation. Chinese financial industry develops rapidly currently, which increases the complexity and uncertainty of research problems. Most specialists consider that boundary value problem of random differential equation will be the hot research direction of boundary value problem in the future.

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