UP-ALGEBRAS CHARACTERIZED BY THEIR ANTI-FUZZY UP-IDEALS AND ANTI-FUZZY UP-SUBALGEBRAS

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Abstract. In this paper, anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras concepts of UP-algebras are introduced and proved some results. We also introduce the notions of Cartesian product and dot product of fuzzy sets, and then we study related properties. Further, we discuss the relation between anti-fuzzy UP-ideals (resp. anti-fuzzy UP-subalgebras) and level subsets of a fuzzy set. Anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras are also applied in the Cartesian product of UP-algebras.

Keywords: UP-algebra, anti-fuzzy UP-ideal, anti-fuzzy UP-subalgebra.

Mathematics Subject Classification: 03G25.

1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [9], BCI-algebras [10], BCH-algebras [7], KU-algebras [27], SU-algebras [17] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [10] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [9], [10] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

A fuzzy subset \( f \) of a set \( S \) is a function from \( S \) to a closed interval \([0, 1]\). The concept of a fuzzy subset of a set was first considered by Zadeh [38] in 1965. The

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fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.


Iampan [8] now introduced a new algebraic structure, called a UP-algebra. The notions of fuzzy subalgebras and fuzzy ideals play an important role in studying the many logical algebras. Somjanta, Thuekaew, Kumpuangkeaw and Iampan [34] introduced and studied fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras, and investigated some of its properties. In this paper, anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras concepts of UP-algebras are introduced and proved some results. Further, we discuss the relation between anti-fuzzy UP-ideals (resp. anti-fuzzy UP-subalgebras) and level subsets of a fuzzy set. Anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras are also applied in the Cartesian product of UP-algebras.
Before we begin our study, we will introduce the definition of a UP-algebra.

**Definition 1.1.** [8] An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a UP-algebra if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = y \cdot x = 0$ implies $x = y$.

**Example 1.2.** [8] Let $X$ be a set. Define a binary operation $\cdot$ on the power set of $X$ by putting $A \cdot B = B \cap A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); \cdot, 0)$ is a UP-algebra.

**Example 1.3.** [8] Let $A = \{0, a, b, c\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>a</th>
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<th>c</th>
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<td>c</td>
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<td>a</td>
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</table>

Then $(A; \cdot, 0)$ is a UP-algebra.

In what follows, let $A$ and $B$ denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

**Proposition 1.4.** [8] In a UP-algebra $A$, the following properties hold: for any $x, y \in A$,

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ and $y \cdot z = 0$ imply $x \cdot z = 0$,

(3) $x \cdot y = 0$ implies $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ implies $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ if and only if $x = y \cdot x$, and

(7) $x \cdot (y \cdot y) = 0$.

On a UP-algebra $A = (A; \cdot, 0)$, we define a binary relation $\leq$ on $A$ [8] as follows: for all $x, y \in A$,

\[
x \leq y \text{ if and only if } x \cdot y = 0.
\] (1.1)
Proposition 1.5. [8] In a UP-algebra $A$, the following properties hold: for any $x, y \in A$,

1. $x \leq x$,
2. $x \leq y$ and $y \leq x$ imply $x = y$,
3. $x \leq y$ and $y \leq z$ imply $x \leq z$,
4. $x \leq y$ implies $z \cdot x \leq z \cdot y$,
5. $x \leq y$ implies $y \cdot z \leq x \cdot z$,
6. $x \leq y \cdot x$,
7. $x \leq y \cdot y$.

Proposition 1.6. [8] An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ with a binary relation $\leq$ defined by (1.1) is a UP-algebra if and only if it satisfies the following conditions: for all $x, y, z \in A$,

1. $(y \cdot z) \leq (x \cdot y) \cdot (x \cdot z)$,
2. $0 \cdot x = x$,
3. $x \leq 0$, and
4. $x \leq y$ and $y \leq x$ imply $x = y$.

Definition 1.7. [8] A nonempty subset $B$ of $A$ is called a UP-ideal of $A$ if it satisfies the following properties:

1. the constant 0 of $A$ is in $B$, and
2. for any $x, y, z \in A$, $x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, $A$ and $\{0\}$ are UP-ideals of $A$.

Theorem 1.8. [8] Let $A$ be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-ideals of $A$. Then $\bigcap_{i \in I} B_i$ is a UP-ideal of $A$.

Definition 1.9. [8] A subset $S$ of $A$ is called a UP-subalgebra of $A$ if the constant 0 of $A$ is in $S$, and $(S; \cdot, 0)$ itself forms a UP-algebra. Clearly, $A$ and $\{0\}$ are UP-subalgebras of $A$.

Applying Proposition 1.4 (1), we can then easily prove the following proposition.

Proposition 1.10. [8] A nonempty subset $S$ of a UP-algebra $A = (A; \cdot, 0)$ is a UP-subalgebra of $A$ if and only if $S$ is closed under the $\cdot$ multiplication on $A$.

Theorem 1.11. [8] Let $A$ be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-subalgebras of $A$. Then $\bigcap_{i \in I} B_i$ is a UP-subalgebra of $A$. 
**Definition 1.12.** [38] A *fuzzy set* in a nonempty set $X$ (or a fuzzy subset of $X$) is an arbitrary function $f: X \to [0,1]$ where $[0,1]$ is the unit segment of the real line.

**Definition 1.13.** A fuzzy set $f$ in $A$ is called an *anti-fuzzy UP-ideal* of $A$ if it satisfies the following properties: for any $x, y, z \in A$,

1. $f(0) \leq f(x)$, and
2. $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$.

**Definition 1.14.** A fuzzy set $f$ in $A$ is called an *anti-fuzzy UP-subalgebra* of $A$ if for any $x, y \in A$,

$$f(x \cdot y) \leq \max\{f(x), f(y)\}.$$ 

**Example 1.15.** Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

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<th>1</th>
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<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>1</td>
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<tr>
<td>1</td>
<td>0</td>
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</table>

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set $f$ in $A$ as follows:

$$f(0) = 0.1 \text{ and } f(1) = 0.2.$$ 

Using this data, we can show that $f$ is an anti-fuzzy UP-ideal of $A$ and an anti-fuzzy UP-subalgebra of $A$.

**Example 1.16.** Let $A = \{0, a, b\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

<table>
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<tr>
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<th>0</th>
<th>a</th>
<th>b</th>
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</thead>
<tbody>
<tr>
<td>0</td>
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<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>a</td>
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<td>b</td>
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<td>0</td>
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</tbody>
</table>

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set $f$ in $A$ as follows:

$$f(0) = 0.1, f(a) = 0.2 \text{ and } f(b) = 0.2.$$ 

Using this data, we can show that $f$ is an anti-fuzzy UP-subalgebra of $A$.

**Definition 1.17.** [34] Let $f$ be a fuzzy set in $A$. The fuzzy set $\overline{f}$ defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in A$ is called the *complement* of $f$ in $A$.

**Remark 1.18.** For all fuzzy set $f$ in $A$, we have $f = \overline{\overline{f}}$.

**Definition 1.19.** [34] Let $f$ be a fuzzy set in $A$. For any $t \in [0,1]$, the sets

$$U(f; t) = \{x \in A \mid f(x) \geq t\} \text{ and } U^+(f; t) = \{x \in A \mid f(x) > t\}$$

are called an *upper $t$-level subset* and an *upper $t$-strong level subset* of $f$, respectively. The sets
$L(f; t) = \{x \in A \mid f(x) \leq t\}$ and $L^-(f; t) = \{x \in A \mid f(x) < t\}$ are called a lower $t$-level subset and a lower $t$-strong level subset of $f$, respectively.

**Definition 1.20.** [21] Let $f$ be a function from a nonempty set $X$ to a nonempty set $Y$. If $\mu$ is a fuzzy set in $X$, then fuzzy set $\beta$ in $Y$ defined by

$$\beta(y) = \begin{cases} \inf \{\mu(t)\}_{t \in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

is said to be the image of $\mu$ under $f$. Similarly, if $\beta$ is a fuzzy set in $Y$, then the fuzzy set $\mu = \beta \circ f$ in $X$ (i.e., the fuzzy set defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the preimage of $\beta$ under $f$.

**Definition 1.21.** [29] A fuzzy set $f$ in $A$ has inf property if for any nonempty subset $T$ of $A$, there exists $t_0 \in T$ such that $f(t_0) = \inf\{f(t)\}_{t \in T}$.

**Definition 1.22.** [5] A fuzzy relation on a nonempty set $X$ is an arbitrary function $f: X \times X \to [0, 1]$ where $[0, 1]$ is the unit segment of the real line.

**Definition 1.23.** [24] Let $f$ and $g$ be fuzzy sets in nonempty sets $A$ and $B$, respectively. The Cartesian product of $f$ and $g$ is $f \times g: A \times B \to [0, 1]$ defined by

$$(f \times g)(x, y) = \max\{f(x), g(y)\} \text{ for all } x \in A \text{ and } y \in B.$$  

The dot product of $f$ and $g$ is $f \cdot g: A \times B \to [0, 1]$ defined by

$$(f \cdot g)(x, y) = \min\{f(x), g(y)\} \text{ for all } x \in A \text{ and } y \in B.$$  

**Definition 1.24.** [24] If $f$ is a fuzzy set in a nonempty set $X$, the strongest fuzzy relation on $X$ is $\mu_f: X \times X \to [0, 1]$ defined by $\mu_f(x, y) = \max\{f(x), f(y)\}$ for all $x, y \in X$. For $x, y \in X$, we have $f(x), f(y) \in [0, 1]$. Thus $\mu_f(x, y) = \max\{f(x), f(y)\} \in [0, 1]$. Hence, $\mu_f$ is a fuzzy relation on $X$. We note that if $f$ is a fuzzy set in a nonempty set $X$, then $f \times f = \mu_f$.

**Definition 1.25.** If $f$ is a fuzzy set in a nonempty set $X$, the weakness fuzzy relation on $X$ is $\beta_f: X \times X \to [0, 1]$ defined by $\beta_f(x, y) = \min\{f(x), f(y)\}$ for all $x, y \in X$. For $x, y \in X$, we have $f(x), f(y) \in [0, 1]$. Thus $\beta_f(x, y) = \min\{f(x), f(y)\} \in [0, 1]$. Hence, $\beta_f$ is a fuzzy relation on $X$. We note that if $f$ is a fuzzy set in a nonempty set $X$, then $f \cdot f = \beta_f$.

**Definition 1.26.** [21] Let $X$ and $Y$ be any two nonempty sets and let $f: X \to Y$ be any function. A fuzzy set $\mu$ in $X$ is called $f$-invariant if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in X$.

**Definition 1.27.** [8] Let $(A; \cdot, 0)$ and $(A'; \cdot', 0')$ be UP-algebras. A mapping $f$ from $A$ to $A'$ is called a UP-homomorphism if

$$f(x \cdot y) = f(x) \cdot' f(y) \text{ for all } x, y \in A.$$  

A UP-homomorphism $f: A \to A'$ is called a
(1) *UP-endomorphism* of $A$ if $A' = A$,
(2) *UP-epimorphism* if $f$ is surjective,
(3) *UP-monomorphism* if $f$ is injective, and
(4) *UP-isomorphism* if $f$ is bijective. Moreover, we say $A$ is *UP-isomorphic* to $A'$, symbolically, $A \cong A'$, if there is a UP-isomorphism from $A$ to $A'$.

Let $f$ be a mapping from $A$ to $A'$, and let $B$ be a nonempty subset of $A$, and $B'$ of $A'$. The set $\{ f(x) \mid x \in B \}$ is called the *image* of $B$ under $f$, denoted by $f(B)$. In particular, $f(A)$ is called the *image* of $f$, denoted by $\text{Im}(f)$. Dually, the set $\{ x \in A \mid f(x) \in B' \}$ is said the *inverse image* of $B'$ under $f$, symbolically, $f^{-1}(B')$. Especially, we say $f^{-1}({0'})$ is the *kernel* of $f$, written by $\text{Ker}(f)$. That is,

\[
\text{Im}(f) = \{ f(x) \in A' \mid x \in A \}
\]

and

\[
\text{Ker}(f) = \{ x \in A \mid f(x) = 0' \}.
\]

**Theorem 1.28.** [8] Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let $f : A \to B$ be a UP-homomorphism. Then the following statements hold:

(1) $f(0_A) = 0_B$, and
(2) for any $x, y \in A$, if $x \leq y$, then $f(x) \leq f(y)$.

**Lemma 1.29.** [34] Let $f$ be a fuzzy set in $A$. Then the following statements hold: for any $x, y \in A$,

(1) $1 - \max\{ f(x), f(y) \} = \min\{ 1 - f(x), 1 - f(y) \}$, and
(2) $1 - \min\{ f(x), f(y) \} = \max\{ 1 - f(x), 1 - f(y) \}$.

**2. Main results**

In this section, we study anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras.

**Theorem 2.1.** Every anti-fuzzy UP-ideal of $A$ is an anti-fuzzy UP-subalgebra of $A$.

**Proof.** Let $f$ be an anti-fuzzy UP-ideal of $A$. Let $x, y \in A$. Then

\[
f(x \cdot y) \leq \max\{ f(x \cdot (y \cdot y)), f(y) \} \quad \text{(Definition 1.13 (2))}
\]

\[
= \max\{ f(x \cdot 0), f(y) \} \quad \text{(Proposition 1.4 (1))}
\]

\[
= \max\{ f(0), f(y) \} \quad \text{(UP-3)}
\]

\[
= f(y) \quad \text{(Definition 1.13 (1))}
\]

\[
\leq \max\{ f(x), f(y) \}.
\]

Hence, $f$ is an anti-fuzzy UP-subalgebra of $A$.  


Lemma 2.2. Let \( f \) be an anti-fuzzy UP-ideal of \( A \). If the inequality \( x \leq y \cdot z \) holds in \( A \) for all \( x, y, z \in A \), then \( f(z) \leq \max\{f(x), f(y)\} \) for all \( x, y, z \in A \).

**Proof.** Assume \( x \leq y \cdot z \) for all \( x, y, z \in A \). Then \( x \cdot (y \cdot z) = 0 \). By Definition 1.13 (2), we have
\[
f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}.
\] (2.1)

By (2.1) and (UP-2), let \( x = 0 \), so
\[
f(z) = f(0 \cdot z) \leq \max\{f(0 \cdot (x \cdot z)), f(x)\} = \max\{f(x \cdot z), f(x)\}.
\] (2.2)

By (2.2) and (2.3), we have
\[
f(z) \leq \max\{f(x \cdot z), f(x)\} \leq \max\{f(y), f(x)\} = \max\{f(x), f(y)\}.
\]

Lemma 2.3. If \( f \) is an anti-fuzzy UP-ideal of \( A \) and if \( x, y \in A \) is such that \( x \leq y \) in \( A \), then \( f(x) \geq f(y) \).

**Proof.** Let \( x, y \in A \) be such that \( x \leq y \) in \( A \). Then \( x \cdot y = 0 \). Thus
\[
f(y) = f(0 \cdot y) \leq \max\{f(0 \cdot (y \cdot y)), f(x)\} = \max\{f(0 \cdot 0), f(x)\} = \max\{f(0), f(x)\} = f(x).
\] ((UP-2))

Lemma 2.4. [26] Let \( f \) be a fuzzy set in \( A \). For any \( t \in [0, 1] \), the following properties hold:

1. \( L(f; t) = U(\overline{f}; 1 - t) \),
2. \( L^{-}(f; t) = U^{+}(\overline{f}; 1 - t) \),
3. \( U(f; t) = L(\overline{f}; 1 - t) \), and
4. \( U^{+}(f; t) = L^{-}(\overline{f}; 1 - t) \).

Lemma 2.5. [20] For any \( a, b \in \mathbb{R} \) such that \( a < b \), \( a < \frac{b + a}{2} < b \).

Theorem 2.6. Let \( f \) be a fuzzy set in \( A \). Then the following statements hold:

1. \( f \) is an anti-fuzzy UP-ideal of \( A \) if and only if it satisfies the condition 
   \( (*) \): for all \( t \in [0, 1] \), \( L(f; t) \neq \emptyset \) implies \( L(f; t) \) is a UP-ideal of \( A \).
(2) \( f \) is an anti-fuzzy UP-ideal of \( A \) if and only if it satisfies the condition
\((\ast)\): for all \( t \in [0, 1] \), \( L^{-}(f; t) \neq \emptyset \) implies \( L^{-}(f; t) \) is a UP-ideal of \( A \),

(3) \( f \) is an anti-fuzzy UP-ideal of \( A \) if and only if it satisfies the condition
\((\ast)\): for all \( t \in [0, 1] \), \( U^{+}(f; t) \neq \emptyset \) implies \( U(f; t) \) is a UP-ideal of \( A \), and

(4) \( f \) is an anti-fuzzy UP-ideal of \( A \) if and only if it satisfies the condition
\((\ast)\): for all \( t \in [0, 1] \), \( U^{+}(f; t) \neq \emptyset \) implies \( U(f; t) \) is a UP-ideal of \( A \).

**Proof.** (1) Assume that \( f \) is an anti-fuzzy UP-ideal of \( A \). Let \( t \in [0, 1] \) be such that \( L(f; t) \neq \emptyset \). Let \( x \in A \) be such that \( x \in L(f; t) \). Then \( f(x) \leq t \). Thus

\[
\begin{align*}
f(0) &= f(x \cdot 0) \\
&\leq \max\{f(x \cdot (x \cdot 0)), f(x)\} \quad \text{(UP-3)} \\
&= \max\{f(x \cdot 0), f(x)\} \quad \text{(Definition 1.13(2))} \\
&= \max\{f(0), f(x)\} \quad \text{(UP-3)} \\
&= f(x) \quad \text{(Definition 1.13(1))} \\
&\leq t.
\end{align*}
\]

Thus \( 0 \in L(f; t) \). Now, let \( x, y, z \in A \) be such that \( x \cdot (y \cdot z) \in L(f; t) \) and \( y \in L(f; t) \). Then \( f(x \cdot (y \cdot z)) \leq t \) and \( f(y) \leq t \). By Definition 1.13 (2), we have \( f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} \leq t \), so \( x \cdot z \in L(f; t) \). Hence, \( L(f; t) \) is a UP-ideal of \( A \).

Conversely, assume that condition \((\ast)\) holds and suppose that Definition 1.13 (1) is false. Then there exists \( x \in A \) such that \( f(0) > f(x) \). Let \( t = \frac{f(x) + f(0)}{2} \). Then \( t \in [0, 1] \) and by Lemma 2.5, we have \( f(0) > t > f(x) \). Thus \( x \in L(f; t) \), so \( L(f; t) \neq \emptyset \). By assumption, \( L(f; t) \) is a UP-ideal of \( A \). Thus \( 0 \in L(f; t) \), so \( f(0) \leq t \). This is a contradiction. Hence, \( f(0) \leq f(x) \) for all \( x \in A \). Suppose that Definition 1.13 (2) is false. Then there exist \( x, y, z \in A \) such that \( f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\} \). Let \( g_0 = \frac{f(x \cdot z) + \max\{f(x \cdot (y \cdot z)), f(y)\}}{2} \). Then \( g_0 \in [0, 1] \) and by Lemma 2.5, we have \( \max\{f(x \cdot (y \cdot z)), f(y)\} < g_0 < f(x \cdot z) \). Thus \( x \cdot z \notin L(f; g_0) \). Since \( \max\{f(x \cdot (y \cdot z)), f(y)\} < g_0 \), we have \( f(y) < g_0 \) and \( f(x \cdot (y \cdot z)) < g_0 \). Thus \( y \in L(f; g_0) \) and \( x \cdot (y \cdot z) \in L(f; g_0) \), so \( L(f; g_0) \neq \emptyset \). By assumption, we have \( L(f; g_0) \) is a UP-ideal of \( A \). Thus \( x \cdot z \in L(f; g_0) \). This is a contradiction. Hence, \( f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} \) for all \( x, y, z \in A \). Therefore, \( f \) is an anti-fuzzy UP-ideal of \( A \).

(2) Assume that \( f \) is an anti-fuzzy UP-ideal of \( A \). Let \( t \in [0, 1] \) be such that \( L^{-}(f; t) \neq \emptyset \). Let \( x \in A \) be such that \( x \in L^{-}(f; t) \). Then \( f(x) < t \). Thus

\[
\begin{align*}
f(0) &= f(x \cdot 0) \\
&\leq \max\{f(x \cdot (x \cdot 0)), f(x)\} \quad \text{(UP-3)} \\
&= \max\{f(x \cdot 0), f(x)\} \quad \text{(Definition 1.13(2))} \\
&= \max\{f(0), f(x)\} \quad \text{(UP-3)} \\
&= f(x) \quad \text{(Definition 1.13(1))} \\
&< t.
\end{align*}
\]
Thus $0 \in L^{-}(f; t)$. Now, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L^{-}(f; t)$ and $y \in L^{-}(f; t)$. Then $f(x \cdot (y \cdot z)) < t$ and $f(y) < t$. By Definition 1.13 (2), we have $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} < t$, so $x \cdot z \in L^{-}(f; t)$. Hence, $L^{-}(f; t)$ is a UP-ideal of $A$.

Conversely, assume that the condition $(\ast)$ holds and suppose that Definition 1.13 (1) is false. Then there exists $x \in A$ such that $f(0) > f(x)$. Let $t = f(x) + f(0)$. Then $t \in [0, 1]$ and by Lemma 2.5, we have $f(0) > t > f(x)$. Thus $x \in L^{-}(f; t)$, so $L^{-}(f; t) \neq \emptyset$. By assumption, $L^{-}(f; t)$ is a UP-ideal of $A$. Thus $0 \in L^{-}(f; t)$, so $f(0) < t$. This is a contradiction. Hence, $f(0) \leq f(x)$ for all $x \in A$. Suppose that Definition 1.13 (2) is false. Then there exist $x, y, z \in A$ such that $f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\}$. Let $g_{0} = \{f(x \cdot z) + \max\{f(x \cdot (y \cdot z)), f(y)\} \}$. Then $g_{0} \in [0, 1]$ and by Lemma 2.5, we have $\max\{f(x \cdot (y \cdot z)), f(y)\} < g_{0} < f(x \cdot z)$. Thus $x \cdot z \notin L^{-}(f; g_{0})$. Since $\max\{f(x \cdot (y \cdot z)), f(y)\} < g_{0}$, we have $f(y) < g_{0}$ and $f(x \cdot (y \cdot z)) < g_{0}$. Thus $y \in L^{-}(f; g_{0})$ and $x \cdot (y \cdot z) \in L^{-}(f; g_{0})$, so $L^{-}(f; g_{0}) \neq \emptyset$. By assumption, we have $L^{-}(f; g_{0})$ is a UP-ideal of $A$. Thus $x \cdot z \in L^{-}(f; g_{0})$. This is a contradiction. Hence, $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$ for all $x, y, z \in A$. Therefore, $f$ is an anti-fuzzy UP-ideal of $A$.

(3) Assume that $\bar{f}$ is an anti-fuzzy UP-ideal of $A$. Let $t \in [0, 1]$ be such that $U(f; t) \neq \emptyset$. Let $x \in A$ be such that $x \in U(f; t)$. Then $f(x) \geq t$. Thus

\[
\bar{f}(0) = \bar{f}(x \cdot 0) \leq \max\{\bar{f}(x \cdot (x \cdot 0)), \bar{f}(x)\} = \max\{\bar{f}(x \cdot 0), \bar{f}(x)\} = \max\{\bar{f}(0), \bar{f}(x)\} = \bar{f}(x).
\]

This implies that $1 - f(0) \leq 1 - f(x)$. Hence, $f(0) \geq f(x) \geq t$, so $0 \in U(f; t)$. Now, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(f; t)$ and $y \in U(f; t)$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$. By Definition 1.13 (2), we have $\bar{f}(x \cdot z) \leq \max\{\bar{f}(x \cdot (y \cdot z)), \bar{f}(y)\}$.

Thus

\[
1 - f(x \cdot z) \leq \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} = 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}.
\]

Thus $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} \geq t$, so $x \cdot z \in U(f; t)$. Hence, $U(f; t)$ is a UP-ideal of $A$.

Conversely, assume that condition $(\ast)$ holds and suppose that Definition 1.13 (1) is false. Then there exists $x \in A$ such that $\bar{f}(0) > \bar{f}(x)$. Thus $1 - f(0) > 1 - f(x)$, so $f(0) < f(x)$.

Let $t = \frac{f(x) + f(0)}{2}$. Then $t \in [0, 1]$ and by Lemma 2.5, we have $f(0) < t < f(x)$. Thus $x \in U(f; t)$, so $U(f; t) \neq \emptyset$. By assumption, $U(f; t)$ is a UP-ideal of $A$. Thus $0 \in U(f; t)$, so $f(0) \geq t$. This is a contradiction. Hence, $\bar{f}(0) \leq \bar{f}(x)$ for all $x \in A$. Suppose that Definition 1.13 (2) is false. Then
there exist \(x, y, z \in A\) such that \(\overline{f}(x \cdot z) > \max\{-1, f(x \cdot (y \cdot z)), f(y)\}\). Thus
\[
1 - f(x \cdot z) > \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\}
\]
\[
= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. \quad \text{(Lemma 1.29 (2))}
\]
Thus \(f(x \cdot z) < \min\{f(x \cdot (y \cdot z)), f(y)\}\). Let \(g_0 = f(x \cdot z) + \min\{f(x \cdot (y \cdot z)), f(y)\}\).

Thus \(g_0 \in [0, 1]\) and by Lemma 2.5, we have \(f(x \cdot z) < g_0 < \min\{f(x \cdot (y \cdot z)), f(y)\}\). Thus \(x \cdot z \notin U(f; g_0)\). Since \(\min\{f(x \cdot (y \cdot z)), f(y)\} > g_0\), we have \(f(y) > g_0\) and \(f(x \cdot (y \cdot z)) > g_0\). Thus \(y \in U(f; g_0)\) and \(x \cdot (y \cdot z) \in U(f; g_0)\), so \(U(f; g_0) \neq \emptyset\).

By assumption, we have \(U(f; g_0)\) is a UP-ideal of \(A\). Thus \(x \cdot z \in U(f; g_0)\). This is a contradiction. Hence, \(\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}\) for all \(x, y, z \in A\).

Therefore, \(\overline{f}\) is an anti-fuzzy UP-ideal of \(A\).

(4) Assume that \(\overline{f}\) is an anti-fuzzy UP-ideal of \(A\). Let \(t \in [0, 1]\) be such that \(\overline{f}(f; t) \neq \emptyset\). Let \(x \in A\) be such that \(x \in \overline{U}(f; t)\). Then \(f(x) > t\). Thus
\[
\overline{f}(0) = \overline{f}(x \cdot 0) \leq \max\{\overline{f}(x \cdot (x \cdot 0)), \overline{f}(x)\} \quad \text{(UP-3)}
\]
\[
= \max\{\overline{f}(x \cdot 0), \overline{f}(x)\} \quad \text{(Definition 1.13 (2))}
\]
\[
= \max\{\overline{f}(0), \overline{f}(x)\} \quad \text{(UP-3)}
\]
\[
= \overline{f}(x). \quad \text{(Definition 1.13 (1))}
\]
This implies that \(1 - f(0) \leq 1 - f(x)\). Hence, \(f(0) \geq f(x) > t\), so \(0 \in \overline{U}(f; t)\).

Now, let \(x, y, z \in A\) be such that \(x \cdot (y \cdot z) \in \overline{U}(f; t)\) and \(y \in \overline{U}(f; t)\). Then \(f(x \cdot (y \cdot z)) > t\) and \(f(y) > t\). By Definition 1.13 (2), we have \(\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}\). Thus
\[
1 - f(x \cdot z) \leq \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\}
\]
\[
= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. \quad \text{(Lemma 1.29 (2))}
\]
Thus \(f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} > t\), so \(x \cdot z \in \overline{U}(f; t)\). Hence, \(\overline{U}(f; t)\) is a UP-ideal of \(A\).

Conversely, assume that condition (\(\star\)) holds and suppose that Definition 1.13 (1) is false. Then there exists \(x \in A\) such that \(\overline{f}(0) > \overline{f}(x)\). Thus \(1 - f(0) > 1 - f(x)\), so \(f(0) < f(x)\). Let \(t = \frac{f(x) + f(0)}{2}\). Then \(t \in [0, 1]\) and by Lemma 2.5, we have \(f(0) < t < f(x)\). Thus \(x \in \overline{U}(f; t)\), so \(\overline{U}(f; t) \neq \emptyset\). By assumption, \(\overline{U}(f; t)\) is a UP-ideal of \(A\). Thus \(0 \in \overline{U}(f; t)\), so \(f(0) > t\). This is a contradiction. Hence, \(\overline{f}(0) \leq \overline{f}(x)\) for all \(x \in A\). Suppose that Definition 1.13 (2) is false. Then there exist \(x, y, z \in A\) such that \(\overline{f}(x \cdot z) > \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}\). Thus
\[
1 - f(x \cdot z) > \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\}
\]
\[
= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. \quad \text{(Lemma 1.29 (2))}
\]
Thus \(f(x \cdot z) < \min\{f(x \cdot (y \cdot z)), f(y)\}\). Let \(g_0 = f(x \cdot z) + \min\{f(x \cdot (y \cdot z)), f(y)\}\). Then \(g_0 \in [0, 1]\) and by Lemma 2.5, we have \(f(x \cdot z) < g_0 < \min\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}\).
Then \(x \cdot z \notin U^+(f; g_0)\). Since \(\min\{f(x \cdot (y \cdot z)), f(y)\} > g_0\), we have \(f(y) > g_0\) and \(f(x \cdot (y \cdot z)) > g_0\). Thus \(y \in U^+(f; g_0)\) and \(x \cdot (y \cdot z) \in U^+(f; g_0)\), so \(U^+(f; g_0) \neq \emptyset\). By assumption, we have \(U^+(f; g_0)\) is a UP-ideal of \(A\). Thus \(x \cdot z \in U^+(f; g_0)\). This is a contradiction. Hence, \(\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}\) for all \(x, y, z \in A\). Therefore, \(\overline{f}\) is an anti-fuzzy UP-ideal of \(A\). 

\[\text{Lemma 2.7.} \quad \text{Let } f \text{ be a fuzzy set in } A. \text{ Then the following statements hold:} \]

1. \(f\) is an anti-fuzzy UP-subalgebra of \(A\) if and only if it satisfies the condition (1): for all \(t \in [0, 1]\), \(L(f; t) \neq \emptyset\) implies \(L(f; t)\) is a UP-subalgebra of \(A\).

2. \(f\) is an anti-fuzzy UP-subalgebra of \(A\) if and only if it satisfies the condition (2): for all \(t \in [0, 1]\), \(L^-(f; t) \neq \emptyset\) implies \(L^-(f; t)\) is a UP-subalgebra of \(A\).

3. \(\overline{f}\) is an anti-fuzzy UP-subalgebra of \(A\) if and only if it satisfies the condition (3): for all \(t \in [0, 1]\), \(U(f; t) \neq \emptyset\) implies \(U(f; t)\) is a UP-subalgebra of \(A\), and

4. \(\overline{f}\) is an anti-fuzzy UP-subalgebra of \(A\) if and only if it satisfies the condition (4): for all \(t \in [0, 1]\), \(U^+(f; t) \neq \emptyset\) implies \(U^+(f; t)\) is a UP-subalgebra of \(A\).

\[\text{Proof. (1) Assume that } f \text{ is an anti-fuzzy UP-subalgebra of } A. \text{ Let } t \in [0, 1] \text{ be such that } L(f; t) \neq \emptyset. \text{ Let } x, y \in L(f; t). \text{ Then } f(x) \leq t \text{ and } f(y) \leq t. \text{ Since } f \text{ is an anti-fuzzy UP-subalgebra of } A, \text{ we have } f(x \cdot y) \leq \max\{f(x), f(y)\} \leq t. \text{ Thus } x \cdot y \in L(f; t). \text{ Hence, } L(f; t) \text{ is a UP-subalgebra of } A.

Conversely, assume that the condition (1) holds. Let \(x, y \in A\) and let \(t = \max\{f(x), f(y)\}, \text{ so } t \in [0, 1]\). Thus \(f(x) \leq t \text{ and } f(y) \leq t\), so \(x, y \in L(f; t)\). Thus \(L(f; t) \neq \emptyset\). By assumption, we have \(L(f; t)\) is a UP-subalgebra of \(A\). Thus \(x \cdot y \in L(f; t), \text{ so } f(x \cdot y) \leq t \leq \max\{f(x), f(y)\}\). Hence, \(f\) is an anti-fuzzy UP-subalgebra of \(A\).

(2) Assume that \(f\) is an anti-fuzzy UP-subalgebra of \(A\). Let \(t \in [0, 1] \text{ be such that } L^-(f; t) \neq \emptyset. \text{ Let } x, y \in L^-(f; t). \text{ Then } f(x) < t \text{ and } f(y) < t. \text{ Since } f \text{ is an anti-fuzzy UP-subalgebra of } A, \text{ we have } f(x \cdot y) \leq \max\{f(x), f(y)\} < t. \text{ Thus } x \cdot y \in L^-(f; t). \text{ Hence, } L^-(f; t) \text{ is a UP-subalgebra of } A.

Conversely, assume that the condition (2) holds. Suppose that there exist \(x, y \in A\) such that \(f(x \cdot y) > \max\{f(x), f(y)\}\). Then \(f(x \cdot y) \in [0, 1]\). Choose \(t = f(x \cdot y)\). Thus \(f(x) < t \text{ and } f(y) < t\), so \(x, y \in L^-(f; t) \neq \emptyset\). By assumption, we have \(L^-(f; t)\) is a UP-subalgebra of \(A\) and so \(x \cdot y \in L^-(f; t). \text{ Thus } f(x \cdot y) < t = f(x \cdot y) \text{ which is a contradiction. Hence, } f(x \cdot y) \leq \max\{f(x), f(y)\} \text{ for all } x, y \in A. \text{ Therefore, } f \text{ is an anti-fuzzy UP-subalgebra of } A.

(3) Assume that \(\overline{f}\) is an anti-fuzzy UP-subalgebra of \(A\). Let \(t \in [0, 1] \text{ be such that } U(f; t) \neq \emptyset. \text{ Let } x, y \in U(f; t). \text{ Then } f(x) \geq t \text{ and } f(y) \geq t. \text{ Since } \overline{f} \text{ is an anti-fuzzy UP-subalgebra of } A, \text{ we have } \overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}. \text{ Thus}

\[1 - f(x \cdot y) \leq \max\{1 - f(x), 1 - f(y)\} = 1 - \min\{f(x), f(y)\}. \quad \text{(Lemma 1.29(2))}\]
Thus $f(x \cdot y) \geq \min\{f(x), f(y)\} \geq t$. Thus $x \cdot y \in U(f; t)$. Hence, $U(f; t)$ is a UP-subalgebra of $A$.

Conversely, assume that the condition $(\ast)$ holds. Let $x, y \in A$ and let $t = \min\{f(x), f(y)\}$, so $t \in [0, 1]$. Thus $f(x) \geq t$ and $f(y) \geq t$, so $x, y \in U(f; t)$. Thus $U(f; t) \neq \emptyset$. By assumption, we have $U(f; t)$ is a UP-subalgebra of $A$. Thus $x \cdot y \in U(f; t)$, so $f(x \cdot y) \geq t = \min\{f(x), f(y)\}$. Thus $1 - f(x \cdot y) \leq 1 - \min\{f(x), f(y)\}$, so

$$\bar{f}(x \cdot y) = 1 - f(x \cdot y) \leq 1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\} \quad \text{(Lemma 1.29(2))}$$

Hence, $\bar{f}$ is an anti-fuzzy UP-subalgebra of $A$.

(4) Assume that $\bar{f}$ is an anti-fuzzy UP-subalgebra of $A$. Let $t \in [0, 1]$ be such that $U^+(f; t) \neq \emptyset$. Let $x, y \in U^+(f; t)$. Then $f(x) > t$ and $f(y) > t$. Since $\bar{f}$ is an anti-fuzzy UP-subalgebra of $A$, we have $\bar{f}(x \cdot y) \leq \max\{\bar{f}(x), \bar{f}(y)\}$. Thus

$$1 - f(x \cdot y) \leq \max\{1 - f(x), 1 - f(y)\} \leq 1 - \min\{f(x), f(y)\}. \quad \text{(Lemma 1.29(2))}$$

Thus $f(x \cdot y) \geq \min\{f(x), f(y)\} > t$, so $x \cdot y \in U^+(f; t)$. Hence, $U^+(f; t)$ is a UP-subalgebra of $A$.

Conversely, assume that the condition $(\ast)$ holds. Suppose that there exist $x, y \in A$ such that $\bar{f}(x \cdot y) > \max\{\bar{f}(x), \bar{f}(y)\}$. Thus

$$1 - f(x \cdot y) > \max\{1 - f(x), 1 - f(y)\} \leq 1 - \min\{f(x), f(y)\} \quad \text{(Lemma 1.29(2))}$$

Thus $f(x \cdot y) < \min\{f(x), f(y)\}$. Now $f(x \cdot y) \in [0, 1]$, we choose $t = f(x \cdot y)$. Thus $f(x) > t$ and $f(y) > t$, so $x, y \in U^+(f; t) \neq \emptyset$. By assumption, we have $U^+(f; t)$ is a UP-subalgebra of $A$ and so $x \cdot y \in U^+(f; t)$. Thus $f(x \cdot y) > t = f(x \cdot y)$ which is a contradiction. Hence, $\bar{f}(x \cdot y) \leq \max\{\bar{f}(x), \bar{f}(y)\}$ for all $x, y \in A$. Therefore, $\bar{f}$ is an anti-fuzzy UP-subalgebra of $A$.

**Proposition 2.8.** If $f$ is an anti-fuzzy UP-ideal of $A$, then $f(x \cdot (x \cdot y)) \leq f(y)$ for all $x, y \in A$.

**Proof.** Let $x, y \in A$. Then

$$f(x \cdot (x \cdot y)) \leq \max\{f(x \cdot ((x \cdot y) \cdot (x \cdot y))), f(x \cdot y)\} \quad \text{(Definition 1.13(2))}$$

$$= \max\{f(x \cdot 0), f(x \cdot y)\} \quad \text{(Proposition 1.4(1))}$$

$$= \max\{f(0), f(x \cdot y)\} \quad \text{(UP-3)}$$

$$= f(x \cdot y) \quad \text{(Definition 1.13 (1))}$$

$$\leq \max\{f(x \cdot (y \cdot y)), f(y)\} \quad \text{(Definition 1.13 (2))}$$

$$= \max\{f(x \cdot 0), f(y)\} \quad \text{(Proposition 1.4 (1))}$$

$$= \max\{f(0), f(y)\} \quad \text{(UP-3)}$$

$$= f(y). \quad \text{(Definition 1.13 (1))}$$
Corollary 2.9. Let $f$ be a fuzzy set in $A$. Then the following statements hold:

(1) if $f$ is an anti-fuzzy UP-ideal of $A$, then for every $t \in \text{Im}(f)$, $L(f; t)$ is a UP-ideal of $A$, and

(2) if $\mathcal{I}$ is an anti-fuzzy UP-ideal of $A$, then for every $t \in \text{Im}(f)$, $U(f; t)$ is a UP-ideal of $A$.

Proof. (1) Assume that $f$ is an anti-fuzzy UP-ideal of $A$ and let $t \in \text{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \leq t$. Thus $x \in L(f; t)$, so $L(f; t) \neq \emptyset$. By Theorem 2.6 (1), we have $L(f; t)$ is a UP-ideal of $A$.

(2) Assume that $\mathcal{I}$ is an anti-fuzzy UP-ideal of $A$ and let $t \in \text{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \geq t$. Thus $x \in U(f; t)$, so $U(f; t) \neq \emptyset$. By Theorem 2.6 (3), we have $U(f; t)$ is a UP-ideal of $A$. □

Corollary 2.10. Let $f$ be a fuzzy set in $A$. Then the following statements hold:

(1) if $f$ is an anti-fuzzy UP-subalgebra of $A$, then for every $t \in \text{Im}(f)$, $L(f; t)$ is a UP-subalgebra of $A$, and

(2) if $\mathcal{I}$ is an anti-fuzzy UP-subalgebra of $A$, then for every $t \in \text{Im}(f)$, $U(f; t)$ is a UP-subalgebra of $A$.

Proof. (1) Assume that $f$ is an anti-fuzzy UP-subalgebra of $A$ and let $t \in \text{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \leq t$. Thus $x \in L(f; t)$, so $L(f; t) \neq \emptyset$. By Theorem 2.7 (1), we have $L(f; t)$ is a UP-subalgebra of $A$.

(2) Assume that $\mathcal{I}$ is an anti-fuzzy UP-subalgebra of $A$ and let $t \in \text{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \geq t$. Thus $x \in U(f; t)$, so $U(f; t) \neq \emptyset$. By Theorem 2.7 (3), we have $U(f; t)$ is a UP-subalgebra of $A$. □

Corollary 2.11. Let $I$ be a UP-ideal of $A$. Then the following statements hold:

(1) for any $k \in (0, 1]$, there exists an anti-fuzzy UP-ideal $f$ of $A$ such that $L(f; t) = I$ for all $t < k$ and $L(f; t) = A$ for all $t \geq k$, and

(2) for any $k \in (0, 1]$, there exists an anti-fuzzy UP-ideal $g$ of $A$ such that $U(g; t) = I$ for all $t > k$ and $U(g; t) = A$ for all $t \leq k$.

Proof. (1) Define a fuzzy set $f: A \to [0, 1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

Case 1: To show that $L(f; t) = I$ for all $t < k$, let $t \in [0, 1]$ be such that $t < k$. Let $x \in L(f; t)$. Then $f(x) \leq t < k$, so $f(x) \neq k$. Thus $f(x) = 0$, so $x \in I$. That is $L(f; t) \subseteq I$. Let $x \in I$. Then $f(x) = 0 \leq t$, so $x \in L(f; t)$. That is $I \subseteq L(f; t)$. Hence, $L(f; t) = I$ for all $t < k$. 

Case 2: To show that \( L(f; t) = A \) for all \( t \geq k \), let \( t \in [0, 1) \) be such that \( t \geq k \). Clearly, \( L(f; t) \subseteq A \). Let \( x \in A \). Then

\[
f(x) = \begin{cases} 
0 & \text{if } x \in I, \\
k & \text{if } x \notin I,
\end{cases}
\]

so \( x \in L(f; t) \). That is \( A \subseteq L(f; t) \). Hence, \( L(f; t) = A \) for all \( t \geq k \).

It follows from Theorem 2.6 (1) that \( f \) is an anti-fuzzy UP-ideal of \( A \).

(2) Define a fuzzy set \( f: A \to [0, 1] \) by

\[
f(x) = \begin{cases} 
1 & \text{if } x \in I, \\
k & \text{if } x \notin I.
\end{cases}
\]

Case 1: To show that \( U(f; t) = I \) for all \( t > k \), let \( t \in [0, 1) \) be such that \( t > k \). Let \( x \in U(f; t) \). Then \( f(x) \geq t > k \), so \( f(x) \neq k \). Thus \( f(x) = 1 \), so \( x \in I \). That is \( U(f; t) \subseteq I \). Let \( x \in I \). Then \( f(x) = 1 \geq t \), so \( x \in U(f; t) \). That is \( I \subseteq U(f; t) \). Hence, \( U(f; t) = I \) for all \( t > k \).

Case 2: To show that \( U(f; t) = A \) for all \( t \leq k \), let \( t \in [0, 1) \) be such that \( t \leq k \). Clearly, \( U(f; t) \subseteq A \). Let \( x \in A \). Then

\[
f(x) = \begin{cases} 
1 & \text{if } x \in I, \\
k & \text{if } x \notin I,
\end{cases}
\]

so \( x \in U(f; t) \). That is \( A \subseteq U(f; t) \). Hence, \( U(f; t) = A \) for all \( t \leq k \).

It follows from Theorem 2.6 (3) that \( \overline{f} \) is an anti-fuzzy UP-ideal of \( A \). By Remark 1.18, we have \( U(\overline{f}; t) = U(f; t) = I \) for all \( t > k \) and \( U(\overline{f}; t) = U(f; t) = A \) for all \( t \leq k \). Putting \( \overline{f} = g \). Then \( g \) is an anti-fuzzy UP-ideal of \( A \) such that \( U(\overline{g}; t) = I \) for all \( t > k \) and \( U(\overline{g}; t) = A \) for all \( t \leq k \).

\[ \blacksquare \]

Corollary 2.12. Let \( I \) be a UP-subalgebra of \( A \). Then the following statements hold:

1. for any \( k \in (0, 1] \), there exists an anti-fuzzy UP-subalgebra \( f \) of \( A \) such that \( L(f; t) = I \) for all \( t < k \) and \( L(f; t) = A \) for all \( t \geq k \), and

2. for any \( k \in [0, 1) \), there exists an anti-fuzzy UP-subalgebra \( g \) of \( A \) such that \( U(\overline{g}; t) = I \) for all \( t > k \) and \( U(\overline{g}; t) = A \) for all \( t \leq k \).

**Proof.** (1) Define a fuzzy set \( f: A \to [0, 1] \) by

\[
f(x) = \begin{cases} 
0 & \text{if } x \in I, \\
k & \text{if } x \notin I.
\end{cases}
\]

In the proof of Corollary 2.11 (1), we have \( L(f; t) = I \) for all \( t < k \) and \( L(f; t) = A \) for all \( t \geq k \).

It follows from Theorem 2.7 (1) that \( f \) is an anti-fuzzy UP-subalgebra of \( A \).

(2) Define a fuzzy set \( f: A \to [0, 1] \) by

\[
f(x) = \begin{cases} 
1 & \text{if } x \in I, \\
k & \text{if } x \notin I.
\end{cases}
\]
In the proof of Corollary 2.11 (2), we have $U(f; t) = I$ for all $t > k$ and $U(f; t) = A$ for all $t \leq k$.

It follows from Theorem 2.7 (3) that $\overline{f}$ is an anti-fuzzy UP-subalgebra of $A$. By Remark 1.18, we have $U(\overline{f}; t) = U(f; t) = I$ for all $t > k$ and $U(\overline{f}; t) = U(f; t) = A$ for all $t \leq k$. Putting $g = \overline{f}$. Then $g$ is an anti-fuzzy UP-subalgebra of $A$ such that $U(g; t) = I$ for all $t > k$ and $U(g; t) = A$ for all $t \leq k$.

**Theorem 2.13.** Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let $f : A \to B$ be a UP-epimorphism. Then the following statements hold:

1. for every anti-fuzzy UP-ideal $\beta$ of $B$, $\mu$ is an anti-fuzzy UP-ideal of $A$, and
2. for every anti-fuzzy UP-subalgebra $\beta$ of $B$, $\mu$ is an anti-fuzzy UP-subalgebra of $A$.

**Proof.** (1) Let $\beta$ be an anti-fuzzy UP-ideal of $B$. Let $x \in A$. Then

$$
\mu(0_A) = (\beta \circ f)(0_A) \\
= \beta(f(0_A)) \\
\leq \beta(f(x)) \quad \text{(Proposition 1.6 (3), Theorem 1.28 (2), Lemma 2.3)} \\
= (\beta \circ f)(x) \\
= \mu(x).
$$

Let $x, y, z \in A$. Then

$$
\mu(x \cdot z) = (\beta \circ f)(x \cdot z) \\
= \beta(f(x \cdot z)) \\
= \beta(f(x) \ast f(z)) \\
\leq \max\{\beta(f(x) \ast (f(y) \ast f(z))), \beta(f(y))\} \quad \text{(Definition 1.13 (2))} \\
= \max\{\beta(f(x \cdot (y \cdot z))), \beta(f(y))\} \\
= \max\{\beta \circ f)(x \cdot (y \cdot z)), (\beta \circ f)(y)\} \\
= \max\{\mu(x \cdot (y \cdot z)), \mu(y)\}.
$$

Hence, $\mu$ is an anti-fuzzy UP-ideal of $A$.

(2) Let $\beta$ be an anti-fuzzy UP-subalgebra of $B$. Let $x, y \in A$. Then

$$
\mu(x \cdot y) = (\beta \circ f)(x \cdot y) \\
= \beta(f(x \cdot y)) \\
= \beta(f(x) \ast f(y)) \\
\leq \max\{\beta(f(x)), \beta(f(y))\} \quad \text{(Definition 1.14)} \\
= \max\{\beta \circ f)(x), (\beta \circ f)(y)\} \\
= \max\{\mu(x), \mu(y)\}.
$$

Hence, $\mu$ is an anti-fuzzy UP-subalgebra of $A$. ■
Lemma 2.14. Let \((A;\cdot,0_A)\) and \((B;\ast,0_B)\) be UP-algebras and let \(f: A \to B\) be a UP-epimorphism. Let \(\mu\) be an \(f\)-invariant fuzzy set in \(A\) with inf property. For any \(a,b \in B\), there exist \(a_0 \in f^{-1}(a)\) and \(b_0 \in f^{-1}(b)\) such that \(\beta(a) = \mu(a_0)\), \(\beta(b) = \mu(b_0)\) and \(\beta(a \ast b) = \mu(a_0 \cdot b_0)\).

Proof. Let \(a,b \in B\). Since \(f\) is surjective, we have \(f^{-1}(a)\), \(f^{-1}(b)\) and \(f^{-1}(a \ast b)\) are nonempty subsets of \(A\). By Definition 1.20, we obtain

\[
\begin{align*}
\beta(a) &= \inf \{\mu(t)\}_{t \in f^{-1}(a)} \\
&= \mu(a_0) \text{ for some } a_0 \in f^{-1}(a), \quad \text{(Definition 1.21)} \\
\beta(b) &= \inf \{\mu(t)\}_{t \in f^{-1}(b)} \\
&= \mu(b_0) \text{ for some } b_0 \in f^{-1}(b) \quad \text{(Definition 1.21)} \\
\beta(a \ast b) &= \inf \{\mu(t)\}_{t \in f^{-1}(a \ast b)} \\
&= \mu(c) \text{ for some } c \in f^{-1}(a \ast b). \quad \text{(Definition 1.21)}
\end{align*}
\]

Since \(f(c) = a \ast b = f(a_0) \ast f(b_0) = f(a_0 \cdot b_0)\) and \(\mu\) is \(f\)-invariant, we have \(\mu(c) = \mu(a_0 \cdot b_0)\). Hence, \(\beta(a \ast b) = \mu(a_0 \cdot b_0)\).

Theorem 2.15. Let \((A;\cdot,0_A)\) and \((B;\ast,0_B)\) be UP-algebras and let \(f: A \to B\) be a UP-epimorphism. Then the following statements hold:

1. for every \(f\)-invariant anti-fuzzy UP-ideal \(\mu\) of \(A\) with inf property, \(\beta\) is an anti-fuzzy UP-ideal of \(B\), and

2. for every \(f\)-invariant anti-fuzzy UP-subalgebra \(\mu\) of \(A\) with inf property, \(\beta\) is an anti-fuzzy UP-subalgebra of \(B\).

Proof. (1) Let \(\mu\) be an \(f\)-invariant anti-fuzzy UP-ideal of \(A\) with inf property. By Definition 1.13 (1), we have \(\mu(0_A) \leq \mu(x)\) for all \(x \in A\). By Theorem 1.28 (1), we have \(0_A \in f^{-1}(0_B)\) and so \(f^{-1}(0_B) \neq \emptyset\). Thus \(\beta(0_B) = \inf \{\mu(t)\}_{t \in f^{-1}(0_B)} \leq \mu(0_A)\).

Let \(y \in B\). Since \(f\) is surjective, we have \(f^{-1}(y) \neq \emptyset\). By Definition 1.13 (1), we have \(\mu(0_A) \leq \mu(t)\) for all \(t \in f^{-1}(y)\). Thus \(\mu(0_A)\) is a lower bound of \(\{\mu(t)\}_{t \in f^{-1}(y)}\), so \(\mu(0_A) \leq \inf \{\mu(t)\}_{t \in f^{-1}(y)} = \beta(y)\). By Proposition 1.5 (3), we have \(\beta(0_B) \leq \beta(y)\). Let \(a, b, c \in B\). By Lemma 2.14, there exist \(a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)\) and \(c_0 \in f^{-1}(c)\) such that \(\beta(b) = \mu(b_0), \beta(a \ast c) = \mu(a_0 \cdot c_0)\) and \(\beta(a \ast (b \ast c)) = \mu(a_0 \cdot (b_0 \cdot c_0))\). Thus

\[
\begin{align*}
\beta(a \ast c) &= \mu(a_0 \cdot c_0) \\
&\leq \max \{\mu(a_0 \cdot (b_0 \cdot c_0)), \mu(b_0)\} \quad \text{(Definition 1.13 (2))} \\
&= \max \{\beta(a \ast (b \ast c)), \beta(b)\}.
\end{align*}
\]

Hence, \(\beta\) is an anti-fuzzy UP-ideal of \(B\).

(2) Let \(\mu\) be an \(f\)-invariant anti-fuzzy UP-subalgebra of \(A\) with inf property. Let \(a, b \in B\). Since \(f\) is surjective, we have \(f^{-1}(a), f^{-1}(b)\) and \(f^{-1}(a \ast b)\) are nonempty subsets of \(A\). By Lemma 2.14, there exist \(a_0 \in f^{-1}(a), b_0 \in f^{-1}(b)\) such that \(\beta(a) = \mu(a_0), \beta(b) = \mu(b_0)\) and \(\beta(a \ast b) = \mu(a_0 \cdot b_0)\). Thus
\[
\beta(a * b) = \mu(a_0 \cdot b_0) \\
\leq \max\{\mu(a_0), \mu(b_0)\} \\
= \max\{\beta(a), \beta(b)\}.
\]

Hence, \( \beta \) is an anti-fuzzy UP-subalgebra of \( B \).

\[\text{Definition 1.14}\]

\section*{Remark 2.16.} [26] Let \((A; \cdot, 0_A)\) and \((B; *, 0_B)\) be UP-algebras. Then \( A \times B \) is a UP-algebra defined by

\[(x_1, x_2) \odot (y_1, y_2) = (x_1 \cdot y_1, x_2 \ast y_2)\]

for all \( x_1, y_1 \in A \) and \( x_2, y_2 \in B \).

\section*{Theorem 2.17.} Let \( f \) be a fuzzy set in \( A \). Then the following statements hold:

\begin{enumerate}
  \item if \( \mu_f \) is an anti-fuzzy UP-ideal of \( A \times A \), then \( f \) is an anti-fuzzy UP-ideal of \( A \), and
  \item if \( \beta_f \) is an anti-fuzzy UP-ideal of \( A \times A \), then \( f \) is an anti-fuzzy UP-ideal of \( A \).
\end{enumerate}

\section*{Proof.}

(1) Assume that \( \mu_f \) is an anti-fuzzy UP-ideal of \( A \times A \). Let \( x \in A \). Then

\[
f(0) = \max\{f(0), f(0)\} \\
= \mu_f(0, 0) \\
\leq \mu_f(x, x) \quad \text{(Definition 1.13 (1))} \\
= \max\{f(x), f(x)\} \\
= f(x).
\]

Let \( x, y, z \in A \). Then

\[
f(x \cdot z) = \max\{f(x \cdot z), f(x \cdot z)\} \\
= \mu_f(x \cdot z, x \cdot z) \\
= \mu_f((x, x) \odot (z, z)) \\
\leq \max\{\mu_f((x, x) \odot ((y, y) \odot (z, z))), \mu_f(y, y)\} \quad \text{(Definition 1.13 (2))} \\
= \max\{\mu_f(x \cdot (y \cdot z), x \cdot (y \cdot z)), \mu_f(y, y)\} \\
= \max\{\max\{f(x \cdot (y \cdot z)), f(x \cdot (y \cdot z))\}, \max\{f(y), f(y)\}\} \\
= \max\{f(x \cdot (y \cdot z)), f(y)\}.
\]

Hence, \( f \) is an anti-fuzzy UP-ideal of \( A \).

(2) Similarly to as in the proof of (1). \hfill \blacksquare

\section*{Proposition 2.18.} Let \( f \) be a fuzzy set in \( A \). Then the following statements hold:

\begin{enumerate}
  \item if \( \mu_f \) is an anti-fuzzy UP-subalgebra of \( A \times A \), then \( f \) is an anti-fuzzy UP-subalgebra of \( A \), and
  \item if \( \beta_f \) is an anti-fuzzy UP-subalgebra of \( A \times A \), then \( f \) is an anti-fuzzy UP-subalgebra of \( A \).
\end{enumerate}
Proof. (1) Assume that $\mu_f$ is an anti-fuzzy UP-subalgebra of $A \times A$. Let $x, y \in A$. Then

$$f(x \cdot z) = \max\{f(x \cdot z), f(x \cdot z)\}$$

$$= \mu_f(x \cdot z, x \cdot z)$$

$$= \mu_f((x, x) \odot (z, z))$$

$$\leq \max\{\mu_f((x, x), \mu_f(y, y))\}$$

$$\leq \max\{\max\{f(x), f(x)\}, \max\{f(y), f(y)\}\}$$

$$= \max\{f(x), f(y)\}.$$  

Hence, $f$ is an anti-fuzzy UP-subalgebra of $A$.

(2) Similarly to as in the proof of (1).

Lemma 2.19. [4] For any $a, b, c, d \in \mathbb{R}$, the following properties hold:

(1) $\max\{\max\{a, b\}, \max\{c, d\}\} = \max\{\max\{a, c\}, \max\{b, d\}\}$, and

(2) $\min\{\min\{a, b\}, \min\{c, d\}\} = \min\{\min\{a, c\}, \min\{b, d\}\}$.

Theorem 2.20. Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras. Then the following statements hold:

(1) if $f$ is an anti-fuzzy UP-ideal of $A$ and $g$ is an anti-fuzzy UP-ideal of $B$, then $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, and

(2) if $f$ is an anti-fuzzy UP-subalgebra of $A$ and $g$ is an anti-fuzzy UP-subalgebra of $B$, then $f \times g$ is an anti-fuzzy UP-subalgebra of $A \times B$.

Proof. (1) Assume that $f$ is an anti-fuzzy UP-ideal of $A$ and $g$ is an anti-fuzzy UP-ideal of $B$. Let $(x, y) \in A \times B$. Then

$$(f \times g)(0, 0) = \max\{f(0), g(0)\}$$

$$\leq \max\{f(x), g(y)\} \quad \text{(Definition 1.13 (1))}$$

$$= (f \times g)(x, y).$$

Now, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in A \times B$. Then

$$(f \times g)((x_1, x_2) \odot (z_1, z_2))$$

$$= (f \times g)(x_1 \cdot z_1, x_2 \ast z_2)$$

$$= \max\{f(x_1 \cdot z_1), g(x_2 \ast z_2)\}$$

$$\leq \max\{\max\{f(x_1 \cdot (y_1 \cdot z_1)), f(y_1)\}, \max\{g(x_2 \ast (y_2 \ast z_2)), g(y_2)\}\}$$

$$\leq \max\{\max\{f(x_1 \cdot (y_1 \cdot z_1)), \{g(x_2 \ast (y_2 \ast z_2))\}\}, \max\{f(y_1), g(y_2)\}\} \quad \text{(Definition 1.13 (2))}$$

$$= \max\{\max\{f(x_1 \cdot (y_1 \cdot z_1)) \cdot (x_2 \ast (y_2 \ast z_2)), (f \times g)(y_1, y_2)\}$$

$$= \max\{(f \times g)((x_1, x_2) \odot ((y_1, y_2) \odot (z_1, z_2))), (f \times g)(y_1, y_2)\}.$$
Hence, \( f \times g \) is an anti-fuzzy UP-ideal of \( A \times B \).

(2) Let \((x_1, x_2), (y_1, y_2) \in A \times B\). Then

\[
(f \times g)((x_1, x_2) \odot (y_1, y_2)) = (f \times g)(x_1 \cdot y_1, x_2 \ast y_2) = \max \{f(x_1 \cdot y_1), g(x_2 \ast y_2)\}
\]

\[
\leq \max \{\max\{f(x_1), f(y_1)\}, \max\{g(x_2), g(y_2)\}\} \quad \text{(Definition 1.14)}
\]

\[
= \max\{\max\{f(x_1), g(x_2)\}, \max\{f(y_1), g(y_2)\}\} \quad \text{(Lemma 2.19 (1))}
\]

\[
= \max\{(f \times g)(x_1, x_2), (f \times g)(y_1, y_2)\}.
\]

Hence, \( f \times g \) is an anti-fuzzy UP-subalgebra of \( A \times B \).

Give examples of conflict that \( f \) and \( g \) are anti-fuzzy UP-ideals (resp. anti-fuzzy UP-subalgebras) of \( A \) but \( f \cdot g \) is not an anti-fuzzy UP-ideal (resp. anti-fuzzy UP-subalgebra) of \( A \times A \).

**Example 2.21.** Let \( A = \{0, 1\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

\[
\begin{array}{ccc}
\cdot & 0 & 1 \\
0 & 0 & 1 \\
1 & 0 & 0 \\
\end{array}
\]

Then \((A; \cdot, 0)\) is a UP-algebra. We define a fuzzy set \( f \) and \( g \) in \( A \) as follows:

\[
\begin{align*}
f(0) &= 0.1, f(1) = 0.5, g(0) = 0.2 \quad \text{and} \quad g(1) = 0.5. \\
\end{align*}
\]

Using this data, we can show that \( f \) and \( g \) are anti-fuzzy UP-ideals of \( A \). Let \( x_1 = 0, x_2 = 0, y_1 = 1, y_2 = 0, z_1 = 1, z_1 = 1. \) Then

\[
(f \cdot g)((x_1, x_2) \odot (z_1, z_2)) = 0.5 \leq 0.2 = \max\{(f \cdot g)((x_1, x_2) \odot ((y_1, y_2) \odot (z_1, z_2))), (f \cdot g)(y_1, y_2)\}.
\]

Thus Definition 1.13 (2) is false. Hence, \( f \cdot g \) is not an anti-fuzzy UP-ideal of \( A \times A \).

**Example 2.22.** Let \( A = \{0, a, b\} \) be a set with a binary operation \( \cdot \) defined by the following Cayley table:

\[
\begin{array}{ccc}
\cdot & 0 & a & b \\
0 & 0 & a & b \\
a & 0 & 0 & a \\
b & 0 & 0 & 0 \\
\end{array}
\]

Then \((A; \cdot, 0)\) is a UP-algebra. We define a fuzzy set \( f \) and \( g \) in \( A \) as follows:

\[
\begin{align*}
f(0) &= 0.1, f(a) = 0.2, f(b) = 0.1, g(0) = 0.1, g(a) = 0.2 \quad \text{and} \quad g(b) = 0.1. \\
\end{align*}
\]

Using this data, we can show that \( f \) and \( g \) are anti-fuzzy UP-subalgebras of \( A \). Let \( x_1 = 0, x_2 = 1, y_1 = 1, y_2 = 2. \) Then
(f \cdot g)((x_1, x_2) \odot (y_1, y_2)) = 0.2 \geq 0.1 = \max\{(f \cdot g)(x_1, x_2), (f \cdot g)(y_1, y_2)\}.

Thus Definition 1.14 is false. Hence, \(f \cdot g\) is not an anti-fuzzy UP-subalgebra of \(A \times A\).

**Theorem 2.23.** Let \((A; \cdot, 0_A)\) and \((B; *, 0_B)\) be UP-algebras and let \(f\) be a fuzzy set in \(A\) and \(g\) be a fuzzy set in \(B\). If \(f \times g\) is an anti-fuzzy UP-ideal of \(A \times B\), then the following statements hold:

1. either \(f(0_A) \leq f(x)\) for all \(x \in A\) or \(g(0_B) \leq g(y)\) for all \(y \in B\),
2. if \(f(0_A) \leq f(x)\) for all \(x \in A\), then either \(g(0_B) \leq g(y)\) for all \(y \in B\) or \(g(0_B) \leq f(x)\) for all \(x \in A\),
3. if \(g(0_B) \leq g(y)\) for all \(y \in B\), then either \(f(0_A) \leq f(x)\) for all \(x \in A\) or \(f(0_A) \leq g(y)\) for all \(y \in B\), and
4. either \(f\) is an anti-fuzzy UP-ideal of \(A\) or \(g\) is an anti-fuzzy UP-ideal of \(B\).

**Proof.** (1) Suppose that \(f(0_A) > f(x)\) for some \(x \in A\) and \(g(0_B) > g(y)\) for some \(y \in B\). Then \((f \times g)(x, y) = \max\{f(x), g(y)\} < \max\{f(0_A), g(0_B)\} = (f \times g)(0_A, 0_B)\) which is a contradiction. Hence, either \(f(0_A) \leq f(x)\) for all \(x \in A\) or \(g(0_B) \leq g(y)\) for all \(y \in B\).

(2) Assume that \(f(0_A) \leq f(x)\) for all \(x \in A\). Suppose that \(g(0_B) > g(y)\) for some \(y \in B\) and \(g(0_B) > f(x)\) for some \(x \in A\). Then \(g(0_B) > f(x) \geq f(0_A)\). Thus

\[
(f \times g)(x, y) = \max\{f(x), g(y)\} \\
< \max\{g(0_B), g(0_B)\} \\
= g(0_B) \\
= \max\{f(0_A), g(0_B)\} \\
= (f \times g)(0_A, 0_B)
\]

which is a contradiction. Hence, either \(g(0_B) \leq g(y)\) for all \(y \in B\) or \(g(0_B) \leq f(x)\) for all \(x \in A\).

(3) Assume that \(g(0_B) \leq g(y)\) for all \(y \in B\). Suppose that \(f(0_A) > f(x)\) for some \(x \in A\) and \(f(0_A) > g(y)\) for some \(y \in B\). Then \(f(0_A) > g(y) \geq g(0_B)\). Thus

\[
(f \times g)(x, y) = \max\{f(x), g(y)\} \\
< \max\{f(0_A), f(0_A)\} \\
= f(0_A) \\
= \max\{f(0_A), g(0_B)\} \\
= (f \times g)(0_A, 0_B)
\]

which is a contradiction. Hence, either \(f(0_A) \leq f(x)\) for all \(x \in A\) or \(f(0_A) \leq g(y)\) for all \(y \in B\).

(4) Suppose that \(f\) is not an anti-fuzzy UP-ideal of \(A\) and \(g\) is not an anti-fuzzy UP-ideal of \(B\). By (1), assume that \(f(0_A) \leq f(x)\) for all \(x \in A\). Then
from (2), either \(g(0_B) \leq g(y)\) for all \(y \in B\) or \(g(0_B) \leq f(x)\) for all \(x \in A\). If \(g(0_B) \leq f(x)\) for all \(x \in A\), then for all \(x \in A\),

\[
(f \times g)(x, 0_B) = \max\{f(x), g(0_B)\} = f(x).
\] (2.4)

Since \(f \times g\) is an anti-fuzzy UP-ideal of \(A \times B\), we have for any \(x, y, z \in A\),

\[
f(x \cdot z) = (f \times g)(x \cdot z, 0_B) \tag{2.4}
\]

\[
= (f \times g)(x \cdot z, 0_B \ast 0_B)
\]

\[
= (f \times g)((x, 0_B) \circ (z, 0_B))
\]

\[
\leq \max\{(f \times g)((x, 0_B) \circ (y, 0_B) \circ (z, 0_B)),
\]

\[
(f \times g)(y, 0_B)\} \tag{UP-3}
\]

\[
= \max\{(f \times g)(x \cdot (y \cdot z), 0_B \ast (0_B \ast 0_B)), (f \times g)(y, 0_B)\}
\]

\[
= \max\{\max\{f(x \cdot (y \cdot z)), g(0_B)\}, \max\{f(y), g(0_B)\}\}
\]

\[
= \max\{f(x \cdot (y \cdot z)), f(y)\}.
\]

Hence, \(f\) is an anti-fuzzy UP-ideal of \(A\) which is a contradiction. Assume that \(g(0_B) \leq g(y)\) for all \(y \in B\). Then from 2.23, either \(f(0_A) \leq f(x)\) for all \(x \in A\) or \(f(0_A) \leq g(y)\) for all \(y \in B\). If \(f(0_A) \leq g(y)\) for all \(y \in B\), then for all \(y \in B\),

\[
(f \times g)(0_A, y) = \max\{f(0_A), g(y)\} = g(y). \tag{2.5}
\]

Since \(f \times g\) is an anti-fuzzy UP-ideal of \(A \times B\), we have for any \(x, y, z \in B\),

\[
g(x \ast z) = (f \times g)(0_A, x \ast z) \tag{2.5}
\]

\[
= (f \times g)(0_A \cdot 0_A, x \ast z)
\]

\[
= (f \times g)((0_A, x) \circ (0_A, z))
\]

\[
\leq \max\{(f \times g)((0_A, x) \circ (0_A, y) \circ (0_A, z)),
\]

\[
(f \times g)(0_A, y)\} \tag{UP-3}
\]

\[
= \max\{(f \times g)(0_A \cdot (0_A \cdot 0_A)), (x \ast (y \ast z)), (f \times g)(0_A, y)\}
\]

\[
= \max\{\max\{f(0_A), x \ast (y \ast z)\}, (f \times g)(0_A, y)\}\] (UP-3)

\[
= \max\{\max\{f(0_A), g(x \ast (y \ast z))\}, \max\{f(0_A), g(y)\}\}
\]

\[
= \max\{g(x \ast (y \ast z)), g(y)\}.
\]

Hence, \(g\) is an anti-fuzzy UP-ideal of \(B\) which is a contradiction. Since \(f\) is not an anti-fuzzy UP-ideal of \(A\) and \(g\) is not an anti-fuzzy UP-ideal of \(B\), and \(f(0_A) \leq f(x)\) for all \(x \in A\) and \(g(0_B) \leq g(y)\) for all \(y \in B\), there exist \(x, y, z \in A\) and \(x', y', z' \in B\) such that

\[
f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\}
\]

and

\[
g(x' \ast z') > \max\{g(x' \ast (y' \ast z')), g(y')\}.
\]
Thus
\[\max\{f(xz), g(x'z')\} > \max\{\max\{f(x'y), f(y)\}, \max\{g(x'y'z'), g(y')\}\}.\]

Since \(f \times g\) is an anti-fuzzy UP-ideal of \(A \times B\), we have
\[
\max\{f(x \cdot z), g(x' \ast z')\} = (f \times g)(x \cdot z, x' \ast z')
= (f \times g)((x, x') \circ (z, z'))
\leq \max\{(f \times g)((x, x') \circ [(y, y') \circ (z, z')]), (f \times g)(y, y')\}
= \max\{(f \times g)(x \cdot (y \cdot z), x' \ast (y' \ast z')), (f \times g)(y, y')\}
= \max\{\max\{f(x \cdot (y \cdot z)), g(x' \ast (y' \ast z'))\},
\max\{f(y), g(y')\}\}.
\]

Thus \(\max\{f(x \cdot z), g(x' \ast z')\} \neq \max\{\max\{f(x \cdot (y \cdot z)), f(y)\}, \max\{g(x' \ast (y' \ast z')), g(y')\}\} \) which is a contradiction. Similarly, by (1), if \(g(0_B) \leq g(y)\) for all \(y \in B\), we have a contradiction. Hence, either \(f\) is an anti-fuzzy UP-ideal of \(A\) or \(g\) is an anti-fuzzy UP-ideal of \(B\).

**Theorem 2.24.** Let \((A; \cdot, 0_A)\) and \((B; \ast, 0_B)\) be UP-algebras and let \(f\) be a fuzzy set in \(A\) and \(g\) be a fuzzy set in \(B\). If \(f \times g\) is an anti-fuzzy UP-subalgebra of \(A \times B\), then either \(f\) is an anti-fuzzy UP-subalgebra of \(A\) or \(g\) is an anti-fuzzy UP-subalgebra of \(B\).

**Proof.** Suppose that \(f\) is not an anti-fuzzy UP-subalgebra of \(A\) and \(g\) is not an anti-fuzzy UP-subalgebra of \(B\). Then there exist \(x, y \in A\) and \(a, b \in B\) such that
\[f(x \cdot y) > \max\{f(x), f(y)\}\]
and
\[g(a \ast b) > \max\{g(a), g(b)\}\]

Thus \(\max\{f(x \cdot y), g(a \ast b)\} > \max\{\max\{f(x), f(y)\}, \max\{g(a), g(b)\}\} \). Since \(f \times g\) is an anti-fuzzy UP-subalgebra of \(A \times B\), we have
\[
\max\{f(x \cdot y), g(a \ast b)\} = (f \times g)(x \cdot y, a \ast b)
= (f \times g)((x, a) \circ (y, b))
\leq \max\{(f \times g)((x, a)), (f \times g)(y, b)\}
= \max\{\max\{f(x), g(a)\}, \max\{f(y), g(b)\}\}
= \max\{\max\{f(x), f(y)\}, \max\{g(a), g(b)\}\}.\]

Thus \(\max\{f(x \cdot y), g(a \ast b)\} \neq \max\{\max\{f(x), f(y)\}, \max\{g(a), g(b)\}\} \) which is a contradiction. Hence, either \(f\) is an anti-fuzzy UP-subalgebra of \(A\) or \(g\) is an anti-fuzzy UP-subalgebra of \(B\).

**Theorem 2.25.** Let \(f\) be a fuzzy set in \(A\). Then the following statements hold:
(1) \( f \) is an anti-fuzzy UP-ideal of \( A \) if and only if \( \mu_f \) is an anti-fuzzy UP-ideal of \( A \times A \), and

(2) \( f \) is an anti-fuzzy UP-subalgebra of \( A \) if and only if \( \mu_f \) is an anti-fuzzy UP-subalgebra of \( A \times A \).

Proof. (1) Assume that \( f \) is an anti-fuzzy UP-ideal of \( A \). By Theorem 2.20 (1), we have \( \mu_f = f \times f \) is an anti-fuzzy UP-ideal of \( A \times A \).

Conversely, assume that \( \mu_f \) is an anti-fuzzy UP-ideal of \( A \times A \). Since \( f \times f = \mu_f \), it follows from Theorem 2.23 (4) that \( f \) is an anti-fuzzy UP-ideal of \( A \).

(2) Assume that \( f \) is an anti-fuzzy UP-subalgebra of \( A \). By Theorem 2.20 (2), we have \( \mu_f = f \times f \) is an anti-fuzzy UP-subalgebra of \( A \times A \).

Conversely, assume that \( \mu_f \) is an anti-fuzzy UP-subalgebra of \( A \times A \). Since \( f \times f = \mu_f \), it follows from Theorem 2.24 that \( f \) is an anti-fuzzy UP-subalgebra of \( A \).

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