UP-ALGEBRAS CHARACTERIZED BY THEIR ANTI-FUZZY UP-IDEALS AND ANTI-FUZZY UP-SUBALGEBRAS[1]

Waraphorn Kaijae
Polatip Poungsumpao
Saranya Arayarangsi
Aiyared Iampan[2]

Department of Mathematics
School of Science
University of Phayao
Phayao 56000
Thailand

Abstract. In this paper, anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras concepts of UP-algebras are introduced and proved some results. We also introduce the notions of Cartesian product and dot product of fuzzy sets, and then we study related properties. Further, we discuss the relation between anti-fuzzy UP-ideals (resp. anti-fuzzy UP-subalgebras) and level subsets of a fuzzy set. Anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras are also applied in the Cartesian product of UP-algebras.

Keywords: UP-algebra, anti-fuzzy UP-ideal, anti-fuzzy UP-subalgebra.

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1. Introduction and preliminaries

Among many algebraic structures, algebras of logic form important class of algebras. Examples of these are BCK-algebras [9], BCI-algebras [10], BCH-algebras [7], KU-algebras [27], SU-algebras [17] and others. They are strongly connected with logic. For example, BCI-algebras introduced by Iséki [10] in 1966 have connections with BCI-logic being the BCI-system in combinatory logic which has application in the language of functional programming. BCK and BCI-algebras are two classes of logical algebras. They were introduced by Imai and Iséki [9], [10] in 1966 and have been extensively investigated by many researchers. It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras.

A fuzzy subset f of a set S is a function from S to a closed interval $[0, 1]$. The concept of a fuzzy subset of a set was first considered by Zadeh [38] in 1965. The

[1]This research is supported by the Group for Young Algebraists in University of Phayao (GYA), Thailand.

[2]Corresponding author. E-mail: aiyared.ia@up.ac.th

fuzzy set theories developed by Zadeh and others have found many applications in the domain of mathematics and elsewhere.

After the introduction of the concept of fuzzy sets by Zadeh [38], several researches were conducted on the generalizations of the notion of fuzzy set and application to many logical algebras such as: In 2001, Lele, Wu, Weke, Mamadou and Njock [19] studied fuzzy ideals and weak ideals in BCK-algebras. Jun [11] introduced the notion of Q-fuzzy subalgebras of BCK/BCI-algebras, and provided some appropriate examples. In 2002, Jun, Roh and Kim [13] studied fuzzy B-algebras in B-algebras. Yonglin and Xiaohong [37] introduced the notion of fuzzy a-ideals in BCI-algebras, and investigated its properties. In 2004, Jun [12] introduced the concept of (α, β)-fuzzy ideals of BCK/BCI-algebras. In 2005, Akram and Dar [1] introduced the notions of T-fuzzy subalgebras and T-fuzzy H-ideals in BCI-algebras and investigated some of their properties. Jun [14] introduced the notion of (α, β)-fuzzy subalgebras of BCK/BCI-algebras. In 2007, Kordi and Moussavi [18] studied fuzzy p-ideals, fuzzy H-ideals and fuzzy BCI-positive implicative ideals of BCI-algebras. Akram and Dar [2] introduced the notion of fuzzy ideals in K-algebras. In 2008, Saeid and Jun [31] introduced the concept of anti fuzzy subalgebras of BCK/BCI-algebras by using the notion of anti fuzzy points. In 2009, Saeid and Rezvani [32] introduced the notion of fuzzy BF-algebra and fuzzy topological BF-algebras. Hadipour [6] introduced the notion of (α, β)-fuzzy BF-algebras. In 2010, Song, Jun and Lee [35] introduced the notion of fuzzy ideals in BE-algebras. Saeid [30] introduced the notion of fuzzy dot BCK-subalgebras. Karamdin [16] characterized anti fuzzy ideals in weak BCC-algebras. In 2011, Mostafa, Abd-Elnaby and Yousef [22] introduced the notion of fuzzy KU-ideals in KU-algebras. Mostafa, Omar and Marie [23] introduced the notion of anti-fuzzy sub-implicative ideal of BCI-algebras. In 2012, Sitharselvam, Priya and Ramachandran [33] introduced the notion of anti Q-fuzzy KU-ideals of KU-algebras. In 2014, Yamini and Kailasavalli [36] introduced the notion of B-ideals and fuzzy B-ideals in B-algebras. Rajam and Chandramouleeswaran [28] introduced the notion of L-fuzzy β-subalgebras on β-algebras, and investigated some of their properties. Al-Shehri [3] introduced the notion of anti fuzzy implicative ideals of BCK-algebras. In 2015, Mostafa, Abdel Naby and Elkabany [25] introduced the notion of α-anti fuzzy new ideals of PU-algebras. Kailasavalli and Yamini [15] introduced the notion of anti fuzzy B-ideals and anti fuzzy B-algebras of B-algebras.

Iampan [8] now introduced a new algebraic structure, called a UP-algebra. The notions of fuzzy subalgebras and fuzzy ideals play an important role in studying the many logical algebras. Somjanta, Thuekaew, Kumpeangkeaw and Iampan [34] introduced and studied fuzzy UP-subalgebras and fuzzy UP-ideals of UP-algebras, and investigated some of its properties. In this paper, anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras concepts of UP-algebras are introduced and proved some results. Further, we discuss the relation between anti-fuzzy UP-ideals (resp. anti-fuzzy UP-subalgebras) and level subsets of a fuzzy set. Anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras are also applied in the Cartesian product of UP-algebras.

Before we begin our study, we will introduce the definition of a UP-algebra.

Definition 1.1. [8] An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ is called a *UP-algebra* if it satisfies the following axioms: for any $x, y, z \in A$,

(UP-1) $(y \cdot z) \cdot ((x \cdot y) \cdot (x \cdot z)) = 0$,

(UP-2) $0 \cdot x = x$,

(UP-3) $x \cdot 0 = 0$, and

(UP-4) $x \cdot y = y \cdot x = 0$ implies $x = y$.

Example 1.2. [8] Let X be a set. Define a binary operation $\cdot$ on the power set of X by putting $A \cdot B = B \cap A'$ for all $A, B \in \mathcal{P}(X)$. Then $(\mathcal{P}(X); \cdot, \emptyset)$ is a UP-algebra.

Example 1.3. [8] Let $A = \{0, a, b, c\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	a	b	c
0	0	a	b	c
a	0	0	0	0
b	0	a	0	c
c	0	a	b	0

Then $(A; \cdot, 0)$ is a UP-algebra.

In what follows, let A and B denote UP-algebras unless otherwise specified. The following proposition is very important for the study of UP-algebras.

Proposition 1.4. [8] *In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \cdot x = 0$,

(2) $x \cdot y = 0$ *and* $y \cdot z = 0$ *imply* $x \cdot z = 0$,

(3) $x \cdot y = 0$ *implies* $(z \cdot x) \cdot (z \cdot y) = 0$,

(4) $x \cdot y = 0$ *implies* $(y \cdot z) \cdot (x \cdot z) = 0$,

(5) $x \cdot (y \cdot x) = 0$,

(6) $(y \cdot x) \cdot x = 0$ *if and only if* $x = y \cdot x$, *and*

(7) $x \cdot (y \cdot y) = 0$.

On a UP-algebra $A = (A; \cdot, 0)$, we define a binary relation $\leq$ on A [8] as follows: for all $x, y \in A$,

$$x \leq y \text{ if and only if } x \cdot y = 0. \tag{1.1}$$

Proposition 1.5. [8] *In a UP-algebra A, the following properties hold: for any $x, y \in A$,*

(1) $x \leq x$,

(2) $x \leq y$ *and* $y \leq x$ *imply* $x = y$,

(3) $x \leq y$ *and* $y \leq z$ *imply* $x \leq z$,

(4) $x \leq y$ *implies* $z \cdot x \leq z \cdot y$,

(5) $x \leq y$ *implies* $y \cdot z \leq x \cdot z$,

(6) $x \leq y \cdot x$, *and*

(7) $x \leq y \cdot y$.

Proposition 1.6. [8] *An algebra $A = (A; \cdot, 0)$ of type $(2, 0)$ with a binary relation $\leq$ defined by (1.1) is a UP-algebra if and only if it satisfies the following conditions: for all $x, y, z \in A$,*

(1) $(y \cdot z) \leq (x \cdot y) \cdot (x \cdot z)$,

(2) $0 \cdot x = x$,

(3) $x \leq 0$, *and*

(4) $x \leq y$ *and* $y \leq x$ *imply* $x = y$.

Definition 1.7. [8] A nonempty subset B of A is called a *UP-ideal* of A if it satisfies the following properties:

(1) the constant 0 of A is in B, and

(2) for any $x, y, z \in A, x \cdot (y \cdot z) \in B$ and $y \in B$ imply $x \cdot z \in B$.

Clearly, A and $\{0\}$ are UP-ideals of A.

Theorem 1.8. [8] *Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-ideals of A. Then $\bigcap_{i \in I} B_i$ is a UP-ideal of A.*

Definition 1.9. [8] A subset S of A is called a *UP-subalgebra* of A if the constant 0 of A is in S, and $(S; \cdot, 0)$ itself forms a UP-algebra. Clearly, A and $\{0\}$ are UP-subalgebras of A.

Applying Proposition 1.4 (1), we can then easily prove the following proposition.

Proposition 1.10. [8] *A nonempty subset S of a UP-algebra $A = (A; \cdot, 0)$ is a UP-subalgebra of A if and only if S is closed under the $\cdot$ multiplication on A.*

Theorem 1.11. [8] *Let A be a UP-algebra and $\{B_i\}_{i \in I}$ a family of UP-subalgebras of A. Then $\bigcap_{i \in I} B_i$ is a UP-subalgebra of A.*

Definition 1.12. [38] A *fuzzy set* in a nonempty set X (or a fuzzy subset of X) is an arbitrary function $f\colon X \to [0,1]$ where $[0,1]$ is the unit segment of the real line.

Definition 1.13. A fuzzy set f in A is called an *anti-fuzzy UP-ideal* of A if it satisfies the following properties: for any $x, y, z \in A$,

(1) $f(0) \leq f(x)$, and

(2) $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$.

Definition 1.14. A fuzzy set f in A is called an *anti-fuzzy UP-subalgebra* of A if for any $x, y \in A$,

$$f(x \cdot y) \leq \max\{f(x), f(y)\}.$$

Example 1.15. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set f in A as follows:

$$f(0) = 0.1 \text{ and } f(1) = 0.2.$$

Using this data, we can show that f is an anti-fuzzy UP-ideal of A and an anti-fuzzy UP-subalgebra of A.

Example 1.16. Let $A = \{0, a, b\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	a	b
0	0	a	b
a	0	0	a
b	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set f in A as follows:

$$f(0) = 0.1, f(a) = 0.2 \text{ and } f(b) = 0.2.$$

Using this data, we can show that f is an anti-fuzzy UP-subalgebra of A.

Definition 1.17. [34] Let f be a fuzzy set in A. The fuzzy set $\overline{f}$ defined by $\overline{f}(x) = 1 - f(x)$ for all $x \in A$ is called the *complement* of f in A.

Remark 1.18. For all fuzzy set f in A, we have $f = \overline{\overline{f}}$.

Definition 1.19. [34] Let f be a fuzzy set in A. For any $t \in [0,1]$, the sets

$$U(f;t) = \{x \in A \mid f(x) \geq t\} \text{ and } U^+(f;t) = \{x \in A \mid f(x) > t\}$$

are called an *upper t-level subset* and an *upper t-strong level subset* of f, respectively. The sets

$$L(f;t) = \{x \in A \mid f(x) \leq t\} \text{ and } L^{-}(f;t) = \{x \in A \mid f(x) < t\}$$

are called a *lower t-level subset* and a *lower t-strong level subset* of f, respectively.

Definition 1.20. [21] Let f be a function from a nonempty set X to a nonempty set Y. If μ is a fuzzy set in X, then fuzzy set β in Y defined by

$$\beta(y) = \begin{cases} \inf\{\mu(t)\}_{t \in f^{-1}(y)} & \text{if } f^{-1}(y) \neq \emptyset, \\ 0 & \text{if } f^{-1}(y) = \emptyset \end{cases}$$

is said to be the *image of μ under f*. Similarly, if β is a fuzzy set in Y, then the fuzzy set $\mu = \beta \circ f$ in X (i.e., the fuzzy set defined by $\mu(x) = \beta(f(x))$ for all $x \in X$) is called the *preimage of β under f*.

Definition 1.21. [29] A fuzzy set f in A has inf *property* if for any nonempty subset T of A, there exists $t_0 \in T$ such that $f(t_0) = \inf\{f(t)\}_{t \in T}$.

Definition 1.22. [5] A *fuzzy relation* on a nonempty set X is an arbitrary function $f\colon X \times X \to [0,1]$ where $[0,1]$ is the unit segment of the real line.

Definition 1.23. [24] Let f and g be fuzzy sets in nonempty sets A and B, respectively. The *Cartesian product* of f and g is $f \times g\colon A \times B \to [0,1]$ defined by

$$(f \times g)(x,y) = \max\{f(x), g(y)\} \text{ for all } x \in A \text{ and } y \in B.$$

The *dot product* of f and g is $f \cdot g\colon A \times B \to [0,1]$ defined by

$$(f \cdot g)(x,y) = \min\{f(x), g(y)\} \text{ for all } x \in A \text{ and } y \in B.$$

Definition 1.24. [24] If f is a fuzzy set in a nonempty set X, the *strongest fuzzy relation* on X is $\mu_f\colon X \times X \to [0,1]$ defined by $\mu_f(x,y) = \max\{f(x), f(y)\}$ for all $x, y \in X$. For $x, y \in X$, we have $f(x), f(y) \in [0,1]$. Thus $\mu_f(x,y) = \max\{f(x), f(y)\} \in [0,1]$. Hence, μ_f is a fuzzy relation on X. We note that if f is a fuzzy set in a nonempty set X, then $f \times f = \mu_f$.

Definition 1.25. If f is a fuzzy set in a nonempty set X, the *weakness fuzzy relation* on X is $\beta_f\colon X \times X \to [0,1]$ defined by $\beta_f(x,y) = \min\{f(x), f(y)\}$ for all $x, y \in X$. For $x, y \in X$, we have $f(x), f(y) \in [0,1]$. Thus $\beta_f(x,y) = \min\{f(x), f(y)\} \in [0,1]$. Hence, β_f is a fuzzy relation on X. We note that if f is a fuzzy set in a nonempty set X, then $f \cdot f = \beta_f$.

Definition 1.26. [21] Let X and Y be any two nonempty sets and let $f\colon X \to Y$ be any function. A fuzzy set μ in X is called *f-invariant* if $f(x) = f(y)$ implies $\mu(x) = \mu(y)$ for all $x, y \in X$.

Definition 1.27. [8] Let $(A;\cdot,0)$ and $(A';\cdot',0')$ be UP-algebras. A mapping f from A to A' is called a *UP-homomorphism* if

$$f(x \cdot y) = f(x) \cdot' f(y) \text{ for all } x, y \in A.$$

A UP-homomorphism $f\colon A \to A'$ is called a

(1) *UP-endomorphism* of A if $A' = A$,

(2) *UP-epimorphism* if f is surjective,

(3) *UP-monomorphism* if f is injective, and

(4) *UP-isomorphism* if f is bijective. Moreover, we say A is *UP-isomorphic* to A', symbolically, $A \cong A'$, if there is a UP-isomorphism from A to A'.

Let f be a mapping from A to A', and let B be a nonempty subset of A, and B' of A'. The set $\{f(x) \mid x \in B\}$ is called the *image* of B under f, denoted by $f(B)$. In particular, $f(A)$ is called the *image* of f, denoted by $\mathrm{Im}(f)$. Dually, the set $\{x \in A \mid f(x) \in B'\}$ is said the *inverse image* of B' under f, symbolically, $f^{-1}(B')$. Especially, we say $f^{-1}(\{0'\})$ is the *kernel* of f, written by $\mathrm{Ker}(f)$. That is,

$$\mathrm{Im}(f) = \{f(x) \in A' \mid x \in A\}$$

and

$$\mathrm{Ker}(f) = \{x \in A \mid f(x) = 0'\}.$$

Theorem 1.28. [8] *Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-homomorphism. Then the following statements hold:*

(1) *$f(0_A) = 0_B$, and*

(2) *for any $x, y \in A$, if $x \leq y$, then $f(x) \leq f(y)$.*

Lemma 1.29. [34] *Let f be a fuzzy set in A. Then the following statements hold: for any $x, y \in A$,*

(1) *$1 - \max\{f(x), f(y)\} = \min\{1 - f(x), 1 - f(y)\}$, and*

(2) *$1 - \min\{f(x), f(y)\} = \max\{1 - f(x), 1 - f(y)\}$.*

2. Main results

In this section, we study anti-fuzzy UP-ideals and anti-fuzzy UP-subalgebras of UP-algebras.

Theorem 2.1. *Every anti-fuzzy UP-ideal of A is an anti-fuzzy UP-subalgebra of A.*

Proof. Let f be an anti-fuzzy UP-ideal of A. Let $x, y \in A$. Then

$$\begin{aligned}
f(x \cdot y) &\leq \max\{f(x \cdot (y \cdot y)), f(y)\} && \text{(Definition 1.13 (2))}\\
&= \max\{f(x \cdot 0), f(y)\} && \text{(Proposition 1.4 (1))}\\
&= \max\{f(0), f(y)\} && \text{(UP-3)}\\
&= f(y) && \text{(Definition 1.13 (1))}\\
&\leq \max\{f(x), f(y)\}.
\end{aligned}$$

Hence, f is an anti-fuzzy UP-subalgebra of A. ∎

Lemma 2.2. *Let f be an anti-fuzzy UP-ideal of A. If the inequality $x \leq y \cdot z$ holds in A for all $x, y, z \in A$, then $f(z) \leq \max\{f(x), f(y)\}$ for all $x, y, z \in A$.*

Proof. Assume $x \leq y \cdot z$ for all $x, y, z \in A$. Then $x \cdot (y \cdot z) = 0$. By Definition 1.13 (2), we have

$$f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}. \tag{2.1}$$

By (2.1) and (UP-2), let $x = 0$, so

$$f(z) = f(0 \cdot z) \leq \max\{f(0 \cdot (x \cdot z)), f(x)\} = \max\{f(x \cdot z), f(x)\}. \tag{2.2}$$

By (2.1) and Definition 1.13 (1), we have

$$f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} = \max\{f(0), f(y)\} = f(y). \tag{2.3}$$

By (2.2) and (2.3), we have

$$f(z) \leq \max\{f(x \cdot z), f(x)\} \leq \max\{f(y), f(x)\} = \max\{f(x), f(y)\}.$$ ∎

Lemma 2.3. *If f is an anti-fuzzy UP-ideal of A and if $x, y \in A$ is such that $x \leq y$ in A, then $f(x) \geq f(y)$.*

Proof. Let $x, y \in A$ be such that $x \leq y$ in A. Then $x \cdot y = 0$. Thus

$$\begin{aligned}
f(y) &= f(0 \cdot y) && \text{((UP-2))} \\
&\leq \max\{f(0 \cdot (x \cdot y)), f(x)\} && \text{(Definition 1.13 (2))} \\
&= \max\{f(0 \cdot 0), f(x)\} \\
&= \max\{f(0), f(x)\} && \text{((UP-2))} \\
&= f(x). && \text{(Definition 1.13 (1))}
\end{aligned}$$

∎

Lemma 2.4. [26] *Let f be a fuzzy set in A. For any $t \in [0,1]$, the following properties hold:*

(1) $L(f;t) = U(\overline{f}; 1-t)$,

(2) $L^-(f;t) = U^+(\overline{f}; 1-t)$,

(3) $U(f;t) = L(\overline{f}; 1-t)$, *and*

(4) $U^+(f;t) = L^-(\overline{f}; 1-t)$.

Lemma 2.5. [20] *For any $a, b \in \mathbb{R}$ such that $a < b$, $a < \dfrac{b+a}{2} < b$.*

Theorem 2.6. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *f is an anti-fuzzy UP-ideal of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $L(f;t) \neq \emptyset$ implies $L(f;t)$ is a UP-ideal of A,*

(2) *f is an anti-fuzzy UP-ideal of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $L^-(f;t) \neq \emptyset$ implies $L^-(f;t)$ is a UP-ideal of A,*

(3) *$\overline{f}$ is an anti-fuzzy UP-ideal of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $U(f;t) \neq \emptyset$ implies $U(f;t)$ is a UP-ideal of A, and*

(4) *$\overline{f}$ is an anti-fuzzy UP-ideal of A if and only if it satisfies the condition $(\star)$: for all $t \in [0,1]$, $U^+(f;t) \neq \emptyset$ implies $U^+(f;t)$ is a UP-ideal of A.*

Proof. (1) Assume that f is an anti-fuzzy UP-ideal of A. Let $t \in [0,1]$ be such that $L(f;t) \neq \emptyset$. Let $x \in A$ be such that $x \in L(f;t)$. Then $f(x) \leq t$. Thus

$$\begin{aligned} f(0) &= f(x \cdot 0) && \text{(UP-3)} \\ &\leq \max\{f(x \cdot (x \cdot 0)), f(x)\} && \text{(Definition 1.13(2))} \\ &= \max\{f(x \cdot 0), f(x)\} && \text{(UP-3)} \\ &= \max\{f(0), f(x)\} && \text{(UP-3)} \\ &= f(x) && \text{(Definition 1.13(1))} \\ &\leq t. \end{aligned}$$

Thus $0 \in L(f;t)$. Now, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L(f;t)$ and $y \in L(f;t)$. Then $f(x \cdot (y \cdot z)) \leq t$ and $f(y) \leq t$. By Definition 1.13 (2), we have $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} \leq t$, so $x \cdot z \in L(f;t)$. Hence, $L(f;t)$ is a UP-ideal of A.

Conversely, assume that condition $(\star)$ holds and suppose that Definition 1.13 (1) is false. Then there exists $x \in A$ such that $f(0) > f(x)$. Let $t = \dfrac{f(x) + f(0)}{2}$. Then $t \in [0,1]$ and by Lemma 2.5, we have $f(0) > t > f(x)$. Thus $x \in L(f;t)$, so $L(f;t) \neq \emptyset$. By assumption, $L(f;t)$ is a UP-ideal of A. Thus $0 \in L(f;t)$, so $f(0) \leq t$. This is a contradiction. Hence, $f(0) \leq f(x)$ for all $x \in A$. Suppose that Definition 1.13 (2) is false. Then there exist $x, y, z \in A$ such that $f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\}$. Let $g_0 = \dfrac{f(x \cdot z) + \max\{f(x \cdot (y \cdot z)), f(y)\}}{2}$. Then $g_0 \in [0,1]$ and by Lemma 2.5, we have $\max\{f(x \cdot (y \cdot z)), f(y)\} < g_0 < f(x \cdot z)$. Thus $x \cdot z \notin L(f;g_0)$. Since $\max\{f(x \cdot (y \cdot z)), f(y)\} < g_0$, we have $f(y) < g_0$ and $f(x \cdot (y \cdot z)) < g_0$. Thus $y \in L(f;g_0)$ and $x \cdot (y \cdot z) \in L(f;g_0)$, so $L(f;g_0) \neq \emptyset$. By assumption, we have $L(f;g_0)$ is a UP-ideal of A. Thus $x \cdot z \in L(f;g_0)$. This is a contradiction. Hence, $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$ for all $x, y, z \in A$. Therefore, f is an anti-fuzzy UP-ideal of A.

(2) Assume that f is an anti-fuzzy UP-ideal of A. Let $t \in [0,1]$ be such that $L^-(f;t) \neq \emptyset$. Let $x \in A$ be such that $x \in L^-(f;t)$. Then $f(x) < t$. Thus

$$\begin{aligned} f(0) &= f(x \cdot 0) && \text{(UP-3)} \\ &\leq \max\{f(x \cdot (x \cdot 0)), f(x)\} && \text{(Definition 1.13(2))} \\ &= \max\{f(x \cdot 0), f(x)\} && \text{(UP-3)} \\ &= \max\{f(0), f(x)\} && \text{(UP-3)} \\ &= f(x) && \text{(Definition 1.13(1))} \\ &< t. \end{aligned}$$

Thus $0 \in L^-(f;t)$. Now, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in L^-(f;t)$ and $y \in L^-(f;t)$. Then $f(x \cdot (y \cdot z)) < t$ and $f(y) < t$. By Definition 1.13 (2), we have $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\} < t$, so $x \cdot z \in L^-(f;t)$. Hence, $L^-(f;t)$ is a UP-ideal of A.

Conversely, assume that the condition $(\star)$ holds and suppose that Definition 1.13 (1) is false. Then there exists $x \in A$ such that $f(0) > f(x)$. Let $t = \dfrac{f(x) + f(0)}{2}$. Then $t \in [0,1]$ and by Lemma 2.5, we have $f(0) > t > f(x)$. Thus $x \in L^-(f;t)$, so $L^-(f;t) \neq \emptyset$. By assumption, $L^-(f;t)$ is a UP-ideal of A. Thus $0 \in L^-(f;t)$, so $f(0) < t$. This is a contradiction. Hence, $f(0) \leq f(x)$ for all $x \in A$. Suppose that Definition 1.13 (2) is false. Then there exist $x, y, z \in A$ such that $f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\}$. Let $g_0 = \dfrac{f(x \cdot z) + \max\{f(x \cdot (y \cdot z)), f(y)\}}{2}$. Then $g_0 \in [0,1]$ and by Lemma 2.5, we have $\max\{f(x \cdot (y \cdot z)), f(y)\} < g_0 < f(x \cdot z)$. Thus $x \cdot z \notin L^-(f;g_0)$. Since $\max\{f(x \cdot (y \cdot z)), f(y)\} < g_0$, we have $f(y) < g_0$ and $f(x \cdot (y \cdot z)) < g_0$. Thus $y \in L^-(f;g_0)$ and $x \cdot (y \cdot z) \in L^-(f;g_0)$, so $L^-(f;g_0) \neq \emptyset$. By assumption, we have $L^-(f;g_0)$ is a UP-ideal of A. Thus $x \cdot z \in L^-(f;g_0)$. This is a contradiction. Hence, $f(x \cdot z) \leq \max\{f(x \cdot (y \cdot z)), f(y)\}$ for all $x, y, z \in A$. Therefore, f is an anti-fuzzy UP-ideal of A.

(3) Assume that $\overline{f}$ is an anti-fuzzy UP-ideal of A. Let $t \in [0,1]$ be such that $U(f;t) \neq \emptyset$. Let $x \in A$ be such that $x \in U(f;t)$. Then $f(x) \geq t$. Thus

$$
\begin{aligned}
\overline{f}(0) &= \overline{f}(x \cdot 0) && \text{(UP-3)}\\
&\leq \max\{\overline{f}(x \cdot (x \cdot 0)), \overline{f}(x)\} && \text{(Definition 1.13 (2))}\\
&= \max\{\overline{f}(x \cdot 0), \overline{f}(x)\} && \text{(UP-3)}\\
&= \max\{\overline{f}(0), \overline{f}(x)\} && \text{(UP-3)}\\
&= \overline{f}(x). && \text{(Definition 1.13 (1))}
\end{aligned}
$$

This implies that $1 - f(0) \leq 1 - f(x)$. Hence, $f(0) \geq f(x) \geq t$, so $0 \in U(f;t)$. Now, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U(f;t)$ and $y \in U(f;t)$. Then $f(x \cdot (y \cdot z)) \geq t$ and $f(y) \geq t$. By Definition 1.13 (2), we have $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. Thus

$$
\begin{aligned}
1 - f(x \cdot z) &\leq \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\}\\
&= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. && \text{(Lemma 1.29 (2))}
\end{aligned}
$$

Thus $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} \geq t$, so $x \cdot z \in U(f;t)$. Hence, $U(f;t)$ is a UP-ideal of A.

Conversely, assume that condition $(\star)$ holds and suppose that Definition 1.13 (1) is false. Then there exists $x \in A$ such that $\overline{f}(0) > \overline{f}(x)$. Thus $1 - f(0) > 1 - f(x)$, so $f(0) < f(x)$. Let $t = \dfrac{f(x) + f(0)}{2}$. Then $t \in [0,1]$ and by Lemma 2.5, we have $f(0) < t < f(x)$. Thus $x \in U(f;t)$, so $U(f;t) \neq \emptyset$. By assumption, $U(f;t)$ is a UP-ideal of A. Thus $0 \in U(f;t)$, so $f(0) \geq t$. This is a contradiction. Hence, $\overline{f}(0) \leq \overline{f}(x)$ for all $x \in A$. Suppose that Definition 1.13 (2) is false. Then there exist $x, y, z \in A$ such that $\overline{f}(x \cdot z) > \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot z) &> \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} \\ &= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. && \text{(Lemma 1.29 (2))} \end{aligned}$$

Thus $f(x \cdot z) < \min\{f(x \cdot (y \cdot z)), f(y)\}$. Let $g_0 = \dfrac{f(x \cdot z) + \min\{f(x \cdot (y \cdot z)), f(y)\}}{2}$. Then $g_0 \in [0, 1]$ and by Lemma 2.5, we have $f(x \cdot z) < g_0 < \min\{f(x \cdot (y \cdot z)), f(y)\}$. Thus $x \cdot z \notin U(f; g_0)$. Since $\min\{f(x \cdot (y \cdot z)), f(y)\} > g_0$, we have $f(y) > g_0$ and $f(x \cdot (y \cdot z)) > g_0$. Thus $y \in U(f; g_0)$ and $x \cdot (y \cdot z) \in U(f; g_0)$, so $U(f; g_0) \neq \emptyset$. By assumption, we have $U(f; g_0)$ is a UP-ideal of A. Thus $x \cdot z \in U(f; g_0)$. This is a contradiction. Hence, $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$ for all $x, y, z \in A$. Therefore, $\overline{f}$ is an anti-fuzzy UP-ideal of A.

(4) Assume that $\overline{f}$ is an anti-fuzzy UP-ideal of A. Let $t \in [0, 1]$ be such that $U^+(f; t) \neq \emptyset$. Let $x \in A$ be such that $x \in U^+(f; t)$. Then $f(x) > t$. Thus

$$\begin{aligned} \overline{f}(0) &= \overline{f}(x \cdot 0) && \text{(UP-3)} \\ &\leq \max\{\overline{f}(x \cdot (x \cdot 0)), \overline{f}(x)\} && \text{(Definition 1.13 (2))} \\ &= \max\{\overline{f}(x \cdot 0), \overline{f}(x)\} && \text{(UP-3)} \\ &= \max\{\overline{f}(0), \overline{f}(x)\} && \text{(UP-3)} \\ &= \overline{f}(x). && \text{(Definition 1.13 (1))} \end{aligned}$$

This implies that $1 - f(0) \leq 1 - f(x)$. Hence, $f(0) \geq f(x) > t$, so $0 \in U^+(f; t)$. Now, let $x, y, z \in A$ be such that $x \cdot (y \cdot z) \in U^+(f; t)$ and $y \in U^+(f; t)$. Then $f(x \cdot (y \cdot z)) > t$ and $f(y) > t$. By Definition 1.13 (2), we have $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot z) &\leq \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} \\ &= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. && \text{(Lemma 1.29 (2))} \end{aligned}$$

Thus $f(x \cdot z) \geq \min\{f(x \cdot (y \cdot z)), f(y)\} > t$, so $x \cdot z \in U^+(f; t)$. Hence, $U^+(f; t)$ is a UP-ideal of A.

Conversely, assume that condition $(\star)$ holds and suppose that Definition 1.13 (1) is false. Then there exists $x \in A$ such that $\overline{f}(0) > \overline{f}(x)$. Thus $1 - f(0) > 1 - f(x)$, so $f(0) < f(x)$. Let $t = \dfrac{f(x) + f(0)}{2}$. Then $t \in [0, 1]$ and by Lemma 2.5, we have $f(0) < t < f(x)$. Thus $x \in U^+(f; t)$, so $U^+(f; t) \neq \emptyset$. By assumption, $U^+(f; t)$ is a UP-ideal of A. Thus $0 \in U^+(f; t)$, so $f(0) > t$. This is a contradiction. Hence, $\overline{f}(0) \leq \overline{f}(x)$ for all $x \in A$. Suppose that Definition 1.13 (2) is false. Then there exist $x, y, z \in A$ such that $\overline{f}(x \cdot z) > \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot z) &> \max\{1 - f(x \cdot (y \cdot z)), 1 - f(y)\} \\ &= 1 - \min\{f(x \cdot (y \cdot z)), f(y)\}. && \text{(Lemma 1.29 (2))} \end{aligned}$$

Thus $f(x \cdot z) < \min\{f(x \cdot (y \cdot z)), f(y)\}$. Let $g_0 = \dfrac{f(x \cdot z) + \min\{f(x \cdot (y \cdot z)), f(y)\}}{2}$. Then $g_0 \in [0, 1]$ and by Lemma 2.5, we have $f(x \cdot z) < g_0 < \min\{f(x \cdot (y \cdot z)), f(y)\}$.

Then $x \cdot z \notin U^+(f; g_0)$. Since $\min\{f(x \cdot (y \cdot z)), f(y)\} > g_0$, we have $f(y) > g_0$ and $f(x \cdot (y \cdot z)) > g_0$. Thus $y \in U^+(f; g_0)$ and $x \cdot (y \cdot z) \in U^+(f; g_0)$, so $U^+(f; g_0) \neq \emptyset$. By assumption, we have $U^+(f; g_0)$ is a UP-ideal of A. Thus $x \cdot z \in U^+(f; g_0)$. This is a contradiction. Hence, $\overline{f}(x \cdot z) \leq \max\{\overline{f}(x \cdot (y \cdot z)), \overline{f}(y)\}$ for all $x, y, z \in A$. Therefore, $\overline{f}$ is an anti-fuzzy UP-ideal of A. ∎

Theorem 2.7. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *f is an anti-fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0, 1]$, $L(f; t) \neq \emptyset$ implies $L(f; t)$ is a UP-subalgebra of A,*

(2) *f is an anti-fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0, 1]$, $L^-(f; t) \neq \emptyset$ implies $L^-(f; t)$ is a UP-subalgebra of A,*

(3) *$\overline{f}$ is an anti-fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0, 1]$, $U(f; t) \neq \emptyset$ implies $U(f; t)$ is a UP-subalgebra of A, and*

(4) *$\overline{f}$ is an anti-fuzzy UP-subalgebra of A if and only if it satisfies the condition $(\star)$: for all $t \in [0, 1]$, $U^+(f; t) \neq \emptyset$ implies $U^+(f; t)$ is a UP-subalgebra of A.*

Proof. (1) Assume that f is an anti-fuzzy UP-subalgebra of A. Let $t \in [0, 1]$ be such that $L(f; t) \neq \emptyset$. Let $x, y \in L(f; t)$. Then $f(x) \leq t$ and $f(y) \leq t$. Since f is an anti-fuzzy UP-subalgebra of A, we have $f(x \cdot y) \leq \max\{f(x), f(y)\} \leq t$. Thus $x \cdot y \in L(f; t)$. Hence, $L(f; t)$ is a UP-subalgebra of A.

Conversely, assume that the condition $(\star)$ holds. Let $x, y \in A$ and let $t = \max\{f(x), f(y)\}$, so $t \in [0, 1]$. Thus $f(x) \leq t$ and $f(y) \leq t$, so $x, y \in L(f; t)$. Thus $L(f; t) \neq \emptyset$. By assumption, we have $L(f; t)$ is a UP-subalgebra of A. Thus $x \cdot y \in L(f; t)$, so $f(x \cdot y) \leq t = \max\{f(x), f(y)\}$. Hence, f is an anti-fuzzy UP-subalgebra of A.

(2) Assume that f is an anti-fuzzy UP-subalgebra of A. Let $t \in [0, 1]$ be such that $L^-(f; t) \neq \emptyset$. Let $x, y \in L^-(f; t)$. Then $f(x) < t$ and $f(y) < t$. Since f is an anti-fuzzy UP-subalgebra of A, we have $f(x \cdot y) \leq \max\{f(x), f(y)\} < t$. Thus $x \cdot y \in L^-(f; t)$. Hence, $L^-(f; t)$ is a UP-subalgebra of A.

Conversely, assume that the condition $(\star)$ holds. Suppose that there exist $x, y \in A$ such that $f(x \cdot y) > \max\{f(x), f(y)\}$. Then $f(x \cdot y) \in [0, 1]$. Choose $t = f(x \cdot y)$. Thus $f(x) < t$ and $f(y) < t$, so $x, y \in L^-(f; t) \neq \emptyset$. By assumption, we have $L^-(f; t)$ is a UP-subalgebra of A and so $x \cdot y \in L^-(f; t)$. Thus $f(x \cdot y) < t = f(x \cdot y)$ which is a contradiction. Hence, $f(x \cdot y) \leq \max\{f(x), f(y)\}$ for all $x, y \in A$. Therefore, f is an anti-fuzzy UP-subalgebra of A.

(3) Assume that $\overline{f}$ is an anti-fuzzy UP-subalgebra of A. Let $t \in [0, 1]$ be such that $U(f; t) \neq \emptyset$. Let $x, y \in U(f; t)$. Then $f(x) \geq t$ and $f(y) \geq t$. Since $\overline{f}$ is an anti-fuzzy UP-subalgebra of A, we have $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$. Thus

$$\begin{aligned} 1 - f(x \cdot y) &\leq \max\{1 - f(x), 1 - f(y)\} \\ &= 1 - \min\{f(x), f(y)\}. && \text{(Lemma 1.29(2))} \end{aligned}$$

Thus $f(x \cdot y) \geq \min\{f(x), f(y)\} \geq t$. Thus $x \cdot y \in U(f;t)$. Hence, $U(f;t)$ is a UP-subalgebra of A.

Conversely, assume that the condition $(\star)$ holds. Let $x, y \in A$ and let $t = \min\{f(x), f(y)\}$, so $t \in [0,1]$. Thus $f(x) \geq t$ and $f(y) \geq t$, so $x, y \in U(f;t)$. Thus $U(f;t) \neq \emptyset$. By assumption, we have $U(f;t)$ is a UP-subalgebra of A. Thus $x \cdot y \in U(f;t)$, so $f(x \cdot y) \geq t = \min\{f(x), f(y)\}$. Thus $1 - f(x \cdot y) \leq 1 - \min\{f(x), f(y)\}$, so

$$\begin{aligned}
\overline{f}(x \cdot y) &= 1 - f(x \cdot y) \\
&\leq 1 - \min\{f(x), f(y)\} \\
&= \max\{1 - f(x), 1 - f(y)\} && \text{(Lemma 1.29(2))} \\
&= \max\{\overline{f}(x), \overline{f}(y)\}.
\end{aligned}$$

Hence, $\overline{f}$ is an anti-fuzzy UP-subalgebra of A.

(4) Assume that $\overline{f}$ is an anti-fuzzy UP-subalgebra of A. Let $t \in [0,1]$ be such that $U^+(f;t) \neq \emptyset$. Let $x, y \in U^+(f;t)$. Then $f(x) > t$ and $f(y) > t$. Since $\overline{f}$ is an anti-fuzzy UP-subalgebra of A, we have $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$. Thus

$$\begin{aligned}
1 - f(x \cdot y) &\leq \max\{1 - f(x), 1 - f(y)\} \\
&= 1 - \min\{f(x), f(y)\}. && \text{(Lemma 1.29 (2))}
\end{aligned}$$

Thus $f(x \cdot y) \geq \min\{f(x), f(y)\} > t$, so $x \cdot y \in U^+(f;t)$. Hence, $U^+(f;t)$ is a UP-subalgebra of A.

Conversely, assume that the condition $(\star)$ holds. Suppose that there exist $x, y \in A$ such that $\overline{f}(x \cdot y) > \max\{\overline{f}(x), \overline{f}(y)\}$. Thus

$$\begin{aligned}
1 - f(x \cdot y) &> \max\{1 - f(x), 1 - f(y)\} \\
&= 1 - \min\{f(x), f(y)\}. && \text{(Lemma 1.29(2))}
\end{aligned}$$

Thus $f(x \cdot y) < \min\{f(x), f(y)\}$. Now $f(x \cdot y) \in [0,1]$, we choose $t = f(x \cdot y)$. Thus $f(x) > t$ and $f(y) > t$, so $x, y \in U^+(f;t) \neq \emptyset$. By assumption, we have $U^+(f;t)$ is a UP-subalgebra of A and so $x \cdot y \in U^+(f;t)$. Thus $f(x \cdot y) > t = f(x \cdot y)$ which is a contradiction. Hence, $\overline{f}(x \cdot y) \leq \max\{\overline{f}(x), \overline{f}(y)\}$ for all $x, y \in A$. Therefore, $\overline{f}$ is an anti-fuzzy UP-subalgebra of A. ∎

Proposition 2.8. *If f is an anti-fuzzy UP-ideal of A, then $f(x \cdot (x \cdot y)) \leq f(y)$ for all $x, y \in A$.*

Proof. Let $x, y \in A$. Then

$$\begin{aligned}
f(x \cdot (x \cdot y)) &\leq \max\{f(x \cdot ((x \cdot y) \cdot (x \cdot y))), f(x \cdot y)\} && \text{(Definition 1.13(2))} \\
&= \max\{f(x \cdot 0), f(x \cdot y)\} && \text{(Proposition 1.4(1))} \\
&= \max\{f(0), f(x \cdot y)\} && \text{(UP-3)} \\
&= f(x \cdot y) && \text{(Definition 1.13 (1))} \\
&\leq \max\{f(x \cdot (y \cdot y)), f(y)\} && \text{(Definition 1.13 (2))} \\
&= \max\{f(x \cdot 0), f(y)\} && \text{(Proposition 1.4 (1))} \\
&= \max\{f(0), f(y)\} && \text{(UP-3)} \\
&= f(y). && \text{(Definition 1.13 (1))}
\end{aligned}$$

Corollary 2.9. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *if f is an anti-fuzzy UP-ideal of A, then for every $t \in \mathrm{Im}(f)$, $L(f;t)$ is a UP-ideal of A, and*

(2) *if $\overline{f}$ is an anti-fuzzy UP-ideal of A, then for every $t \in \mathrm{Im}(f)$, $U(f;t)$ is a UP-ideal of A.*

Proof. (1) Assume that f is an anti-fuzzy UP-ideal of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \leq t$. Thus $x \in L(f;t)$, so $L(f;t) \neq \emptyset$. By Theorem 2.6 (1), we have $L(f;t)$ is a UP-ideal of A.

(2) Assume that $\overline{f}$ is an anti-fuzzy UP-ideal of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \geq t$. Thus $x \in U(f;t)$, so $U(f;t) \neq \emptyset$. By Theorem 2.6 (3), we have $U(f;t)$ is a UP-ideal of A. ∎

Corollary 2.10. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *if f is an anti-fuzzy UP-subalgebra of A, then for every $t \in \mathrm{Im}(f)$, $L(f;t)$ is a UP-subalgebra of A, and*

(2) *if $\overline{f}$ is an anti-fuzzy UP-subalgebra of A, then for every $t \in \mathrm{Im}(f)$, $U(f;t)$ is a UP-subalgebra of A.*

Proof. (1) Assume that f is an anti-fuzzy UP-subalgebra of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \leq t$. Thus $x \in L(f;t)$, so $L(f;t) \neq \emptyset$. By Theorem 2.7 (1), we have $L(f;t)$ is a UP-subalgebra of A.

(2) Assume that $\overline{f}$ is an anti-fuzzy UP-subalgebra of A and let $t \in \mathrm{Im}(f)$. Then $t = f(x)$ for some $x \in A$, so $f(x) \geq t$. Thus $x \in U(f;t)$, so $U(f;t) \neq \emptyset$. By Theorem 2.7 (3), we have $U(f;t)$ is a UP-subalgebra of A. ∎

Corollary 2.11. *Let I be a UP-ideal of A. Then the following statements hold:*

(1) *for any $k \in (0,1]$, there exists an anti-fuzzy UP-ideal f of A such that $L(f;t) = I$ for all $t < k$ and $L(f;t) = A$ for all $t \geq k$, and*

(2) *for any $k \in [0,1)$, there exists an anti-fuzzy UP-ideal g of A such that $U(\overline{g};t) = I$ for all $t > k$ and $U(\overline{g};t) = A$ for all $t \leq k$.*

Proof. (1) Define a fuzzy set $f\colon A \to [0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

Case 1: To show that $L(f;t) = I$ for all $t < k$, let $t \in [0,1]$ be such that $t < k$. Let $x \in L(f;t)$. Then $f(x) \leq t < k$, so $f(x) \neq k$. Thus $f(x) = 0$, so $x \in I$. That is $L(f;t) \subseteq I$. Let $x \in I$. Then $f(x) = 0 \leq t$, so $x \in L(f;t)$. That is $I \subseteq L(f;t)$. Hence, $L(f;t) = I$ for all $t < k$.

Case 2: To show that $L(f;t) = A$ for all $t \geq k$, let $t \in [0,1]$ be such that $t \geq k$. Clearly, $L(f;t) \subseteq A$. Let $x \in A$. Then

$$f(x) = \begin{cases} 0 < t & \text{if } x \in I, \\ k \leq t & \text{if } x \notin I, \end{cases}$$

so $x \in L(f;t)$. That is $A \subseteq L(f;t)$. Hence, $L(f;t) = A$ for all $t \geq k$.

It follows from Theorem 2.6 (1) that f is an anti-fuzzy UP-ideal of A.

(2) Define a fuzzy set $f \colon A \to [0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

Case 1: To show that $U(f;t) = I$ for all $t > k$, let $t \in [0,1]$ be such that $t > k$. Let $x \in U(f;t)$. Then $f(x) \geq t > k$, so $f(x) \neq k$. Thus $f(x) = 1$, so $x \in I$. That is $U(f;t) \subseteq I$. Let $x \in I$. Then $f(x) = 1 \geq t$, so $x \in U(f;t)$. That is $I \subseteq U(f;t)$. Hence, $U(f;t) = I$ for all $t > k$.

Case 2: To show that $U(f;t) = A$ for all $t \leq k$, let $t \in [0,1]$ be such that $t \leq k$. Clearly, $U(f;t) \subseteq A$. Let $x \in A$. Then

$$f(x) = \begin{cases} 1 > t & \text{if } x \in I, \\ k \geq t & \text{if } x \notin I, \end{cases}$$

so $x \in U(f;t)$. That is $A \subseteq U(f;t)$. Hence, $U(f;t) = A$ for all $t \leq k$.

It follows from Theorem 2.6 (3) that $\overline{f}$ is an anti-fuzzy UP-ideal of A. By Remark 1.18, we have $U(\overline{\overline{f}};t) = U(f;t) = I$ for all $t > k$ and $U(\overline{\overline{f}};t) = U(f;t) = A$ for all $t \leq k$. Putting $\overline{f} = g$. Then g is an anti-fuzzy UP-ideal of A such that $U(\overline{g};t) = I$ for all $t > k$ and $U(\overline{g};t) = A$ for all $t \leq k$. ■

Corollary 2.12. *Let I be a UP-subalgebra of A. Then the following statements hold:*

(1) *for any $k \in (0,1]$, there exists an anti-fuzzy UP-subalgebra f of A such that $L(f;t) = I$ for all $t < k$ and $L(f;t) = A$ for all $t \geq k$, and*

(2) *for any $k \in [0,1)$, there exists an anti-fuzzy UP-subalgebra g of A such that $U(\overline{g};t) = I$ for all $t > k$ and $U(\overline{g};t) = A$ for all $t \leq k$.*

Proof. (1) Define a fuzzy set $f \colon A \to [0,1]$ by

$$f(x) = \begin{cases} 0 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

In the proof of Corollary 2.11 (1), we have $L(f;t) = I$ for all $t < k$ and $L(f;t) = A$ for all $t \geq k$.

It follows from Theorem 2.7 (1) that f is an anti-fuzzy UP-subalgebra of A.

(2) Define a fuzzy set $f \colon A \to [0,1]$ by

$$f(x) = \begin{cases} 1 & \text{if } x \in I, \\ k & \text{if } x \notin I. \end{cases}$$

In the proof of Corollary 2.11 (2), we have $U(f;t) = I$ for all $t > k$ and $U(f;t) = A$ for all $t \leq k$.

It follows from Theorem 2.7 (3) that $\overline{f}$ is an anti-fuzzy UP-subalgebra of A. By Remark 1.18, we have $U(\overline{\overline{f}};t) = U(f;t) = I$ for all $t > k$ and $U(\overline{\overline{f}};t) = U(f;t) = A$ for all $t \leq k$. Putting $\overline{f} = g$. Then g is an anti-fuzzy UP-subalgebra of A such that $U(\overline{g};t) = I$ for all $t > k$ and $U(\overline{g};t) = A$ for all $t \leq k$. ∎

Theorem 2.13. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let $f\colon A \to B$ be a UP-epimorphism. Then the following statements hold:*

(1) *for every anti-fuzzy UP-ideal β of B, μ is an anti-fuzzy UP-ideal of A, and*

(2) *for every anti-fuzzy UP-subalgebra β of B, μ is an anti-fuzzy UP-subalgebra of A.*

Proof. (1) Let β be an anti-fuzzy UP-ideal of B. Let $x \in A$. Then

$$\begin{aligned}
\mu(0_A) &= (\beta \circ f)(0_A) \\
&= \beta(f(0_A)) \\
&\leq \beta(f(x)) && \text{(Proposition 1.6 (3), Theorem 1.28 (2), Lemma 2.3)} \\
&= (\beta \circ f)(x) \\
&= \mu(x).
\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}
\mu(x \cdot z) &= (\beta \circ f)(x \cdot z) \\
&= \beta(f(x \cdot z)) \\
&= \beta(f(x) * f(z)) \\
&\leq \max\{\beta(f(x) * (f(y) * f(z))), \beta(f(y))\} && \text{(Definition 1.13 (2))} \\
&= \max\{\beta(f(x \cdot (y \cdot z))), \beta(f(y))\} \\
&= \max\{(\beta \circ f)(x \cdot (y \cdot z)), (\beta \circ f)(y)\} \\
&= \max\{\mu(x \cdot (y \cdot z)), \mu(y)\}.
\end{aligned}$$

Hence, μ is an anti-fuzzy UP-ideal of A.

(2) Let β be an anti-fuzzy UP-subalgebra of B. Let $x, y \in A$. Then

$$\begin{aligned}
\mu(x \cdot y) &= (\beta \circ f)(x \cdot y) \\
&= \beta(f(x \cdot y)) \\
&= \beta(f(x) * f(y)) \\
&\leq \max\{\beta(f(x)), \beta(f(y))\} && \text{(Definition 1.14)} \\
&= \max\{(\beta \circ f)(x), (\beta \circ f)(y)\} \\
&= \max\{\mu(x), \mu(y)\}.
\end{aligned}$$

Hence, μ is an anti-fuzzy UP-subalgebra of A. ∎

Lemma 2.14. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let $f\colon A\to B$ be a UP-epimorphism. Let μ be an f-invariant fuzzy set in A with* inf *property. For any $a,b\in B$, there exist $a_0\in f^{-1}(a)$ and $b_0\in f^{-1}(b)$ such that $\beta(a)=\mu(a_0)$, $\beta(b)=\mu(b_0)$ and $\beta(a\cdot b)=\mu(a_0\cdot b_0)$.*

Proof. Let $a,b\in B$. Since f is surjective, we have $f^{-1}(a), f^{-1}(b)$ and $f^{-1}(a*b)$ are nonempty subsets of A. By Definition 1.20, we obtain

$$\begin{aligned}\beta(a)&=\inf\{\mu(t)\}_{t\in f^{-1}(a)}\\&=\mu(a_0)\text{ for some }a_0\in f^{-1}(a), &&\text{(Definition 1.21)}\\ \beta(b)&=\inf\{\mu(t)\}_{t\in f^{-1}(b)}\\&=\mu(b_0)\text{ for some }b_0\in f^{-1}(b) &&\text{(Definition 1.21)}\\ \beta(a\cdot b)&=\inf\{\mu(t)\}_{t\in f^{-1}(a\cdot b)}\\&=\mu(c)\text{ for some }c\in f^{-1}(a*b). &&\text{(Definition 1.21)}\end{aligned}$$

Since $f(c)=a*b=f(a_0)*f(b_0)=f(a_0\cdot b_0)$ and μ is f-invariant, we have $\mu(c)=\mu(a_0\cdot b_0)$. Hence, $\beta(a\cdot b)=\mu(a_0\cdot b_0)$. ■

Theorem 2.15. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let $f\colon A\to B$ be a UP-epimorphism. Then the following statements hold:*

(1) *for every f-invariant anti-fuzzy UP-ideal μ of A with* inf *property, β is an anti-fuzzy UP-ideal of B, and*

(2) *for every f-invariant anti-fuzzy UP-subalgebra μ of A with* inf *property, β is an anti-fuzzy UP-subalgebra of B.*

Proof. (1) Let μ be an f-invariant anti-fuzzy UP-ideal of A with inf property. By Definition 1.13 (1), we have $\mu(0_A)\le\mu(x)$ for all $x\in A$. By Theorem 1.28 (1), we have $0_A\in f^{-1}(0_B)$ and so $f^{-1}(0_B)\neq\emptyset$. Thus $\beta(0_B)=\inf\{\mu(t)\}_{t\in f^{-1}(0_B)}\le\mu(0_A)$. Let $y\in B$. Since f is surjective, we have $f^{-1}(y)\neq\emptyset$. By Definition 1.13 (1), we have $\mu(0_A)\le\mu(t)$ for all $t\in f^{-1}(y)$. Thus $\mu(0_A)$ is a lower bound of $\{\mu(t)\}_{t\in f^{-1}(y)}$, so $\mu(0_A)\le\inf\{\mu(t)\}_{t\in f^{-1}(y)}=\beta(y)$. By Proposition 1.5 (3), we have $\beta(0_B)\le\beta(y)$. Let $a,b,c\in B$. By Lemma 2.14, there exist $a_0\in f^{-1}(a), b_0\in f^{-1}(b)$ and $c_0\in f^{-1}(c)$ such that $\beta(b)=\mu(b_0),\beta(a*c)=\mu(a_0\cdot c_0)$ and $\beta(a*(b*c))=\mu(a_0\cdot(b_0\cdot c_0))$. Thus

$$\begin{aligned}\beta(a*c)&=\mu(a_0\cdot c_0)\\&\le\max\{\mu(a_0\cdot(b_0\cdot c_0)),\mu(b_0)\} &&\text{(Definition 1.13 (2))}\\&=\max\{\beta(a*(b*c)),\beta(b)\}.\end{aligned}$$

Hence, β is an anti-fuzzy UP-ideal of B.

(2) Let μ be an f-invariant anti-fuzzy UP-subalgebra of A with inf property. Let $a,b\in B$. Since f is surjective, we have $f^{-1}(a),f^{-1}(b)$ and $f^{-1}(a*b)$ are nonempty subsets of A. By Lemma 2.14, there exist $a_0\in f^{-1}(a), b_0\in f^{-1}(b)$ such that $\beta(a)=\mu(a_0),\beta(b)=\mu(b_0)$ and $\beta(a*b)=\mu(a_0\cdot b_0)$. Thus

$$\begin{aligned}\beta(a * b) &= \mu(a_0 \cdot b_0) \\ &\leq \max\{\mu(a_0), \mu(b_0)\} && \text{(Definition 1.14)} \\ &= \max\{\beta(a), \beta(b)\}.\end{aligned}$$

Hence, β is an anti-fuzzy UP-subalgebra of B. ∎

Remark 2.16. [26] Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras. Then $A \times B$ is a UP-algebra defined by

$$(x_1, x_2) \diamond (y_1, y_2) = (x_1 \cdot y_1, x_2 * y_2)$$

for all $x_1, y_1 \in A$ and $x_2, y_2 \in B$.

Theorem 2.17. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *if μ_f is an anti-fuzzy UP-ideal of $A \times A$, then f is an anti-fuzzy UP-ideal of A, and*

(2) *if β_f is an anti-fuzzy UP-ideal of $A \times A$, then f is an anti-fuzzy UP-ideal of A.*

Proof. (1) Assume that μ_f is an anti-fuzzy UP-ideal of $A \times A$. Let $x \in A$. Then

$$\begin{aligned}f(0) &= \max\{f(0), f(0)\} \\ &= \mu_f(0, 0) \\ &\leq \mu_f(x, x) && \text{(Definition 1.13 (1))} \\ &= \max\{f(x), f(x)\} \\ &= f(x).\end{aligned}$$

Let $x, y, z \in A$. Then

$$\begin{aligned}f(x \cdot z) &= \max\{f(x \cdot z), f(x \cdot z)\} \\ &= \mu_f(x \cdot z, x \cdot z) \\ &= \mu_f((x, x) \diamond (z, z)) \\ &\leq \max\{\mu_f((x, x) \diamond ((y, y) \diamond (z, z))), \mu_f(y, y)\} && \text{(Definition 1.13 (2))} \\ &= \max\{\mu_f(x \cdot (y \cdot z), x \cdot (y \cdot z)), \mu_f(y, y)\} \\ &= \max\{\max\{f(x \cdot (y \cdot z)), f(x \cdot (y \cdot z))\}, \max\{f(y), f(y)\} \\ &= \max\{f(x \cdot (y \cdot z)), f(y)\}.\end{aligned}$$

Hence, f is an anti-fuzzy UP-ideal of A.

(2) Similarly to as in the proof of (1). ∎

Proposition 2.18. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *if μ_f is an anti-fuzzy UP-subalgebra of $A \times A$, then f is an anti-fuzzy UP-subalgebra of A, and*

(2) *if β_f is an anti-fuzzy UP-subalgebra of $A \times A$, then f is an anti-fuzzy UP-subalgebra of A.*

Proof. (1) Assume that μ_f is an anti-fuzzy UP-subalgebra of $A \times A$. Let $x, y \in A$. Then

$$\begin{aligned} f(x \cdot z) &= \max\{f(x \cdot z), f(x \cdot z)\} \\ &= \mu_f(x \cdot z, x \cdot z) \\ &= \mu_f((x, x) \diamond (z, z)) \\ &\leq \max\{\mu_f((x, x), \mu_f(y, y)\} && \text{(Definition 1.14)} \\ &\leq \max\{\max\{f(x), f(x)\}, \max\{f(y), f(y)\}\} \\ &= \max\{f(x), f(y)\}. \end{aligned}$$

Hence, f is an anti-fuzzy UP-subalgebra of A.

(2) Similarly to as in the proof of (1). ■

Lemma 2.19. [4] *For any $a, b, c, d \in \mathbb{R}$, the following properties hold:*

(1) $\max\{\max\{a, b\}, \max\{c, d\}\} = \max\{\max\{a, c\}, \max\{b, d\}\}$, *and*

(2) $\min\{\min\{a, b\}, \min\{c, d\}\} = \min\{\min\{a, c\}, \min\{b, d\}\}$.

Theorem 2.20. *Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras. Then the following statements hold:*

(1) *if f is an anti-fuzzy UP-ideal of A and g is an anti-fuzzy UP-ideal of B, then $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, and*

(2) *if f is an anti-fuzzy UP-subalgebra of A and g is an anti-fuzzy UP-subalgebra of B, then $f \times g$ is an anti-fuzzy UP-subalgebra of $A \times B$.*

Proof. (1) Assume that f is an anti-fuzzy UP-ideal of A and g is an anti-fuzzy UP-ideal of B. Let $(x, y) \in A \times B$. Then

$$\begin{aligned} (f \times g)(0, 0) &= \max\{f(0), g(0)\} \\ &\leq \max\{f(x), g(y)\} && \text{(Definition 1.13 (1))} \\ &= (f \times g)(x, y). \end{aligned}$$

Now, let $(x_1, x_2), (y_1, y_2), (z_1, z_2) \in A \times B$. Then

$$\begin{aligned} &(f \times g)((x_1, x_2) \diamond (z_1, z_2)) \\ &\quad = (f \times g)(x_1 \cdot z_1, x_2 * z_2) \\ &\quad = \max\{f(x_1 \cdot z_1), g(x_2 * z_2)\} \\ &\quad \leq \max\{\max\{f(x_1 \cdot (y_1 \cdot z_1)), f(y_1)\}, \\ &\quad\quad \max\{g(x_2 * (y_2 * z_2)), g(y_2)\}\} && \text{(Definition 1.13 (2))} \\ &\quad = \max\{\max\{f(x_1 \cdot (y_1 \cdot z_1)), \{g(x_2 * (y_2 * z_2))\}\}, \\ &\quad\quad \max\{f(y_1), g(y_2)\}\} && \text{(Lemma 2.19 (1))} \\ &\quad = \max\{(f \times g)(x_1 \cdot (y_1 \cdot z_1), x_2 * (y_2 * z_2)), (f \times g)(y_1, y_2)\} \\ &\quad = \max\{(f \times g)((x_1, x_2) \diamond ((y_1, y_2) \diamond (z_1, z_2))), (f \times g)(y_1, y_2)\}. \end{aligned}$$

Hence, $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$.

(2) Let $(x_1, x_2), (y_1, y_2) \in A \times B$. Then

$$\begin{aligned}
&(f \times g)((x_1, x_2) \diamond (y_1, y_2)) \\
&\quad = (f \times g)(x_1 \cdot y_1, x_2 * y_2) \\
&\quad = \max\{f(x_1 \cdot y_1), g(x_2 * y_2)\} \\
&\quad \leq \max\{\max\{f(x_1), f(y_1)\}, \max\{g(x_2), g(y_2)\}\} && \text{(Definition 1.14)} \\
&\quad = \max\{\max\{f(x_1), g(x_2)\}, \max\{f(y_1), g(y_2)\}\} && \text{(Lemma 2.19 (1))} \\
&\quad = \max\{(f \times g)(x_1, x_2), (f \times g)(y_1, y_2)\}.
\end{aligned}$$

Hence, $f \times g$ is an anti-fuzzy UP-subalgebra of $A \times B$. ∎

Give examples of conflict that f and g are anti-fuzzy UP-ideals (resp. anti-fuzzy UP-subalgebras) of A but $f \cdot g$ is not an anti-fuzzy UP-ideal (resp. anti-fuzzy UP-subalgebra) of $A \times A$.

Example 2.21. Let $A = \{0, 1\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	1
0	0	1
1	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set f and g in A as follows:

$$f(0) = 0.1, f(1) = 0.5, g(0) = 0.2 \text{ and } g(1) = 0.5.$$

Using this data, we can show that f and g are anti-fuzzy UP-ideals of A. Let $x_1 = 0, x_2 = 0, y_1 = 1, y_2 = 0, z_1 = 1, z_1 = 1$. Then

$$(f \cdot g)((x_1, x_2) \diamond (z_1, z_2)) = 0.5 \leq 0.2 =$$
$$\max\{(f \cdot g)((x_1, x_2) \diamond ((y_1, y_2) \diamond (z_1, z_2))), (f \cdot g)(y_1, y_2)\}.$$

Thus Definition 1.13 (2) is false. Hence, $f \cdot g$ is not an anti-fuzzy UP-ideal of $A \times A$.

Example 2.22. Let $A = \{0, a, b\}$ be a set with a binary operation $\cdot$ defined by the following Cayley table:

$\cdot$	0	a	b
0	0	a	b
a	0	0	a
b	0	0	0

Then $(A; \cdot, 0)$ is a UP-algebra. We define a fuzzy set f and g in A as follows:

$$f(0) = 0.1, f(a) = 0.2, f(b) = 0.1, g(0) = 0.1, g(a) = 0.2 \text{ and } g(b) = 0.1.$$

Using this data, we can show that f and g are anti-fuzzy UP-subalgebras of A. Let $x_1 = 0, x_2 = 1, y_1 = 1, y_2 = 2$. Then

$$(f \cdot g)((x_1, x_2) \diamond (y_1, y_2)) = 0.2 \geq 0.1 = \max\{(f \cdot g)(x_1, x_2), (f \cdot g)(y_1, y_2)\}.$$

Thus Definition 1.14 is false. Hence, $f \cdot g$ is not an anti-fuzzy UP-subalgebra of $A \times A$.

Theorem 2.23. *Let $(A; \cdot, 0_A)$ and $(B; *, 0_B)$ be UP-algebras and let f be a fuzzy set in A and g be a fuzzy set in B. If $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, then the following statements hold:*

(1) *either $f(0_A) \leq f(x)$ for all $x \in A$ or $g(0_B) \leq g(y)$ for all $y \in B$,*

(2) *if $f(0_A) \leq f(x)$ for all $x \in A$, then either $g(0_B) \leq g(y)$ for all $y \in B$ or $g(0_B) \leq f(x)$ for all $x \in A$,*

(3) *if $g(0_B) \leq g(y)$ for all $y \in B$, then either $f(0_A) \leq f(x)$ for all $x \in A$ or $f(0_A) \leq g(y)$ for all $y \in B$, and*

(4) *either f is an anti-fuzzy UP-ideal of A or g is an anti-fuzzy UP-ideal of B.*

Proof. (1) Suppose that $f(0_A) > f(x)$ for some $x \in A$ and $g(0_B) > g(y)$ for some $y \in B$. Then $(f \times g)(x, y) = \max\{f(x), g(y)\} < \max\{f(0_A), g(0_B)\} = (f \times g)(0_A, 0_B)$ which is a contradiction. Hence, either $f(0_A) \leq f(x)$ for all $x \in A$ or $g(0_B) \leq g(y)$ for all $y \in B$

(2) Assume that $f(0_A) \leq f(x)$ for all $x \in A$. Suppose that $g(0_B) > g(y)$ for some $y \in B$ and $g(0_B) > f(x)$ for some $x \in A$. Then $g(0_B) > f(x) \geq f(0_A)$. Thus

$$\begin{aligned}(f \times g)(x, y) &= \max\{f(x), g(y)\} \\ &< \max\{g(0_B), g(0_B)\} \\ &= g(0_B) \\ &= \max\{f(0_A), g(0_B)\} \\ &= (f \times g)(0_A, 0_B)\end{aligned}$$

which is a contradiction. Hence, either $g(0_B) \leq g(y)$ for all $y \in B$ or $g(0_B) \leq f(x)$ for all $x \in A$.

(3) Assume that $g(0_B) \leq g(y)$ for all $y \in B$. Suppose that $f(0_A) > f(x)$ for some $x \in A$ and $f(0_A) > g(y)$ for some $y \in B$. Then $f(0_A) > g(y) \geq g(0_B)$. Thus

$$\begin{aligned}(f \times g)(x, y) &= \max\{f(x), g(y)\} \\ &< \max\{f(0_A), f(0_A)\} \\ &= f(0_A) \\ &= \max\{f(0_A), g(0_B)\} \\ &= (f \times g)(0_A, 0_B)\end{aligned}$$

which is a contradiction. Hence, either $f(0_A) \leq f(x)$ for all $x \in A$ or $f(0_A) \leq g(y)$ for all $y \in B$.

(4) Suppose that f is not an anti-fuzzy UP-ideal of A and g is not an anti-fuzzy UP-ideal of B. By (1), assume that $f(0_A) \leq f(x)$ for all $x \in A$. Then

from (2), either $g(0_B) \leq g(y)$ for all $y \in B$ or $g(0_B) \leq f(x)$ for all $x \in A$. If $g(0_B) \leq f(x)$ for all $x \in A$, then for all $x \in A$,

$$(f \times g)(x, 0_B) = \max\{f(x), g(0_B)\} = f(x). \tag{2.4}$$

Since $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, we have for any $x, y, z \in A$,

$$\begin{aligned}
f(x \cdot z) &= (f \times g)(x \cdot z, 0_B) && ((2.4))\\
&= (f \times g)(x \cdot z, 0_B * 0_B) && \text{(UP-3)}\\
&= (f \times g)((x, 0_B) \diamond (z, 0_B))\\
&\leq \max\{(f \times g)((x, 0_B) \diamond [(y, 0_B) \diamond (z, 0_B)]),\\
&\quad (f \times g)(y, 0_B)\} && \text{(Definition 1.13 (2))}\\
&= \max\{(f \times g)(x \cdot (y \cdot z), 0_B * (0_B * 0_B)), (f \times g)(y, 0_B)\}\\
&= \max\{(f \times g)(x \cdot (y \cdot z), 0_B), (f \times g)(y, 0_B)\} && \text{(UP-3)}\\
&= \max\{\max\{f(x \cdot (y \cdot z)), g(0_B)\}, \max\{f(y), g(0_B)\}\}\\
&= \max\{f(x \cdot (y \cdot z)), f(y)\}.
\end{aligned}$$

Hence, f is an anti-fuzzy UP-ideal of A which is a contradiction. Assume that $g(0_B) \leq g(y)$ for all $y \in B$. Then from 2.23, either $f(0_A) \leq f(x)$ for all $x \in A$ or $f(0_A) \leq g(y)$ for all $y \in B$. If $f(0_A) \leq g(y)$ for all $y \in B$, then for all $y \in B$,

$$(f \times g)(0_A, y) = \max\{f(0_A), g(y)\} = g(y). \tag{2.5}$$

Since $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, we have for any $x, y, z \in B$,

$$\begin{aligned}
g(x * z) &= (f \times g)(0_A, x * z) && ((2.5))\\
&= (f \times g)(0_A \cdot 0_A, x * z) && \text{(UP-3)}\\
&= (f \times g)((0_A, x) \diamond (0_A, z))\\
&\leq \max\{(f \times g)((0_A, x) \diamond [(0_A, y) \diamond (0_A, z)]),\\
&\quad (f \times g)(0_A, y)\} && \text{(Definition 1.13 (2))}\\
&= \max\{(f \times g)(0_A \cdot (0_A \cdot 0_A)), (x * (y * z)), (f \times g)(0_A, y)\}\\
&= \max\{(f \times g)(0_A, x * (y * z)), (f \times g)(0_A, y)\} && \text{(UP-3)}\\
&= \max\{\max\{f(0_A), g(x * (y * z))\}, \max\{f(0_A), g(y)\}\}\\
&= \max\{g(x * (y * z)), g(y)\}.
\end{aligned}$$

Hence, g is an anti-fuzzy UP-ideal of B which is a contradiction. Since f is not an anti-fuzzy UP-ideal of A and g is not an anti-fuzzy UP-ideal of B, and $f(0_A) \leq f(x)$ for all $x \in A$ and $g(0_B) \leq g(y)$ for all $y \in B$, there exist $x, y, z \in A$ and $x', y', z' \in B$ such that

$$f(x \cdot z) > \max\{f(x \cdot (y \cdot z)), f(y)\}$$

and

$$g(x' * z') > \max\{g(x' * (y' * z')), g(y')\}.$$

Thus

$$\max\{f(x\cdot z), g(x'*z')\} > \max\{\max\{f(x\cdot(y\cdot z)), f(y)\}, \max\{g(x'*(y'*z')), g(y')\}\}.$$

Since $f \times g$ is an anti-fuzzy UP-ideal of $A \times B$, we have

$$\begin{aligned}
\max\{f(x\cdot z), g(x'*z')\} &= (f\times g)(x\cdot z, x'*z')\\
&= (f\times g)((x,x')\diamond(z,z'))\\
&\leq \max\{(f\times g)((x,x')\diamond[(y,y')\diamond(z,z')]),\\
&\quad (f\times g)(y,y')\} \qquad \text{(Definition 1.13 (2))}\\
&= \max\{(f\times g)(x\cdot(y\cdot z), x'*(y'*z')), (f\times g)(y,y')\}\\
&= \max\{\max\{f(x\cdot(y\cdot z)), g(x'*(y'*z'))\},\\
&\quad \max\{f(y), g(y')\}\}.
\end{aligned}$$

Thus $\max\{f(x\cdot z), g(x'*z')\} \not> \max\{\max\{f(x\cdot(y\cdot z)), f(y)\}, \max\{g(x'*(y'*z')), g(y')\}\}$ which is a contradiction. Similarly, by (1), if $g(0_B) \leq g(y)$ for all $y \in B$, we have a contradiction. Hence, either f is an anti-fuzzy UP-ideal of A or g is an anti-fuzzy UP-ideal of B. ∎

Theorem 2.24. *Let $(A;\cdot,0_A)$ and $(B;*,0_B)$ be UP-algebras and let f be a fuzzy set in A and g be a fuzzy set in B. If $f\times g$ is an anti-fuzzy UP-subalgebra of $A\times B$, then either f is an anti-fuzzy UP-subalgebra of A or g is an anti-fuzzy UP-subalgebra of B.*

Proof. Suppose that f is not an anti-fuzzy UP-subalgebra of A and g is not an anti-fuzzy UP-subalgebra of B. Then there exist $x, y \in A$ and $a, b \in B$ such that

$$f(x\cdot y) > \max\{f(x), f(y)\}$$

and

$$g(a*b) > \max\{g(a), g(b)\}.$$

Thus $\max\{f(x\cdot y), g(a*b)\} > \max\{\max\{f(x), f(y)\}, \max\{g(a), g(b)\}\}$. Since $f\times g$ is an anti-fuzzy UP-subalgebra of $A\times B$, we have

$$\begin{aligned}
\max\{f(x\cdot y), g(a*b)\} &= (f\times g)(x\cdot y, a*b)\\
&= (f\times g)((x,a)\diamond(y,b))\\
&\leq \max\{(f\times g)(x,a), (f\times g)(y,b)\} \qquad \text{(Definition 1.14)}\\
&= \max\{\max\{f(x), g(a)\}, \max\{f(y), g(b)\}\}\\
&= \max\{\max\{f(x), f(y)\}, \max\{g(a), g(b)\}\}. \qquad \text{(Lemma 2.19 (1))}
\end{aligned}$$

Thus $\max\{f(x\cdot y), g(a*b)\} \not> \max\{\max\{f(x), f(y)\}, \max\{g(a), g(b)\}\}$ which is a contradiction. Hence, either f is an anti-fuzzy UP-subalgebra of A or g is an anti-fuzzy UP-subalgebra of B. ∎

Theorem 2.25. *Let f be a fuzzy set in A. Then the following statements hold:*

(1) *f is an anti-fuzzy UP-ideal of A if and only if μ_f is an anti-fuzzy UP-ideal of $A \times A$, and*

(2) *f is an anti-fuzzy UP-subalgebra of A if and only if μ_f is an anti-fuzzy UP-subalgebra of $A \times A$.*

Proof. (1) Assume that f is an anti-fuzzy UP-ideal of A. By Theorem 2.20 (1), we have $\mu_f = f \times f$ is an anti-fuzzy UP-ideal of $A \times A$.

Conversely, assume that μ_f is an anti-fuzzy UP-ideal of $A \times A$. Since $f \times f = \mu_f$, it follows from Theorem 2.23 (4) that f is an anti-fuzzy UP-ideal of A.

(2) Assume that f is an anti-fuzzy UP-subalgebra of A. By Theorem 2.20 (2), we have $\mu_f = f \times f$ is an anti-fuzzy UP-subalgebra of $A \times A$.

Conversely, assume that μ_f is an anti-fuzzy UP-subalgebra of $A \times A$. Since $f \times f = \mu_f$, it follows from Theorem 2.24 that f is an anti-fuzzy UP-subalgebra of A. ∎

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