ON P-DERIVATIONS AND P-JORDAN DERIVATIONS OF A RING

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Abstract. In this paper, we generalize ordinary derivation and Jordan derivation of a ring to the concepts of P-derivation and P-Jordan derivation of a ring respectively. Some properties, e.g. the composition of P-derivations and P-Jordan derivations, are given. Moreover, it proved that if $R/P(R)$ is 2-torsion free for a ring R and its prime radical $P(R)$, P-Jordan derivations of R are P-derivations. More mainly, some conclusions on the commutativity of R/I and $R/P(R)$ are obtained via P-derivations and P-Jordan derivations, for a prime ideal I of R.

Keywords: P-derivation, P-Jordan derivation, torsion free, prime radical, commutativity.

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1. Introduction

Let R be an associative ring. Let $P(R)$ denote the *prime radical* of R, i.e., the intersection of all prime ideals of R. For any $a, b \in R$, we shall write $[a, b]$ for $ab - ba$. A ring R is said to be *n-torsion free*, where n is a positive number, if whenever $na = 0$, with $a \in R$, then $a = 0$. For a ring R, if there is a smallest positive number n such that $nR = 0$, we say R to have *characteristic* n, denoted as $charR = n$. Otherwise, we say that R has characteristic zero. A ring R is called a *semiprime ring* if $P(R) = 0$, or equivalently, $aRa = (0)$ implies $a = 0$. An additive mapping $d : R \longrightarrow R$ is called a *derivation* of R, if $d(ab) = d(a)b + ad(b)$ holds for all pairs $a, b \in R$. An additive mapping $d : R \longrightarrow R$ is called a *Jordan derivation* of R, if $d(a^2) = d(a)a + ad(a)$ holds for all $a \in R$. Obviously, every derivation is a Jordan derivation. The converse is, in general, not true. The following example

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gives a Jordan derivation, which is not a derivation. We are thankful for the first person who constructed it, though we haven't found who did this work:

Let k be an algebraic closed field and let $S = k[x]$ with the relation $x^2 = 0$. Let $R = \begin{pmatrix} S & S \\ I & S \end{pmatrix}$ where $I = kx$, the ideal of S. It is easy to see that R, with matrix addition and multiplication, is a ring. For any $r = \begin{pmatrix} a & b \\ c & f \end{pmatrix}$ where $a, b, f \in R$ and $c \in I$, define

$$d : R \longrightarrow R,$$

$$\begin{pmatrix} a & b \\ c & f \end{pmatrix} \longmapsto \begin{pmatrix} 0 & c \\ 0 & 0 \end{pmatrix}$$

Clearly, d is additive. Observe that for any $c \in I$, $c^2 = 0$, we have

$$d(r^2) = \begin{pmatrix} 0 & (a+f)c \\ 0 & 0 \end{pmatrix} = d(r)r + rd(r)$$

Thus d is a Jordan derivation. Meanwhile, let $r_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, r_2 = \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$. Then $d(r_1 r_2) = 0$ and $d(r_1)r_2 + r_1 d(r_2) = \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix} \neq 0$, hence d is not a derivation.

It has been an increasingly interested problem that wether Jordan derivations degenerate to ordinary derivations on a particular ring or algebra. Herstein [7] showed that it is true on a 2-torsion free prime ring, and another proof is given by M. Bresar in [4], moreover, the same author generalized the results to 2-torsion free semiprime rings in [3]. In recent years, a series of more general concepts of derivations and Jordan derivations are introduced and studied on semiprime rings (see [1], [2], [5]–[7], [10], [11]).

Motivated above, we may assume that for a Jordan derivation d of a particular ring R, where R is not necessarily a prime or semiprime ring, $d(ab) - d(a)b - ad(b)$ may be in an ideal of R, for any $a, b \in R$, obviously, if the ideal is zero ideal, d is an ordinary derivation. So it is interesting to find a relatively small ideal satisfying this condition. In this paper, we will verify that the prime radical satisfies this condition. Let us first introduce the following definitions:

Definition 1.1. Let R be a ring. An additive mapping $d : R \longrightarrow R$ is called a *P-derivation* if $(d(ab) - d(a)b - ad(b)) \in P(R)$ holds for all pairs $a, b \in R$.

Definition 1.2. An additive mapping $d : R \longrightarrow R$ is called a *P-Jordan derivation* if $(d(a^2) - d(a)a - ad(a)) \in P(R)$ holds for all $a \in R$.

With these definitions, we will investigate properties of rings with P-derivations and P-Jordan derivations. This paper is arranged as follow:

In Section 2, we prove that over a ring R satisfying $R/P(R)$ is 2-torsion free, its P-Jordan derivations and P-derivations coincide. Thus whenever $R/P(R)$ is 2-torsion free, the Jordan derivation d of R satisfies $d(ab) - d(a)b - ad(b) \in P(R)$,

$\forall a, b \in R$, which is a generalization of M. Breasar's result in [3]. For the proof of the main result in this section, some characterizations of P-derivations and P-Jordan derivations are also given. In Section 3, some properties of P-derivations and P-Jordan derivations are given. We provide a necessary condition for that the composition of two P-(Jordan) derivations is again a P-(Jordan) derivation. And the results of this section will be used for the proof of the conclusions in the next section. In Section 4, we consider the commutativity of the quotient rings $R/P(R)$ and R/I, where I is a prime ideal of R, as an application of conclusions in Section 2 and Section 3. In [9], Posner proved that in a prime ring R, a non-zero derivation satisfying a particular polynomial identity forces R to be commutative. We will generalize Posner's results by omitting the prime condition in [9], and give some sufficient conditions for the commutativity of R/I via P-derivations. Moreover, the main results of Section 2 enable us to discuss the commutativity of the quotient ring $R/P(R)$ and R/I via P-(Jordan) derivations of R when $R/P(R)$ or R/I is 2-torsion free.

2. Characterization of P-derivation and $P-$Jordan derivation

Lemma 2.1. *Let R be a ring with a P-Jordan derivation d, and $R/P(R)$ is 2-torsion free. Then for all $a, b, c \in R$ the following statements hold:*

(i) $d(ab + ba) - (d(a)b + ad(b) + d(b)a + bd(a)) \in P(R)$,

(ii) $d(aba) - (d(a)ba + ad(b)a + abd(a)) \in P(R)$,

(iii) $d(abc + cba) - (d(a)bc + ad(b)c + abd(c) + d(c)ba + cd(b)a + cbd(a)) \in P(R)$.

Proof. (i) Since $d(a^2) - (ad(a) + d(a)a) \in P(R)$. Then $d((a + b)^2) - (d(a + b)(a + b) + (a + b)d(a + b)) \in P(R)$, we write this as $(d(a^2) - ad(a) - d(a)a) + (d(b^2) - bd(b) - d(b)b) + (d(ab + ba) - d(a)b - ad(b) - d(b)a - bd(a)) \in P(R)$, thus $d(ab + ba) - (d(a)b + ad(b) + d(b)a + bd(a)) \in P(R), \forall a, b \in R$.

(ii) Consider $d(a(ab+ba)+(ab+ba)a) = d((a^2b+ba^2)+2aba) = d(a^2b+ba^2)+2d(aba)$, $d(a(ab + ba) + (ab + ba)a) - (d(a)(ab + ba) + ad(ab + ba) + d(ab + ba)a + (ab + ba)d(a)) \in P(R)$, by (i), we get $d(a)(ab + ba) + ad(ab + ba) + d(ab + ba)a + (ab+ba)d(a) - (d(a)ab+d(a)ba+a(d(a)b+ad(b)+d(b)a+bd(a))+(d(a)b+ad(b)+d(b)a + bd(a))a + (ab + ba)d(a)) \in P(R)$, i.e. $d(a)ab + d(a)ba + a(d(a)b + ad(b) + d(b)a+bd(a))+(d(a)b+ad(b)+d(b)a+bd(a))a+(ab+ba)d(a)-(d(a)ab+2d(a)ba+ad(a)b+a^2d(b)+2abd(a)+2ad(b)a+bad(a)+d(b)a^2+bd(a)a) = 0 \in P(R)$. On the other hand, $d(a^2b+ba^2)-d(a^2)b-a^2d(b)-d(b)a^2-bd(a^2) \in P(R)$, then by (i), we have $d((a^2b+ba^2)+2aba)-(d(a^2)b+a^2d(b)+d(b)a^2+bd(a^2)+2d(aba)) \in P(R)$ thus $d(a^2)b + a^2d(b) + d(b)a^2 + bd(a^2) + 2d(aba) - (d(a)ab + ad(a)b + a^2d(b) + d(b)a^2 + bad(a) + bd(a)a + 2d(aba)) \in P(R)$, then from (3) and (5) we get $2d(aba) - 2(d(a)ba + ad(b)a + abd(a)) = 2(d(aba) - (d(a)ba + ad(b)a + abd(a))) \in P(R)$, thus $d(aba) - (d(a)ba + ad(b)a + abd(a)) \in P(R)$.

(iii) Consider $d((a+c)b(a+c))$. $d((a+c)b(a+c)) = d(aba)+d(cbc)+d(abc+cba)$ $d(aba) - (d(a)ba + ad(b)a + abd(a)) \in P(R)$, $d(cbc) - (d(c)bc + cd(b)c + cbd(c)) \in$

$P(R)$, $d((a+c)b(a+c))-(d(a+c)b(a+c)+(a+c)d(b)(a+c)+(a+c)bd((a+c)))\in P(R)$, then we have $(d(aba)-(d(a)ba+ad(b)a+abd(a)))+((cbc)-(d(c)bc+cd(b)c+cbd(c)))+(d(abc+cba)-(d(a)bc+ad(b)c+abd(c)+d(c)ba+cd(b)a+cbd(a)))\in P(R)$, thus $d(abc+cba)-(d(a)bc+ad(b)c+abd(c)+d(c)ba+cd(b)a+cbd(a))\in P(R)$. ■

Remark 2.2. We should notice that if $R/P(R)$ is 2-torsion free, a linear map d of R is a P-Jordan derivation if and only if $d(ab+ba)-(d(a)b+ad(b)+d(b)a+bd(a))\in P(R)$, $\forall a,b\in R$.

Proof. In fact, Lemma 2.1 has already verified the "only if" part. For the "if" part, we get $2d(a^2)-2d(a)a-2ad(a)\in P(R)$ by replacing b to a in $d(ab+ba)-(d(a)b+ad(b)+d(b)a+bd(a))\in P(R)$, since $R/P(R)$ is 2-torsion free, it follows $d(a^2)-d(a)a-ad(a)\in P(R)$. ■

Lemma 2.3. *Let R be a ring with a P-Jordan derivation d, and $R/P(R)$ is 2-torsion free. Then $(d(ab)-d(a)b-ad(b))x[a,b]+[a,b]x(d(ab)-d(a)b-ad(b)))\in P(R)$, $\forall a,b,x\in R$,*

Proof. First consider $d(abxba+baxab)=d(abxba)+d(baxab)$. By Lemma 2.1(ii), we have $d(abxba)-d(a)bxba-ad(bxb)a-abxbd(a)\in P(R)$, $d(bxb)-d(b)xb-bd(x)b-bxd(b)\in P(R)$, thus, $d(abxba)-d(a)bxba-ad(b)xba-abd(x)ba-abxd(b)a-abxbd(a)\in P(R)$, similarly, we have $d(baxab)-d(b)axab-bd(a)xab-bad(x)ab-baxd(a)b-baxad(b)\in P(R)$, and by Lemma 2.1(iii), $d(abxba+baxab)-d(ab)xba-abd(x)ba-abxd(ba)-d(ba)xab-bad(x)ab-baxd(ab)\in P(R)$, then it follows that $(d(ab)-d(a)b-ad(b))x[a,b]+[a,b]x(d(ab)-d(a)b-ad(b))\in P(R)$. ■

Lemma 2.4. *Let R be a ring with two elements $a,b\in R$ and $R/P(R)$ is 2-torsion free. If for all $x\in R$ the relation $axb+bxa\in P(R)$ holds. Then $axb,bxa\in P(R)$ is fulfilled for all $x\in R$.*

Proof. Let $x,y\in R$. Then, by the assumption, $axbyaxb+bxayaxb\in P(R)$, $bxayaxb+axaybxb\in P(R)$, $axaybxb+axbyaxb\in P(R)$. Thus, we have $2axbyaxb\in P(R)$, since $R/P(R)$ is 2-torsion free, it follows that $axbyaxb\in P(R)$, thus $axb\in P(R),\forall a,b,x\in R$. Similarly $bxa\in P(R),\forall a,b,x\in R$. ■

Corollary 2.5. *Let R be a ring and $R/P(R)$ is 2-torsion free. Then for any $a,b,x\in R$, $(d(ab)-d(a)b-ad(b))x[a,b]\in P(R)$, and $[a,b]x(d(ab)-d(a)b-ad(b))\in P(R)$.*

Proof. It follows from Lemma 2.3 and Lemma 2.4 immediately. ■

Recall that a ring is 6-torsion free if and only if it is both 2-torsion free and 3-torsion free. The next theorem states that the converse of Lemma 2.1(2) is right when $R/P(R)$ is 6-torsion free ring:

Theorem 2.6. *Let R be a ring with a linear map d, and $R/P(R)$ is 6-torsion free. Then d is a P-Jordan derivation if and only if $d(aba)-(d(a)ba+ad(b)a+abd(a))\in P(R),\forall a,b\in R$.*

Proof. It suffices to verify the "if" part. By the assumption, $d(x^2yx^2)-d(x^2)yx^2-x^2d(y)x^2-xyd(x^2)\in P(R),\forall x,y\in R$, On the other hand, $d(x^2yx^2)-d(x)xyx^2-xd(xyx)x-x^2yxd(x)\in P(R),\forall x,y\in R$, thus we have $d(x^2yx^2)-d(x)yx^2-xd(x)yx^2-x^2d(y)x^2-x^2yd(x)x-x^2yxd(x),\forall x,y\in R$. Let $A(x)=d(x^2)-d(x)x-xd(x)$, we have therefore $A(x)yx^2+x^2yA(x)=0,\forall x,y\in R$. By Lemma 2.4, we have $A(x)x^2,x^2A(x)\in P(R),\forall x\in R$. Let $B(x,y)=d(xy+yx)-d(x)y-d(y)x-xd(y)-yd(x)$, then from $A(x+y)(x+y)^2\in P(R),\forall x,y\in R$, we have $A(x)y^2+A(y)x^2+B(x,y)x^2+B(x,y)y^2+A(x)(xy+yx)+A(y)(xy+yx)+B(x,y)(xy+yx)\in P(R),\forall x,y\in R$, then substituting $-x$ for x in the above relation we have $A(x)y^2+A(y)x^2-B(x,y)x^2-B(x,y)y^2-A(x)(xy+yx)-A(y)(xy+yx)+B(x,y)(xy+yx)\in P(R),\forall x,y\in R$, thus $B(x,y)x^2+B(x,y)y^2+A(x)(xy+yx)+A(y)(xy+yx)\in P(R),\forall x,y\in R$, also $4B(x,y)x^2+B(x,y)y^2+4A(x)(xy+yx)+A(y)(xy+yx)\in P(R),\forall x,y\in R$, it follows that $3A(x)(xy+yx)+3B(x,y)x^2\in P(R),\forall x,y\in R$, since $R/P(R)$ is 6-torsion free. Thus it is 3-torsion free. Then $A(x)(xy+yx)+B(x,y)x^2\in P(R),\forall x,y\in R$, right multiplication by $A(x)x$, we have $A(x)xyA(x)x+A(x)yxA(x)x\in P(R),\forall x,y\in R$, notice that $xA(x)xyxA(x)x+xA(x)yx^2A(x)x\in P(R),\forall x,y\in R$, by $x^2A(x)\in P(R),\forall x\in R$, we have $xA(x)xyxA(x)x\in P(R),\forall x,y\in R$, therefore $xA(x)x\in P(R),\forall x\in R$, since $A(x)xyA(x)x+A(x)yxA(x)x\in P(R),\forall x,y\in R$, then we have $A(x)xyA(x)x\in P(R),\forall x\in R$, it follows $A(x)x\in P(R),\forall x\in R$, by the relation $A(x)(xy+yx)+B(x,y)x^2\in P(R),\forall x,y\in R$, we have $A(x)yx+B(x,y)x^2\in P(R),\forall x,y\in R$, thus $xA(x)yxA(x)+xB(x,y)x^2A(x)\in P(R),\forall x,y\in R$, so $xA(x)yxA(x)\in P(R)$ which gives $xA(x)\in P(R),\forall x,y\in R$. Notice that $A(x+y)=A(x)+A(y)+B(x,y)$, then it follows from $(x+y)A(x+y)\in P(R)$, $\forall x,y\in R$ that $xA(y)+yA(x)+(x+y)B(x,y)\in P(R)$, $\forall x,y\in R$, replace x by $-x$ we have $-xA(y)+yA(x)+(x-y)B(x,y)\in P(R),\forall x,y\in R$, therefore $yA(x)+xB(x,y)\in P(R),\forall x,y\in R$, left multiplying with $A(x)$ follows $A(x)yA(x)\in P(R),\forall x,y\in R$, thus $A(x)\in P(R)$, i.e d is a P-Jordan derivation. ∎

Now, we prove our main result of this section:

Theorem 2.7. *Let R be a ring with a P-Jordan derivation d, and $R/P(R)$ is 2-torsion free. Then d is a P-derivation.*

Proof. Our aim is to show that $(d(ab)-d(a)b-ad(b))\in P(R),\forall a,b\in R$. By Corollary 2.5, we have $(d(a(b+c))-d(a)(b+c)-ad(b+c))x[a,b+c]\in P(R)$, $\forall a,b,c\in R$, then $(d(ab)-d(a)b-ad(b))x[a,c]+(d(ac)-d(a)c-ad(c))x[a,b]\in P(R)$. By Corollary 2.5 again, we have $(d(ab)-d(a)b-ad(b))x[a,c]y(d(ac)-d(a)c-ad(c))x[a,b]\in P(R)$, $\forall a,b,c,x,y\in R$. Then, we have $((d(ab)-d(a)b-ad(b))x[a,c])y((d(ab)-d(a)b-ad(b))x[a,c])\in P(R)$. By the definition of $P(R)$, $(d(ab)-d(a)b-ad(b))x[a,c]\in P(R),\forall a,b,c,x\in R$. Thus, replacing a by $a+z$, we get $(d((a+z)b)-d(a+z)b-(a+z)d(b))x[a+z,c]\in P(R)$, then $(d(ab)-d(a)b-ad(b))x[z,c]+(d(zb)-d(z)b-zd(b))x[a,c]\in P(R)$, and $(d(ab)-d(a)b-ad(b))x[z,c]y(d(zb)-d(z)b-zd(b))x[a,c]\in P(R)$. It follows that $((d(ab)-d(a)b-ad(b))x[z,c])y((d(ab)-d(a)b-ad(b))x[z,c])\in P(R)$, thus $(d(ab)-d(a)b-ad(b))x[z,c]\in P(R)$, $\forall a,b,c,x,z\in R$.

Now, we consider $[(d(ab)-d(a)b-ad(b)),c]x[(d(ab)-d(a)b-ad(b)),c]$, by Corollary 2.5, $(d(ab)-d(a)b-ad(b))(cx)[(d(ab)-d(a)b-ad(b)),c]\in P(R)$, and $c(d(ab)-d(a)b-ad(b))x[(d(ab)-d(a)b-ad(b)),c]\in P(R)$, thus $[(d(ab)-d(a)b-ad(b)),c]x[(d(ab)-d(a)b-ad(b)),c]\in P(R)$, hence $[(d(ab)-d(a)b-ad(b)),c]\in P(R)$ $\forall a,b,c\in R$. On the other hand, $((d(ab)-d(a)b-ad(b))[z,c])x((d(ab)-d(a)b-ad(b))[z,c])\in P(R)$, hence $(d(ab)-d(a)b-ad(b))[z,c]\in P(R)$, $\forall a,b,c,z\in R$. Since $d(ab+ba)-d(a)b-ad(b)-d(b)a-bd(a)\in P(R)$, i.e. $(d(ab)-d(a)b-ad(b))+(d(ba)-d(b)a-ad(b))\in P(R)$, and $(d(ab)-d(a)b-ad(b))-(d(ba)-d(b)a-ad(b))=d[a,b]+[b,d(a)]+[d(b),a]$, multiply with $(d(ab)-d(a)b-ad(b))$, we have $(d(ab)-d(a)b-ad(b))[b,d(a)]\in P(R)$, $(d(ab)-d(a)b-ad(b))[d(b),a]\in P(R)$, thus $2(d(ab)-d(a)b-ad(b))^2-(d(ab)-d(a)b-ad(b))d[a,b]\in P(R)$. Now, consider $d((d(ab)-d(a)b-ad(b))[a,b](d(ab)-d(a)b-ad(b)))$ since $(d(ab)-d(a)b-ad(b))[a,b]\in P(R)$, and $[a,b](d(ab)-d(a)b-ad(b))\in P(R)$, then $d(d(ab)-d(a)b-ad(b))[a,b](d(ab)-d(a)b-ad(b))\in P(R)$, $(d(ab)-d(a)b-ad(b))[a,b]d(d(ab)-d(a)b-ad(b))\in P(R)$, and since $2(d(ab)-d(a)b-ad(b))^3-(d(ab)-d(a)b-ad(b))^2d[a,b]\in P(R)$, thus $2(d(ab)-d(a)b-ad(b))^3\in P(R)$, hence $(d(ab)-d(a)b-ad(b))\in P(R)$. ∎

Corollary 2.8. *Let R be a ring with a Jordan derivation d, and $R/P(R)$ is 2-torsion free. Then d is a P-derivation.*

Corollary 2.9. *Let R be a ring with a P-Jordan derivation d and a prime ideal I. If R/I is 2-torsion free. Then $d(ab)-d(a)b-ad(b)\in I$.*

Proof. By the proof of Theorem 2.7 we may find that, there exits an integer n such that $2^n(d(ab)-d(a)b-ad(b))\in P(R)\subseteq I$. Since R/I is 2-torsion free, then $d(ab)-d(a)b-ad(b)\in I$. ∎

3. The composition of two P-(Jordan)derivations

In this section, we continue with the properties of P-derivations and P-Jordan derivations. We give a necessary condition for that the composition of two P-(Jordan) derivations is again a P-(Jordan) derivation. These results will be used in the next section, while they are also independently interesting.

Lemma 3.1. *Let R be a ring with a prime ideal I and a P-derivation of d. Let a be an element of R such that $ad(x)\in I$ for all $x\in R$. Then either $a\in I$ or* $\mathrm{Im}d\subseteq I$.

Proof. Since d is a P-derivation, we have $d(xy)-d(x)y-xd(y)\in P(R)\subseteq I$. From the assumptions, $ad(xy)\in I$, for any $x,y\in R$, i.e. $ad(x)y+axd(y)\in I$, since $ad(x)y\in I$. Then $axd(y)\in I$ for all $x,y\in R$, by the definition of prime ideal, we finish the proof. ∎

Corollary 3.2. *Let R be a ring with a prime ideal I and a P-Jordan derivation of d. $R/P(R)$ is 2-torsion free. Let a be an element of R such that $ad(x)\in I$ for all $x\in R$. Then either $a\in I$ or* $\mathrm{Im}d\subseteq I$.

Proof. It follows immediately from Theorem 2.7 and Lemma 3.1. ∎

Lemma 3.3. *Let R be a ring with a prime ideal I. Let p, q, r be elements of R such that $paqar \in I$ for all $a \in R$. Then at least one of p, q, r is in I.*

Proof. If $p \in I$ or $r \in I$, then we complete this proof. Now let $p, r \notin I$. Our aim is to prove $q \in I$. Since $paqar \in I$, we have $p(a+b)q(a+b)r \in I, \forall a, b \in R$, we write this as $paqbr + pbqar \in I, \forall a, b \in R$. On the other hand, $paqart \in I, \forall t \in I$, it follows $p(aqart)qbr + pbq(aqart)r \in I, \forall b \in R$, since $p(aqart)qbr \in I, \forall a, t \in R$, we have $pbq(aqart)r \in I, \forall a, b, t \in R$, by the arbitration of b and t, we have $qaqar \in I, \forall a \in R$. Similarly, $paqaq \in I, \forall a \in R$. From $paqar \in I, \forall a \in R$, $p(a+b)q(a+b)r \in I$, and $paqbr + pbqar \in I$, then $paq(tqbqb)r + p(tqbqb)qar \in I$. Then $p(tqbqb)qar \in I$, thus we have $qbqbq \in I$ for any $b \in R$. Also $q(a+aqb)q(a+aqb)q \in I, \forall a, b \in R$, i.e. $qaqbqaq \in I, \forall a, b \in R$, thus $qaq \in I, \forall a \in R$, and it follows $q \in I$. ∎

The next theorem will provide a necessary condition for the composition of two P-derivations is again a P-derivation.

Theorem 3.4. *Let R be a ring. d_1, d_2 are P-derivations of R such that $d_1d_2(ab) - d_1d_2(a)b - ad_1d_2(b) \in I$. Then for any prime ideal I, either $\mathrm{Im}2d_1 \subseteq I$ or $\mathrm{Im}2d_2 \subseteq I$.*

Proof. Note that for any prime ideal I, $P(R) \subseteq I$, then $d_1d_2(a)b + ad_1d_2(b) - d_1d_2(ab) \in I$, $\forall a, b \in R$. $d_1d_2(ab) - d_1d_2(a)b - ad_1d_2(b) - d_1(a)d_2(b) - d_2(a)d_1(b)$, $\forall a, b \in R$, then $d_1(a)d_2(b) + d_2(a)d_1(b) \in I$, $\forall a, b \in R$, then $d_1(ad_1(c))d_2(b) + d_2(ad_1(c))d_1(b) \in I$, $\forall a, b \in R$, thus $d_2(a)d_1(c)d_1(b) + d_1(a)d_1(c)d_2(b) \in I$, $\forall a, b, c \in R$. Notice that $d_1(c)d_2(b) + d_2(c)d_1(b) \in I, \forall b, c \in R$. Then $d_2(a)d_1(c)d_1(b) - d_1(a)d_2(c)d_1(b) \in I$, $\forall a, b, c \in R$. By Lemma 3.1, $d_2(a)d_1(c) - d_1(a)d_2(c) \in I$, $\forall a, c \in R$. Along with $d_1(a)d_2(c) + d_2(a)d_1(c) \in I, \forall a, c \in R$, it follows $2d_2(a)d_1(c) \in I$, if $\mathrm{Im}d_1 \nsubseteq I$. By Lemma 3.1 again, we have $2d_2(a) \in I$, $\forall a \in R$. Similarly, if $\mathrm{Im}d_2 \nsubseteq I$. Then $2d_1(c) \in I$, $\forall c \in R$. Therefore, we finish the proof. ∎

Corollary 3.5. *Let R be a ring, and I an prime ideal, and R/I is 2-torsion free. d_1, d_2 are P-derivations of R such that d_1d_2 is also a P-derivation. Then either $\mathrm{Im}d_1 \subseteq I$ or $\mathrm{Im}d_2 \subseteq I$.*

Proof. It can be deduced immediately from Theorem 2.7 and Theorem 3.4. ∎

Corollary 3.6. *Let R be a ring, and $R/P(R)$ is 2-torsion free and $n (\geq 2)$ is an integer. d is a P-(Jordan) derivation of R such that d^n is also a P-(Jordan) derivation. Then $\mathrm{Im}d \subseteq P(R)$.*

4. On commutativity of quotient ring via P-(Jordan) derivations

In this section, we discuss the commutativity of $R/P(R)$ as well as R/I, where I is a prime ideal of R. Posner showed in [9] that some properties of a derivation determine the commutativity of a prime ring. In other word, the author provided some sufficient conditions for the commutativity of a prime ring via its derivation.

We will generalize these results by providing sufficient conditions for R/I to be commutative via the P-derivations of R. Moreover, if $R/P(R)$ is 2-torsion free, we consider the commutativity of $R/P(R)$ via P-(Jordan) derivations. These can be regarded as an application of Section 2 and Section 3.

Theorem 4.1. *Let R be a ring with a prime ideal I. If there exists a P-derivation d of R such that:*

(i) $[a, d(a)] \in I$ *for all* $a \in R$.

(ii) $\mathrm{Im}d \nsubseteq I$.

Then R/I is commutative.

Proof. We consider the characteristic of R/I. In the case that $\mathrm{char}R/I \neq 2$, we have $[a+ax, d(a+ax)] \in I, \forall a, x \in R$, we write this as $ad(ax)-d(a)ax+axd(a)-d(ax)a \in I, \forall a, x \in R$, thus we have $a^2d(x) - d(a)xa - ad(x)a + axd(a) \in I$, $\forall a, x \in R$. Similarly, from $[a + xa, d(a + xa)] \in I$, we have $d(x)a^2 - ad(x)a - axd(a) + d(a)xa \in I, \forall a, x \in R$, so $a^2d(x) + d(x)a^2 - 2ad(x)a \in I, \forall a, x \in R$, it follows $[[d(x), a], a] \in I, \forall a, x \in R$, then $[[d(x), a + d(x)], a + d(x)] \in I, \forall a, x \in R$, therefore $[[d(x), a], d(x)] \in I$. Notice that the function $f_{d(x)} : a \mapsto [d(x), a]$ is a derivation and $f_{d(x)}(f_{d(x)}(a)) \in I, \forall a, x \in R$, thus by Corollary 3.5, $[d(x), a] \in I$, i.e., $f_a(d(x)) \in I, \forall a, x \in R$. Since $\mathrm{Im}d \nsubseteq I$, by Corollary 3.5 again, we have $f_a(R) \subseteq I, \forall a \in R$ (i.e., R/I is commutative).

Now, we consider the case that $\mathrm{char}R/I = 2$. Then from the previous discussion, we have $[d(x), a^2] \in I, \forall a, x \in R$, thus $[d(x), a + b] \in I, \forall a, b, x \in R$, and therefore $[d(x), ab + ba] \in I, \forall a, b, x \in R$, then we get $[d(x), aad(x) + ad(x)a] \in I$, $\forall a, b, x \in R$, it follows that $ad(x)ad(x) - d(x)ad(x)a \in I, \forall a, b, x \in R$. Since $\mathrm{char}R/I = 2$, we have $ad(x)ad(x) + d(x)ad(x)a \in I, \forall a, b, x \in R$. On the other hand we have $[d(x), ad(x) + d(x)a] \in I, \forall a, x \in R$, then $[d(x)^2, a] \in I, \forall a, x \in R$. Above all, we have $(ad(x) + d(x)a)^2 = ad(x)ad(x) + d(x)ad(x)a + ad(x)^2a + d(x)a^2d(x) \in I, \forall a, x \in R$. Let $z = ad(x) + d(x)a$. Then we have $[d(x), z(bz)^2 + bzbz^2] \in I, \forall b, x \in R$. Since $z^2 \in I$, we have $[d(x), z(bz)^2] \in I, \forall b, x \in R$, it follows that $(zd(x) + d(x)z)bzbz \in I, \forall b, x \in R$. By Lemma 3.3, we have $z \in I$ or $zd(x) + d(x)z \in I$, any way, $zd(x) + d(x)z \in I$ holds for all $a, x \in R$. Similarly, $[d(x), zw(bz)^2] \in I, \forall a, x, w \in R$, i.e., $(zwd(x) + d(x)zw)bzbz \in I$, $\forall b, x, w \in R$. Notice that $[z, d(x)] \in I$, we have $z[d(x), w] \in I, \forall b, x, w \in R$, namely, by Lemma 3.1, we have $z \in I$ or $[d(x), w] \in I, \forall x, w \in R$, any way $[d(x), y] \in I$, $\forall x, y \in R$. Observe that $d(b^2) = d(b)b + bd(b) \in I$. Then $d(ab^2) - d(a)b^2 \in I$ and $[d(a)b^2, x] \in I, \forall a, x \in I$. Since $\mathrm{Im}d \nsubseteq I$, i.e $\exists a \in R$ such that $d(a) \notin I$. Then from $[d(a)b^2, x] \in I$ and $[d(a), b^2] \in I$, we have $d(a)[b^2, x] \in I, \forall a, b, x \in R$. By Lemma 3.1 again, since $d(a) \notin I$, it follows that $[b^2, x] \in I, \forall a, b, x \in R$. Thus, by the discussion above, we have $[b, x] \in I, \forall b, x \in R$. Hence R/I is commutative. ∎

Corollary 4.2. *Let R be a ring with a prime ideal I. If there exists a derivation d of R such that*

(i) $[a, d(a)] \in I$ *for all* $a \in R$.

(ii) Im$d \not\subseteq I$.

Then R/I is commutative.

According to Polynomial Identity(PI) theory, the condition $[a, d(a)] \in I$ in Theorem 4.1 is actually a polynomial identity of the P-derivation d modulo the prime ideal I. It is interesting to check whether the commutativity still holds if we weaken the polynomial identity of Theorem 4.1. The next theorem will show that, under the condition $R/P(R)$ is 2-torsion free, the conclusion could be consolidated.

Theorem 4.3. *Let R be a ring and I a prime ideal, and R/I is 2-torsion free. If there exists a P-derivation d of R such that:*

(i) $[[d(a), a], x] \in P(R), \forall a, x \in R$,

(ii) *for any prime ideal I of R,* Im$d \not\subseteq I$.

Then $R/P(R)$ is commutative.

Proof. Let us assume that for any prime ideal I, Im$d \not\subseteq I$. We prove that for any prime ideal I, $[a, x] \in I, \forall a, x \in R$, thus $R/P(R)$ is commutative. By assumption we have $[[d(x), x], y] \in P(R), \forall x, y \in R$. Replace x by $a + [a, x]$ it follows $[[d(a + [a, x]), a + [a, x]], y] \in P(R), \forall a, x, y \in R$. Since $[[d(a), a], y] \in P(R), [[d([a, x]), [a, x]], y] \in P(R)$ we then get the following $[[d(a), [a, x]], y] + [[[d(a), x], a], y] + [[[d(x), a], a], y] \in P(R), \forall a, x, y \in R$ write this as $[[d(a), [a, x]], y] + [[a, [x, d(a)]], y] + [[[d(x), a], a], y] \in P(R), \forall a, x, y \in R$. Observe that $[d(a), [a, x]] + [a, [x, d(a)]] + [x, [d(a), a]] \in P(R), \forall a, x \in R$, by the assumption, $[x, [d(a), a]] \in P(R), \forall a, x \in R$, thus $[[[d(x), a], a], y] \in P(R), \forall a, x, y \in R$, i.e. $[a^2 d(x) + d(x)a^2 - 2ad(x)a, y] \in P(R), \forall a, x, y \in R$, replacing y by a, we have $3ad(x)a^2 + a^3 d(x) - (3a^2 d(x)a + d(x)a^3) \in P(R) \subseteq I, \forall a, x \in R$. Now consider the case that charR/I=3. Then we have $[a^3, d(x)] \in I, \forall a, x \in R$. Since for any prime ideal I, Im$d \not\subseteq I$. Then by Corollary 3.5, $[a^3, x] \in I, \forall a, x \in R$. $I \ni [(a+b)^3 - a^3 - b^3, x] = [a^2 b + aba + ba^2 + b^2 a + bab + ab^2, x]$, $\forall a, b, x \in R$, also we have $[a^2 b + aba + ba^2 - b^2 a - bab - ab^2, x] \in I$, $\forall a, b, x \in R$, and then it follows $[a^2 b + aba + ba^2, x] \in I$, $\forall a, b, x \in R$, thus $[a^2(ab) + a(ab)a + (ab)a^2, x] = [a(a^2 b + aba + ba^2), x] \in I$, $\forall a, b, x \in R$. Observe that $[a(a^2 b + aba + ba^2), x] - (a^2 b + aba + ba^2)[a, x] \in I$, $\forall a, b, x \in R$, then $[a, x](a^2 b + aba + ba^2) \in I$, $\forall a, b, x \in R$. Since $[a^2 b + aba + ba^2, x] \in I$, $\forall a, b, x \in R$, it follows $[a, x]R(a^2 b + aba + ba^2) \subseteq I$, $\forall a, b, x \in R$. If $[a, x] \in I$, $\forall a, x \in R$. Then we finish the proof. Let us assume that there exists $a', x' \in R$ such that $[a', x'] \notin I$. Thus $a'^2 b + a'ba' + ba'^2 \in I$, $\forall b \in R$, by the definition of prime ideal. Note that $[a', [a', b]] = a'^2 b - 2a'ba' + ba'^2$ and since char$R/I = 3$, then $[a', [a', b]] \in I, \forall b \in R$. By Corollary 3.5 we have $[a', y] \in I$, $\forall y \in R$. It is a contradiction when $y = x'$.

Next, consider the case that char$R/I \neq 3$. Since we have already got $3ad(x)a^2 + a^3 d(x) - (3a^2 d(x)a + d(x)a^3) \in P(R) \subseteq I$, $\forall a, x \in R$. Thus $3ad(a)a^2 + a^3 d(a) - (3a^2 d(a)a + d(a)a^3) \in I, \forall a \in R$. It follows that $[a^3, d(a)] - 3a[a, d(a)]a \in I$, $\forall a \in R$. Since $[[d(a), a], x] \in I$, $\forall a, x \in R$, we have $[a^3, d(a)] - 3[a, d(a)]a^2 \in I$,

$\forall a \in R$. Since $[a, d(a)]a = ad(a)a - d(a)a^2, a[a, d(a)] = a^2d(a) - ad(a)a$, observe that $[[a, d(a)], a] \in I$, thus $2[a, d(a)]a - [a^2, d(a)] \in I$, $\forall a \in R$.

On the other hand, replacing x by $ad(x)$ in $3ad(x)a^2 + a^3d(x) - (3a^2d(x)a + d(x)a^3) \in I, \forall a, x \in R$, we have $3a^2d(d(x))a^2+a^4d(d(x))-3a^3d(d(x))a-ad(d(x))a^3 - a(3ad(d(x)))a^2 + a^3d(d(x)) - 3a^2d(d(x)) - d(d(x))a^3 \in I$, $\forall a, x \in R$, observe that $a(3ad(d(x)))a^2 + a^3d(d(x)) - 3a^2d(d(x)) - d(d(x))a^3 \in I$, $\forall a, x \in R$, thus $3a^2d(d(x))a^2 + a^4d(d(x)) - 3a^3d(d(x))a - ad(d(x))a^3 \in I, \forall a, x \in R$. Similarly, we have $3ad(a) + a^3d(a)d(x) - 3a^2d(a)d(x)a - d(a)d(x)a^3 \in I, \forall a, x \in R$, comparing with $3d(a)ad(x)a^2 + d(a)a^3d(x) - 3d(a)a^2d(x)a - d(a)d(x)a^3 \in I$, we have $3[a, d(a)]d(x)a^2 + [a^3, d(a)]d(x) - 3[a^2, d(a)]d(x)a \in I, \forall a, x \in R$, since $[a^3, d(a)] - 3[a, d(a)]a^2, 2[a, d(a)]a-[a^2, d(a)]\in I$, then $3[a, d(a)](d(x)a^2+a^2d(x)-2ad(x)a) \in I$, $\forall a, x \in R$. Because $\mathrm{char} R/I \neq 3$, it follows $[a, d(a)](d(x)a^2+a^2d(x)-2ad(x)a) \in I$, $\forall a, x \in R$. Since $[[a, d(a)], x] \in I$, $\forall a, x \in R$, so $[a, d(a)]R(d(x)a^2 + a^2d(x) - 2ad(x)a) \subseteq I$, $\forall a, x \in R$. Therefore, $[a, d(a)]\in I$ or $(d(x)a^2+a^2d(x)-2ad(x)a)\in I$. Assume $\exists a \in R$ such that $[a, d(a)] \notin I$, then $(d(x)a^2+a^2d(x)-2ad(x)a) \in I, \forall x \in R$. Replacing x by ax, we have $ad(x)a^2+a^3d(x)-2a^2d(x)a+d(a)xa^2+a^2d(a)x - 2ad(a)xa \in I, \forall x \in R$. Since $d(x)a^2+a^2d(x)-2ad(x)a \in I$, thus $ad(x)a^2+a^3d(x) - 2a^2d(x)a = a(d(x)a^2 + a^2d(x) - 2ad(x)a) \in I$, $\forall x \in R$. Therefore, $d(a)xa^2 + a^2d(a)x - 2ad(a)xa \in I, \forall x \in R$. We also have $d(a)a^2+a^2d(a)-2ad(a)a\in I$. Then $(d(a)a^2+a^2d(a)-2ad(a)a)x \in I$, $\forall x \in R$. Thus $(d(a)xa^2+a^2d(a)x-2ad(a)xa) - ((d(a)a^2 + a^2d(a) - 2ad(a)a)x) = d(a)a[x, a^2] - 2ad(a)a[a, x] \in I, \forall x \in R$. Replacing x by ax, we have $d(a)a[x, a^2] - 2ad(a)a[x, a] \in I$, $\forall x \in R$. On the other hand, $ad(x)[x, a^2] - 2a^2d(a)[a, x] \in I$, $\forall x \in R$. Thus we have $[a, d(a)][x, a^2] - 2a[a, d(a)][x, a] \in I$, $\forall x \in R$, then it follows $[a, d(a)]R([x, a^2] - 2a[x, a]) \in I$, $\forall x \in R$, therefore, $[x, a^2] - 2a[x, a] = xa^2 + a^2x - 2axa = [[x, a], a] \in I$, $\forall x \in R$. By Corollary 3.5, $[x, a] \in I$, $\forall x \in R$. ∎

From Theorem 4.1 and Theorem 4.3, we can see that, if we restrict R to the condition that $R/P(R)$ is 2-torsion free, the condition: $[a, d(a)] \in I, \forall a \in R$ could be weaken correspondingly. It's an interesting question that whether or not the conclusion of Theorem 4.3 is true with the condition: $R/P(R)$ is 2-torsion free omitted? Unfortunately, we haven't found the way approaching to the answer. On the other hand, we can state this question in a more universal form: could we continue to weaken the condition: $[a, d(a)] \in I, \forall a \in R$ in Theorem 4.1? This may be an interesting question for researchers of PI theory.

An immediate corollary of Theorem 4.3 could be used for the commutativity of 2-torsion free semiprime ring:

Corollary 4.4. *If a derivation d of a 2-torsion free semiprime ring R satisfies:*

(i) $[a, d(a)]$ *is in the central of* R,

(ii) *any prime ideal* I, $\mathrm{Im} d \nsubseteq I$.

Then R is commutative.

Also, a sufficient condition for the commutativity of R/I, where I is a prime ideal of R, can also be got from Theorem 4.3.

Corollary 4.5. *Let R be a ring and $R/P(R)$ is 2-torsion free. For any prime ideal I of R, if there exists a P-(Jordan) derivation d of R such that*

(i) $[[d(a), a], x] \in I, \forall a, x \in R$,

(ii) $\mathrm{Im} d \nsubseteq I$.

Then R/I is commutative.

We know that Jordan derivations are P-Jordan derivations, Corollary 4.5 also enables us to discuss the commutativity of R/I via the Jordan derivations of R.

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