SOME NEW INTEGRAL INEQUALITIES OF HADAMARD-SIMPSON TYPE FOR EXTENDED (s, m)-PREINVEX FUNCTIONS

Yujiao Li
Tingsong Du
Bo Yu[1]

College of Science
China Three Gorges University
Yichang 443002
P.R. China
e-mails: yujiaolictgu@163.com
tingsongdu@ctgu.edu.cn
hbycdts@163.com

Abstract. The (s, m)-preinvex function and the extended (s, m)-preinvex function are firstly introduced. An integral identity for differentiable functions is then derived. Based on this integral identity, some Hadamard-Simpson type integral inequalities are investigated, which generalizes the existing similar type integral inequalities.

Keywords: (s, m)-preinvex functions, extended (s, m)-preinvex functions, Hadamard type integral inequality, Simpson type integral inequality.

2010 AMS Classification: Primary 26D15; Secondary 26A51, 26B12, 41A55.

1. Introduction

The following notation is used throughout this paper. I is an interval on the real line $\mathbb{R}$. $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space, $\mathbb{R}^n_0$ denotes an n-dimensional nonnegative vector space, and $\mathbb{R}^n_+$ denotes an n-dimensional positive vector space. For any subset $K \subseteq \mathbb{R}^n$, K° is used to denote the interior of K. $\mathbb{R}_0 = [0, \infty)$. $L_1[a, b]$ is the set of integrable functions over the interval $[a, b]$. Let us firstly recall some definitions of various convex type functions.

Definition 1.1 ([6]) A function $f : I \subseteq \mathbb{R} \to \mathbb{R}_0$ is said to be a Godunova-Levin function if f is nonnegative and for all $x, y \in I$, $\lambda \in (0, 1)$ we have that

$$f\big(\lambda x + (1 - \lambda)y\big) \le \frac{f(x)}{\lambda} + \frac{f(y)}{1 - \lambda}.$$

[1]Corresponding author. E-mail: hbycdts@163.com

Definition 1.2 ([5]) For some $(s,m)\in(0,1]^2$, a function $f:[0,b]\to\mathbb{R}$ is said to be (s,m)-convex in the second sense if, for every $x,y\in[0,b]$ and $\lambda\in[0,1]$, we have that

$$f\big(\lambda x+m(1-\lambda)y\big)\le\lambda^s f(x)+m(1-\lambda)^s f(y).$$

Definition 1.3 ([34]) For some $s\in[-1,1]$ and $m\in(0,1]$, a function $f:[0,b]\to\mathbb{R}_0$ is said to be extended (s,m)-convex if for all $x,y\in[0,b]$ and $\lambda\in(0,1)$ we have that

$$f\big(\lambda x+m(1-\lambda)y\big)\le\lambda^s f(x)+m(1-\lambda)^s f(y).$$

Definition 1.4 ([1]) A set $K\subseteq\mathbb{R}^n$ is said to be invex with respect to the map $\eta:K\times K\to\mathbb{R}^n$, if $x+t\eta(y,x)\in K$ for every $x,y\in K$ and $t\in[0,1]$.

Notice that every convex set is invex with respect to the map $\eta(y,x)=y-x$, but the converse is not necessarily true. For more details please refer to [1], [35] and the references therein.

Definition 1.5 ([1]) Let $K\subseteq\mathbb{R}^n$ be an invex set with respect to $\eta:K\times K\to\mathbb{R}^n$, for every $x,y\in K$, the η-path $P_{x\nu}$ joining the points x and $\nu=x+\eta(y,x)$ is defined by

$$P_{x\nu}=\big\{z|z=x+t\eta(y,x),\ t\in[0,1]\big\}.$$

Definition 1.6 ([25]) The function f defined on the invex set $K\subseteq\mathbb{R}^n$ is said to be preinvex with respect to η if for every $x,y\in K$ and $t\in[0,1]$ we have that

$$f\Big(x+t\eta(y,x)\Big)\le(1-t)f(x)+tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the map $\eta(y,x)=y-x$ but the converse is not true.

Definition 1.7 ([12]) The function f defined on the invex set $K\subseteq[0,b^*]$ with $b^*>0$ is said to be m-preinvex with respect to η if for all $x,y\in K$, $t\in[0,1]$ and for some fixed $m\in(0,1]$, we have that

$$f\Big(x+t\eta(y,x)\Big)\le(1-t)f(x)+mtf\Big(\frac{y}{m}\Big).$$

Remark 1.1. Notice that if $y\in[0,b^*]$, then for any $0<m<1$, $\frac{y}{m}$ could be greater than b^*, which is not in the domain of f. Thus, the right hand side of the inequality in this definition could be meaningless. To fix this flaw, we suggest to replace $[0,b^*]$ by the half real line $\mathbb{R}_0$.

Definition 1.8 ([15]) Let $K\subseteq\mathbb{R}_0$ be an invex set with respect to η. A function $f:K\to\mathbb{R}$ is said to be s-preinvex with respect to η, if for all $x,y\in K$, $t\in[0,1]$ and some fixed $s\in(0,1]$ we have that

$$f\Big(x+t\eta(y,x)\Big)\le(1-t)^s f(x)+t^s f(y).$$

Now, we are ready to review some inequalities of Hermite-Hadamard and Simpson type for these kinds of convex functions mentioned above.

Theorem 1.1 ([21]) *Let $f : K = [a, a + \eta(b, a)] \to (0, \infty)$ be a preinvex function on the interval of the real numbers K° and $a, b \in K^\circ$ with $a < a + \eta(b, a)$. Then the following inequality holds:*

$$f\left(\frac{2a+\eta(b,a)}{2}\right) \le \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \le \frac{f(a)+f(b)}{2}. \tag{1.1}$$

Inequality (1.1) is named by the Hermite-Hadamard-Noor type inequality for preinvex functions. This result is analogous to the original Hermite-Hadamard inequalities. If $\eta(b, a) = b - a$ then inequality (1.1) reduces to the remarkable Hermite-Hadamard's inequality.

Theorem 1.2 ([4]) *If $f : [a, b] \to \mathbb{R}$ is a four times continuously differentiable mapping on the interval (a, b) and $||f^{(4)}||_\infty = \sup_{x\in(a,b)} |f^{(4)}(x)| < \infty$ then the following inequality holds:*

$$\left|\frac{1}{3}\left[\frac{f(a)+f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \le \frac{1}{2880}||f^{(4)}||_\infty (b-a)^4. \tag{1.2}$$

Theorem 1.3 ([28]) *Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a, b]$, where $a, b \in I^\circ$ with $a < b$. If $\frac{1}{p} + \frac{1}{q} = 1$ and $|f'|^q$ is convex on $[a, b]$, $q > 1$, then the following inequality holds:*

$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le \frac{b-a}{12}\left(\frac{1+2^{p+1}}{3(p+1)}\right)^{\frac{1}{p}} \left\{\left(\frac{3|f'(b)|^q+|f'(a)|^q}{4}\right)^{\frac{1}{q}} + \left(\frac{|f'(b)|^q+3|f'(a)|^q}{4}\right)^{\frac{1}{q}}\right\}. \tag{1.3}$$

Theorem 1.4 ([27]) *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I° such that $f' \in L_1[a, b]$, where $a, b \in I$ with $a < b$. If $|f'|$ is s-convex on $[a, b]$, for some fixed $s \in (0, 1]$, then*

$$\left|\frac{1}{6}\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le \frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+2}(s+1)(s+2)}(b-a)\Big[|f'(a)|+|f'(b)|\Big]. \tag{1.4}$$

Theorem 1.5 ([33]) *Let $f : I \subseteq \mathbb{R}_0 \to \mathbb{R}$ be a differentiable mapping on I°, $a, b \in I$ with $a < b$, $f' \in L_1[a, b]$ and $0 \le \lambda, \mu \le 1$. If $|f'(x)|^q$ for $q \ge 1$ is extended s-convex on $[a, b]$ for some $s \in [-1, 1]$, then*

1. *when* $s=-1$, *the following inequality holds:*

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\le\frac{b-a}{2^{3-\frac{2}{q}}}\left\{\Big[(2\ln 2-1)|f'(a)|^q+|f'(b)|^q\Big]^{\frac{1}{q}}+\Big[|f'(a)|^q+(2\ln 2-1)|f'(b)|^q\Big]^{\frac{1}{q}}\right\};\tag{1.5}$$

2. *when* $q=1$ *and* $s=-1$, *the following inequality holds:*

$$\left|f\Big(\frac{a+b}{2}\Big)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right|\le(b-a)\ln 2\Big(|f'(a)|+|f'(b)|\Big).\tag{1.6}$$

Theorem 1.6 ([2], [26]) *Let* $K\subseteq\mathbb{R}$ *be an open invex subset with respect to* $\eta:K\times K\to\mathbb{R}$. *Suppose that* $f:K\to\mathbb{R}$ *is a differentiable function. If* $|f'|$ *is preinvex on* K *then for every* $a,b\in K$ *with* $\eta(b,a)\neq 0$ *we have that:*

$$\begin{aligned}&\left|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2}-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\&\le\frac{|\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big)\end{aligned}\tag{1.7}$$

and

$$\begin{aligned}&\left|f\left(\frac{2a+\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\&\le\frac{|\eta(b,a)|}{8}\Big(|f'(a)|+|f'(b)|\Big).\end{aligned}\tag{1.8}$$

Theorem 1.7 ([31]) *Let* $A\subseteq\mathbb{R}$ *be an open invex subset with respect to* $\eta:A\times A\to\mathbb{R}$. *Suppose that* $f:A\to\mathbb{R}$ *is a differentiable function. If* $q>1$, $q\ge r$, $s\ge 0$ *and* $|f'|$ *is preinvex on* A, *then for every* $a,b\in A$ *with* $\eta(a,b)\neq 0$, *we have that*

$$\begin{aligned}&\left|f\left(\frac{2b+\eta(a,b)}{2}\right)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\mathrm{d}x\right|\\&\le\frac{|\eta(a,b)|}{4}\left\{\left(\frac{1}{r+1}\right)^{\frac{1}{q}}\left(\frac{q-1}{2q-r-1}\right)^{1-\frac{1}{q}}\left[\frac{(r+1)|f'(a)|^q+(r+3)|f'(b)|^q}{2(r+2)}\right]^{\frac{1}{q}}\right.\\&\left.+\left(\frac{1}{s+1}\right)^{\frac{1}{q}}\left(\frac{q-1}{2q-s-1}\right)^{1-\frac{1}{q}}\left[\frac{(s+3)|f'(a)|^q+(s+1)|f'(b)|^q}{2(s+2)}\right]^{\frac{1}{q}}\right\}.\end{aligned}$$

Corollary 1.1 ([31]) *Under the conditions of Theorem* 1.7, *when* $r=s=0$, *the following inequality holds:*

$$\left|f\left(\frac{2b+\eta(a,b)}{2}\right)-\frac{1}{\eta(a,b)}\int_b^{b+\eta(a,b)}f(x)\mathrm{d}x\right|$$
$$\le\left(\frac{q-1}{2q-1}\right)^{1-\frac{1}{q}}\frac{|\eta(a,b)|}{4}\left[\left(\frac{1}{4}|f'(a)|^q+\frac{3}{4}|f'(b)|^q\right)^{\frac{1}{q}}+\left(\frac{3}{4}|f'(a)|^q+\frac{1}{4}|f'(b)|^q\right)^{\frac{1}{q}}\right].\tag{1.9}$$

Currently, the Hadamard type and Simpson type inequalities concerning different kinds of preinvex functions are still interesting topics to many researchers in the field of convex analysis. For more information please refer to [3], [7]-[14], [17]-[19], [22]-[24], [29]-[30], [32] and references cited therein.

Motivated by the inspiring idea in [20], [33] and based on our previous works [16], [17], [36], in this paper we are going to introduce the (s, m)-preinvex function and the extended (s, m)-preinvex function, and then we will establish some Hadamard-Simpson-like type integral inequalities for functions related to these two new kinds of functions. In Section 2, we will introduce new definitions and an integral identity. Section 3 will be devoted of presenting the main results.

2. New definitions and an integral identity

We now introduce definitions of the (s, m)-preinvex function and the extended (s, m)-preinvex function.

Definition 2.1 Let $K = [0, b^*]$ with $b^* > 0$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}_+$. For $f : K \to \mathbb{R}$ and some fixed $(s, m) \in (0, 1] \times (0, 1]$, if

$$f\Big(x + t\eta(y, x)\Big) \le (1-t)^s f(x) + mt^s f\Big(\frac{y}{m}\Big) \tag{2.1}$$

is valid for all $x, y \in K$, $t \in [0, 1]$, with $\frac{y}{m} \le b^*$, then we say that $f(x)$ is an (s, m)-preinvex function with respect to η.

Definition 2.2 Let $K = [0, b^*]$ with $b^* > 0$ be an invex set with respect to $\eta : K \times K \to \mathbb{R}_+$. For $f : K \to \mathbb{R}_0$ and some fixed $(s, m) \in [-1, 1] \times (0, 1]$, if

$$f\Big(x + t\eta(y, x)\Big) \le (1-t)^s f(x) + mt^s f\Big(\frac{y}{m}\Big) \tag{2.2}$$

is valid for all $x, y \in K$, $t \in (0, 1)$, with $\frac{y}{m} \le b^*$, then we say that $f(x)$ is an extended (s, m)-preinvex function with respect to η.

Remark 2.1. In Definition 2.1, if $s = 1$ then one obtains the definition of m-preinvex function. If $m = 1$ then one obtains the definition of s-preinvex function. It is also worthwhile to note that every (s, m)-preinvex function is (s, m)-convex and every extended (s, m)-preinvex functions is extended (s, m)-convex with respect to $\eta(y, x) = y - x$ respectively.

In order to establish some new Hadamard-Simpson-like type integral inequalities, we present the following integral identity.

Lemma 2.1 *Let $K \subseteq \mathbb{R}$ be an invex subset with respect to $\eta : K \times K \to \mathbb{R}$ and $a, b \in K$ with $\eta(b, a) > 0$. If $0 \le \lambda,\ \mu < 1$, $f : K \to \mathbb{R}$ is a differentiable function and f' is integrable on the η-path P_{ac} for $c = a + \eta(b, a)$, then we have that*

$$
\begin{aligned}
&-\left[(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f(a+\eta(b,a))\right]\\
&+\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \qquad (2.3)\\
&=\eta(b,a)\left[\int_0^{\frac{1}{2}}(1-\lambda-t)f'(a+t\eta(b,a))\mathrm{d}t+\int_{\frac{1}{2}}^{1}(1-\mu-t)f'(a+t\eta(b,a))\mathrm{d}t\right].
\end{aligned}
$$

Proof. Set

$$
J=\eta(b,a)\left[\int_0^{\frac{1}{2}}(1-\lambda-t)f'\Big(a+t\eta(b,a)\Big)\mathrm{d}t+\int_{\frac{1}{2}}^{1}(1-\mu-t)f'\Big(a+t\eta(b,a)\Big)\mathrm{d}t\right].
$$

Since $a,b\in K$ and K is an invex set with respect to η , for every $t\in[0,1]$ we have $a+t\eta(b,a)\in K$. Integrating by part yields that

$$
\begin{aligned}
J &= \eta(b,a)\left\{\frac{1}{\eta(b,a)}\left[(1-\lambda-t)f\Big(a+t\eta(b,a)\Big)\Big|_0^{\frac{1}{2}}+\int_0^{\frac{1}{2}} f\Big(a+t\eta(b,a)\Big)\mathrm{d}t\right]\right.\\
&\quad\left.+\frac{1}{\eta(b,a)}\left[(1-\mu-t)f\Big(a+t\eta(b,a)\Big)\Big|_{\frac{1}{2}}^{1}+\int_{\frac{1}{2}}^{1} f\Big(a+t\eta(b,a)\Big)\mathrm{d}t\right]\right\}\\
&= -(1-\lambda)f(a)-(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)-\mu f\Big(a+\eta(b,a)\Big)\\
&\quad+\int_0^{\frac{1}{2}} f\Big(a+t\eta(b,a)\Big)\mathrm{d}t+\int_{\frac{1}{2}}^{1} f\Big(a+t\eta(b,a)\Big)\mathrm{d}t\\
&= -\left[(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\Big(a+\eta(b,a)\Big)\right]\\
&\quad+\int_0^{1} f\Big(a+t\eta(b,a)\Big)\mathrm{d}t.
\end{aligned}
$$

Let $x=a+t\eta(b,a)$, then $\mathrm{d}x=\eta(b,a)\mathrm{d}t$ and we have

$$
\begin{aligned}
J &= -\left[(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\Big(a+\eta(b,a)\Big)\right]\\
&\quad+\frac{1}{\eta(a,b)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x,
\end{aligned}
$$

which is required.

3. Some Hadamard-Simpson-like type integral inequalities

First of all, we state the following results for (s, m)-preinvex mapping in the second sense.

Theorem 3.1 *Let $A = [0, b^*]$ with $b^* > 0$ be an invex subset with respect to $\eta : A \times A \to \mathbb{R}_0$ and $a, b \in A$ with $\eta(b, a) > 0$. Let $0 \leq \lambda,\ \mu < 1$. Suppose that $f : A \to \mathbb{R}_0$ is a differentiable function and f' is integrable on the η-path P_{ac} for $c = a + \eta(b, a)$. If $|f'|$ is (s, m)-preinvex on $\left[a, \frac{b}{m}\right]$ for some fixed $s \in (0, 1]$, $0 \leq a < b$, and $m \in (0, 1]$ with $\frac{b}{m} \leq b^*$, then the following inequality holds:*

$$\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\Big(a+\eta(b,a)\Big)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|$$

$$\begin{aligned}
&\leq \frac{|\eta(b,a)|}{(s+1)(s+2)}\Bigg\{\left[2\lambda^{s+2}-\frac{2s\lambda+4\lambda-s-1}{2^{s+2}}+s-s\lambda-2\lambda+1\right]\Big|f'(a)\Big| \\
&+m\left[2(1-\lambda)^{s+2}+\frac{2s\lambda+4\lambda-s-3}{2^{s+2}}\right]\left|f'\left(\frac{b}{m}\right)\right| \\
&+\left[2\mu^{s+2}-\frac{2s\mu+4\mu-s-1}{2^{s+2}}\right]\Big|f'(a)\Big| \\
&+m\left[2(1-\mu)^{s+2}+\frac{2s\mu+4\mu-s-3}{2^{s+2}}+s\mu+2\mu-1\right]\left|f'\left(\frac{b}{m}\right)\right|\Bigg\}.
\end{aligned} \tag{3.1}$$

Proof. Since A is an invex subset with respect to η, for every $t \in [0, 1]$, we have $a + t\eta(b, a) \in A$. Using Lemma 2.1 and the (s, m)-preinvexity of $|f'|$ on $\left[a, \frac{b}{m}\right]$ obtains that

$$\begin{aligned}
&\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\Big(a+\eta(b,a)\Big)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right| \\
&\leq \Big|\eta(b,a)\Big|\left[\int_0^{\frac{1}{2}}|1-\lambda-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t+\int_{\frac{1}{2}}^1|1-\mu-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t\right] \\
&\leq \Big|\eta(b,a)\Big|\Bigg\{\int_0^{\frac{1}{2}}|1-\lambda-t|(1-t)^s|f'(a)|\mathrm{d}t+m\int_0^{\frac{1}{2}}|1-\lambda-t|t^s\left|f'\left(\frac{b}{m}\right)\right|\mathrm{d}t \\
&\quad+\int_{\frac{1}{2}}^1|1-\mu-t|(1-t)^s|f'(a)|\mathrm{d}t+m\int_{\frac{1}{2}}^1|1-\mu-t|t^s\left|f'\left(\frac{b}{m}\right)\right|\mathrm{d}t\Bigg\}.
\end{aligned}$$

Direct computation yields that

$$\int_0^{\frac{1}{2}}|1-\lambda-t|t^s\mathrm{d}t=\frac{1}{(s+1)(s+2)}\left[2(1-\lambda)^{s+2}+\frac{2s\lambda+4\lambda-s-3}{2^{s+2}}\right],$$

$$\int_0^{\frac{1}{2}}|1-\lambda-t|(1-t)^s\mathrm{d}t=\frac{1}{(s+1)(s+2)}\left[2\lambda^{s+2}-\frac{2s\lambda+4\lambda-s-1}{2^{s+2}}+s-s\lambda-2\lambda+1\right].$$

Similarly, we have

$$\int_{\frac{1}{2}}^{1} |1-\mu-t| t^s \mathrm{d}t = \frac{1}{(s+1)(s+2)} \left[2(1-\mu)^{s+2} + \frac{2s\mu+4\mu-s-3}{2^{s+2}} + s\mu+2\mu-1 \right],$$

$$\int_{\frac{1}{2}}^{1} |1-\mu-t|(1-t)^s \mathrm{d}t = \frac{1}{(s+1)(s+2)} \left[2\mu^{s+2} - \frac{2s\mu+4\mu-s-1}{2^{s+2}} \right].$$

Plugging these four equalities into the right hand side of the above inequality establishes the desired inequality and the proof is completed.

By elementary calculation, it is easy to get the following results.

Corollary 3.1 *Under the conditions of Theorem* 3.1*,*

1. *if* $\lambda = 1, \mu = 0$*, and* $s = m = 1$*, we have*

$$\left| f\left(\frac{2a+\eta(b,a)}{2}\right) - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \right| \le \frac{|\eta(b,a)|}{8} \Big(|f'(a)| + |f'(b)| \Big); \tag{3.2}$$

2. *if* $\lambda = \mu = \dfrac{1}{2}$ *and* $s = m = 1$*, we have*

$$\left| \frac{f(a) + f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)} \int_a^{a+\eta(b,a)} f(x)\mathrm{d}x \right| \le \frac{|\eta(b,a)|}{8} \Big(|f'(a)| + |f'(b)| \Big); \tag{3.3}$$

3. *if* $\eta(b,a) = b-a, \lambda = \dfrac{5}{6}, \mu = \dfrac{1}{6}$ *and* $m = 1$*, we have*

$$\left| \left[f(a) + f(b) + 4f\Big(\frac{a+b}{2}\Big) \right] - \frac{6}{b-a} \int_a^b f(x)\mathrm{d}x \right| \le \frac{(s-4)6^{s+1} + 2\times 5^{s+2} - 2\times 3^{s+2} + 2}{6^{s+1}(s+1)(s+2)} (b-a)(|f'(a)| + |f'(b)|). \tag{3.4}$$

Remark 3.1. Inequality (3.2) is the same as inequality of (1.8) established by Sarikaya in [26]. Inequality (3.3) is the same as inequality of (1.7) obtained by Barani in [2]. Inequality (3.4) is the same as inequality of (1.4) presented by Sarikaya in [27]. Thus, inequality (3.1) is a generalization of these Hadamard type inequalities.

Using Lemma 2.1 again, we are going to present and prove new integral inequalities of Hadamard-Simpson-like type for differentiable extended (s,m)-preinvex functions.

Theorem 3.2 *Let $A=[0,b^*]$ with $b^*>0$ be an invex subset with respect to $\eta: A\times A\to\mathbb{R}_0$ and $a,b\in A$ with $\eta(b,a)>0$. Assume that $0\le\lambda,\ \mu<1$. Suppose that $f:A\to\mathbb{R}_0$ is a differentiable function and f' is integrable on the η-path P_{ac} for $c=a+\eta(b,a)$. If $|f'|^q$ for $q\ge1$ is extended (s,m)-preinvex on $[a,\frac{b}{m}]$ for some fixed $s\in[-1,1]$, $0\le a<b$, and $m\in(0,1]$ with $\frac{b}{m}\le b^*$, then*

1. *when $-1<s\le1$, we have*

$$\begin{aligned}
&\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f(a+\eta(b,a))-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{|\eta(b,a)|}{[(s+1)(s+2)]^{1/q}}\left\{\left(\lambda^2-\frac{3}{2}\lambda+\frac{5}{8}\right)^{1-\frac{1}{q}}\left[\left(2\lambda^{s+2}-\frac{2s\lambda+4\lambda-s-1}{2^{s+2}}+s-s\lambda-2\lambda+1\right)\left|f'(a)\right|^q\right.\right.\\
&\left.\quad+m\left(2(1-\lambda)^{s+2}+\frac{2s\lambda+4\lambda-s-3}{2^{s+2}}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} \qquad (3.5)\\
&\quad+\left(\mu^2-\frac{1}{2}\mu+\frac{1}{8}\right)^{1-\frac{1}{q}}\left[\left(2\mu^{s+2}-\frac{2s\mu+4\mu-s-1}{2^{s+2}}\right)\left|f'(a)\right|^q\right.\\
&\left.\left.\quad+m\left(2(1-\mu)^{s+2}+\frac{2s\mu+4\mu-s-3}{2^{s+2}}+s\mu+2\mu-1\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\};
\end{aligned}$$

2. *when $s=-1$, $\lambda=1$ and $\mu=0$, we have*

$$\begin{aligned}
&\left|f\left(a+\frac{\eta(b,a)}{2}\right)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le\frac{|\eta(b,a)|}{8^{1-\frac{1}{q}}}\left\{\left[\left(\ln2-\frac{1}{2}\right)\left|f'(a)\right|^q+\frac{m}{2}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right. \qquad (3.6)\\
&\quad\left.+\left[\frac{1}{2}\left|f'(a)\right|^q+m\left(\ln2-\frac{1}{2}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}$$

Proof. 1. When $-1<s\le1$, by Lemma 2.1, we have that

$$\begin{aligned}
&\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f\big(a+\eta(b,a)\big)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|\\
&\le|\eta(b,a)|\left[\int_0^{\frac{1}{2}}|1-\lambda-t|\big|f'(a+t\eta(b,a))\big|\mathrm{d}t+\int_{\frac{1}{2}}^1|1-\mu-t||f'(a+t\eta(b,a))|\mathrm{d}t\right].
\end{aligned}$$

Using the power-mean integral inequality, we have

$$\int_0^{\frac{1}{2}} |1-\lambda-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t \le \Big[\int_0^{\frac{1}{2}} |1-\lambda-t|\mathrm{d}t\Big]^{1-\frac{1}{q}} \Big(\int_0^{\frac{1}{2}} |1-\lambda-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}$$

and

$$\int_{\frac{1}{2}}^1 |1-\mu-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t \le \Big[\int_{\frac{1}{2}}^1 |1-\mu-t|\mathrm{d}t\Big]^{1-\frac{1}{q}} \Big(\int_{\frac{1}{2}}^1 |1-\mu-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}.$$

Now, the extended (s,m)-preinvexity of $|f'|^q$ in $[a,\frac{b}{m}]$ implies that

$$\Big(\int_0^{\frac{1}{2}} |1-\lambda-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \le \Big[\int_0^{\frac{1}{2}} |1-\lambda-t|\Big((1-t)^s|f'(a)|^q+mt^s\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)\mathrm{d}t\Big]^{\frac{1}{q}}$$

and

$$\Big(\int_{\frac{1}{2}}^1 |1-\mu-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}} \le \Big[\int_{\frac{1}{2}}^1 |1-\mu-t|\Big((1-t)^s|f'(a)|^q+mt^s\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)\mathrm{d}t\Big]^{\frac{1}{q}}.$$

Finally, the combination of these inequalities and an elementary computation establish the desired inequality.

2. When $s=-1$, $\lambda=1$ and $\mu=0$, Lemma 2.1, the power-mean integral inequality and the extended $(-1,m)$-preinvexity of $|f'(x)|^q$ on $[a,\frac{b}{m}]$ can be used in a similar way to obtain that

$$\begin{aligned}
&\Big|f\Big(a+\frac{\eta(b,a)}{2}\Big)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\Big|\\
&\le \big|\eta(b,a)\big|\Big[\int_0^{\frac{1}{2}}|-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t+\int_{\frac{1}{2}}^1|1-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t\Big]\\
&\le \big|\eta(b,a)\big|\Bigg\{\Big[\int_0^{\frac{1}{2}}|-t|\mathrm{d}t\Big]^{1-\frac{1}{q}}\Big[\int_0^{\frac{1}{2}}\Big(t(1-t)^{-1}\Big|f'(a)\Big|^q+mtt^{-1}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)\mathrm{d}t\Big]^{\frac{1}{q}}\\
&\quad+\Big[\int_{\frac{1}{2}}^1|1-t|\mathrm{d}t\Big]^{1-\frac{1}{q}}\Big[\int_{\frac{1}{2}}^1\Big((1-t)(1-t)^{-1}\Big|f'(a)\Big|^q+m(1-t)t^{-1}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)\mathrm{d}t\Big]^{\frac{1}{q}}\Bigg\}
\end{aligned}$$

$$= \big|\eta(b,a)\big|\Big(\frac{1}{8}\Big)^{1-\frac{1}{q}}\Bigg\{\Bigg[\int_0^{\frac{1}{2}}\Big(\frac{t}{1-t}\Big|f'(a)\Big|^q + m\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)\mathrm{d}t\Bigg]^{\frac{1}{q}}$$

$$+\Bigg[\int_{\frac{1}{2}}^1\Big(\Big|f'(a)\Big|^q + \frac{m(1-t)}{t}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)\mathrm{d}t\Bigg]^{\frac{1}{q}}\Bigg\}$$

$$= \big|\eta(b,a)\big|\Big(\frac{1}{8}\Big)^{1-\frac{1}{q}}\Bigg\{\Bigg[(\ln 2 - \frac{1}{2})\Big|f'(a)\Big|^q + \frac{m}{2}\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Bigg]^{\frac{1}{q}}$$

$$+\Bigg[\frac{1}{2}\Big|f'(a)\Big|^q + m\Big(\ln 2 - \frac{1}{2}\Big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Bigg]^{\frac{1}{q}}\Bigg\}.$$

The proof is completed. ∎

Remark 3.2. If $\eta(b,a) = b-a$ and $m=1$, then inequality (3.6) reduces to inequality (1.5). Thus, inequality (3.6) is an extension of inequality (1.5).

Direct computation yields the following corollary.

Corollary 3.2 *Under the conditions of Theorem* 3.2,

1. *if* $q=1$ *and* $-1 < s \leq 1$, *then*

$$\Bigg|(1-\lambda)f(a)+(\lambda-\mu)f\Big(a+\frac{\eta(b,a)}{2}\Big)+\mu f(a+\eta(b,a)) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\Bigg|$$

$$\leq \frac{|\eta(b,a)|}{(s+1)(s+2)}\Bigg\{\Bigg[2\lambda^{s+2} - \frac{2s\lambda+4\lambda-s-1}{2^{s+2}} + s - s\lambda - 2\lambda + 1\Bigg]\Big|f'(a)\Big|$$

$$+m\Bigg[2(1-\lambda)^{s+2} + \frac{2s\lambda+4\lambda-s-3}{2^{s+2}}\Bigg]\Big|f'\Big(\frac{b}{m}\Big)\Big| \tag{3.7}$$

$$+\Bigg[2\mu^{s+2} - \frac{2s\mu+4\mu-s-1}{2^{s+2}}\Bigg]\Big|f'(a)\Big|$$

$$+m\Bigg[2(1-\mu)^{s+2} + \frac{2s\mu+4\mu-s-3}{2^{s+2}} + s\mu + 2\mu - 1\Bigg]\Big|f'\Big(\frac{b}{m}\Big)\Big|\Bigg\};$$

2. *if* $\eta(b,a) = b-a$, $s=-1$, $\lambda = 1$, $\mu = 0$ *and* $q=1$, *then*

$$\Bigg|f\Big(\frac{a+b}{2}\Big) - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Bigg| \leq (b-a)\ln 2\Bigg[|f'(a)| + m\Big|f'\Big(\frac{b}{m}\Big)\Big|\Bigg]; \tag{3.8}$$

3. *if* $\lambda = \mu = \dfrac{1}{2}$, $s=1$ *and* $q=1$, *then*

$$\begin{aligned}&\Bigg|\frac{f(a)+f\big(a+\eta(b,a)\big)}{2} - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\Bigg| \\ &\qquad\leq \frac{|\eta(b,a)|}{8}\Bigg[|f'(a)| + m\Big|f'\Big(\frac{b}{m}\Big)\Big|\Bigg];\end{aligned} \tag{3.9}$$

4. *if* $\eta(b,a)=b-a$, $\lambda=\dfrac{5}{6}, \mu=\dfrac{1}{6}, -1<s\leq 1$ *and* $q=1$, *then*

$$\left|\left[f(a)+f(b)+4f\left(\frac{a+b}{2}\right)\right]-\frac{6}{b-a}\int_a^b f(x)\mathrm{d}x\right|$$
$$\leq \frac{(s-4)6^{s+1}+2\times 5^{s+2}-2\times 3^{s+2}+2}{6^{s+1}(s+1)(s+2)}(b-a)\left[|f'(a)|+m\Big|f'\Big(\frac{b}{m}\Big)\Big|\right]. \tag{3.10}$$

Remark 3.3. In inequalities (3.8), (3.9) and (3.10), if we let $m=1$ then we will get inequalities (1.6), (1.7) and (1.4) respectively. Thus, Theorem 3.2 and its consequences generalize the main results in [33], [2] and [27].

Theorem 3.3 *Let* $A=[0,b^*]$ *with* $b^*>0$ *be an invex subset with respect to* $\eta: A\times A\to\mathbb{R}_0$ *and* $a,b\in A$ *with* $\eta(b,a)>0$. *Assume that* $0\leq\lambda,\ \mu<1$. *Suppose that* $f:A\to\mathbb{R}_0$ *is a differentiable function and* f' *is integrable on the* η*-path* P_{ac} *for* $c=a+\eta(b,a)$. *If* $\frac{1}{p}+\frac{1}{q}=1$ *and* $|f'|^q$ *for* $q>1$ *is extended* (s,m)*-preinvex on* $\left[a,\frac{b}{m}\right]$ *for some fixed* $s\in(-1,1]$, $0\leq a<b$, *and* $m\in[0,1]$ *with* $\frac{b}{m}\leq b^*$, *then*

$$\left|(1-\lambda)f(a)+(\lambda-\mu)f\Big(a+\frac{\eta(b,a)}{2}\Big)+\mu f(a+\eta(b,a))-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|$$
$$\leq\frac{|\eta(b,a)|}{(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\Big(\frac{1}{2^{s+1}}\Big)^{\frac{1}{q}}\left\{\left[(1-\lambda)^{p+1}+\Big(\lambda-\frac{1}{2}\Big)^{p+1}\right]^{\frac{1}{p}}\left[(2^{s+1}-1)|f'(a)|^q+m\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\right]^{\frac{1}{q}}\right.$$
$$\left.+\left[\Big(\frac{1}{2}-\mu\Big)^{p+1}+\mu^{p+1}\right]^{\frac{1}{p}}\left[\Big|f'(a)\Big|^q+m\big(2^{s+1}-1\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\right]^{\frac{1}{q}}\right\}. \tag{3.11}$$

Proof. Lemma 2.1, the Hölder integral inequality, and the extended (s,m)-preinvexity of $|f'(x)|^q$ on $[a,\frac{b}{m}]$ can be used sequently to obtain that

$$\left|(1-\lambda)f(a)+(\lambda-\mu)f\Big(a+\frac{\eta(b,a)}{2}\Big)+\mu f\Big(a+\eta(b,a)\Big)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)}f(x)\mathrm{d}x\right|$$
$$\leq\Big|\eta(b,a)\Big|\left[\int_0^{\frac{1}{2}}|1-\lambda-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t+\int_{\frac{1}{2}}^1|1-\mu-t|\Big|f'\Big(a+t\eta(b,a)\Big)\Big|\mathrm{d}t\right]$$
$$\leq\Big|\eta(b,a)\Big|\left[\Big(\int_0^{\frac{1}{2}}|1-\lambda-t|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_0^{\frac{1}{2}}\Big|f'\Big(a+t\eta(b,a)\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\right.$$
$$\left.+\Big(\int_{\frac{1}{2}}^1|1-\mu-t|^p\mathrm{d}t\Big)^{\frac{1}{p}}\Big(\int_{\frac{1}{2}}^1\Big|f'\Big(a+t\eta(b,a)\Big)\Big|^q\mathrm{d}t\Big)^{\frac{1}{q}}\right]$$
$$\leq\Big|\eta(b,a)\Big|\left\{\Big(\int_0^{\frac{1}{2}}|1-\lambda-t|^p\mathrm{d}t\Big)^{\frac{1}{p}}\left[\int_0^{\frac{1}{2}}\Big((1-t)^s\Big|f'(a)\Big|^q+mt^s\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)\mathrm{d}t\right]^{\frac{1}{q}}\right.$$
$$\left.+\Big(\int_{\frac{1}{2}}^1|1-\mu-t|^p\mathrm{d}t\Big)^{\frac{1}{p}}\left[\int_{\frac{1}{2}}^1\Big((1-t)^s\Big|f'(a)\Big|^q+mt^s\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Big)\mathrm{d}t\right]^{\frac{1}{q}}\right\}.$$

Using the fact that

$$\int_0^{\frac{1}{2}} |1-\lambda-t|^p \mathrm{d}t = \frac{(1-\lambda)^{p+1} + \left(\lambda - \frac{1}{2}\right)^{p+1}}{p+1}$$

and

$$\int_{\frac{1}{2}}^{1} |1-\mu-t|^p \mathrm{d}t = \frac{\left(\frac{1}{2}-\mu\right)^{p+1} + \mu^{p+1}}{p+1},$$

we have

$$\begin{aligned}
&\left|(1-\lambda)f(a)+(\lambda-\mu)f\left(a+\frac{\eta(b,a)}{2}\right)+\mu f(a+\eta(b,a)) - \frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\right| \\
&\le \frac{\left|\eta(b,a)\right|}{(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left\{\left[(1-\lambda)^{p+1}+\left(\lambda-\frac{1}{2}\right)^{p+1}\right]^{\frac{1}{p}}\right. \\
&\qquad \times\left[\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'(a)\right|^q+\frac{m}{2^{s+1}}\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} \\
&\quad \left.+\left[\left(\frac{1}{2}-\mu\right)^{p+1}+\mu^{p+1}\right]^{\frac{1}{p}}\left[\frac{\left|f'(a)\right|^q}{2^{s+1}}+m\left(1-\left(\frac{1}{2}\right)^{s+1}\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\} \\
&= \frac{\left|\eta(b,a)\right|}{(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\left\{\left[(1-\lambda)^{p+1}+\left(\lambda-\frac{1}{2}\right)^{p+1}\right]^{\frac{1}{p}}\right. \\
&\qquad \times\left[\left(2^{s+1}-1\right)\left|f'(a)\right|^q+m\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}} \\
&\quad \left.+\left[\left(\frac{1}{2}-\mu\right)^{p+1}+\mu^{p+1}\right]^{\frac{1}{p}}\left[\left|f'(a)\right|^q+m\left(2^{s+1}-1\right)\left|f'\left(\frac{b}{m}\right)\right|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

which completes the proof. ∎

Elementary calculation provides the following corollary.

Corollary 3.3 *Under the conditions of Theorem* 3.3,

1. *when* $\eta(b,a) = b-a$, $s=1$ *and* $m=1$, *we have*

$$\begin{aligned}
&\left|(1-\lambda)f(b)+\mu f(a)+(\lambda-\mu)f\left(\frac{a+b}{2}\right)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{(b-a)}{(p+1)^{\frac{1}{p}}}\left\{\left[(1-\lambda)^{p+1}+\left(\lambda-\frac{1}{2}\right)^{p+1}\right]^{\frac{1}{p}}\left[\frac{3}{8}|f'(a)|^q+\frac{1}{8}|f'(b)|^q\right]^{\frac{1}{q}}\right. \\
&\quad \left.+\left[\left(\frac{1}{2}-\mu\right)^{p+1}+\mu^{p+1}\right]^{\frac{1}{p}}\left[\frac{1}{8}|f'(a)|^q+\frac{3}{8}|f'(b)|^q\right]^{\frac{1}{q}}\right\};
\end{aligned} \tag{3.12}$$

2. *if* $\eta(b,a)=b-a$, $\lambda=\mu, s=1$ *and* $m=1$, *we have*

$$\begin{aligned}
&\Big|(1-\lambda)f(b)+\lambda f(a)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big| \\
&\leq \frac{(b-a)}{(p+1)^{\frac{1}{p}}}\Bigg\{\Bigg[(1-\lambda)^{p+1}+\Big(\lambda-\frac{1}{2}\Big)^{p+1}\Bigg]^{\frac{1}{p}}\Bigg[\frac{3}{8}|f'(a)|^q+\frac{1}{8}|f'(b)|^q\Bigg]^{\frac{1}{q}} \\
&\quad+\Bigg[\Big(\frac{1}{2}-\lambda\Big)^{p+1}+\lambda^{p+1}\Bigg]^{\frac{1}{p}}\Bigg[\frac{1}{8}|f'(a)|^q+\frac{3}{8}|f'(b)|^q\Bigg]^{\frac{1}{q}}\Bigg\}; \qquad (3.13)
\end{aligned}$$

3. *if* $\eta(b,a)=b-a$, $\lambda=\mu$, *we have*

$$\begin{aligned}
&\Big|(1-\lambda)f(b)+\lambda f(a)-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big| \\
&\leq \frac{(b-a)}{(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\Big(\frac{1}{2^{s+1}}\Big)^{\frac{1}{q}}\Bigg\{\Bigg[(1-\lambda)^{p+1}+\Big(\lambda-\frac{1}{2}\Big)^{p+1}\Bigg]^{\frac{1}{p}} \\
&\quad\times\Bigg[(2^{s+1}-1)\Big|f'(a)\Big|^q+m\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Bigg]^{\frac{1}{q}} \\
&\quad+\Bigg[\Big(\frac{1}{2}-\lambda\Big)^{p+1}+\lambda^{p+1}\Bigg]^{\frac{1}{p}}\Bigg[\Big|f'(a)\Big|^q+m\big(2^{s+1}-1\big)\Big|f'\Big(\frac{b}{m}\Big)\Big|^q\Bigg]^{\frac{1}{q}}\Bigg\}; \qquad (3.14)
\end{aligned}$$

4. *if* $\eta(b,a)=b-a$, $\lambda=\mu=\frac{1}{2}, s=1$ *and* $m=1$, *we have*

$$\begin{aligned}
&\Big|\frac{f(a)+f(b)}{2}-\frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\Big| \\
&\leq \frac{(b-a)}{2(2p+2)^{\frac{1}{p}}}\Bigg[\Big(\frac{3}{8}|f'(a)|^q+\frac{1}{8}|f'(b)|^q\Big)^{\frac{1}{q}}+\Big(\frac{1}{8}|f'(a)|^q+\frac{3}{8}|f'(b)|^q\Big)^{\frac{1}{q}}\Bigg]; \qquad (3.15)
\end{aligned}$$

5. *if* $\lambda=1,\mu=0$ *and* $s=m=1$, *we have*

$$\begin{aligned}
&\Big|f\Big(\frac{2a+\eta(b,a)}{2}\Big)-\frac{1}{\eta(b,a)}\int_a^{a+\eta(b,a)} f(x)\mathrm{d}x\Big| \\
&\leq \frac{|\eta(b,a)|}{2(2p+2)^{\frac{1}{p}}}\Bigg[\Big(\frac{3}{8}|f'(a)|^q+\frac{1}{8}|f'(b)|^q\Big)^{\frac{1}{q}}+\Big(\frac{1}{8}|f'(a)|^q+\frac{3}{8}|f'(b)|^q\Big)^{\frac{1}{q}}\Bigg]; \qquad (3.16)
\end{aligned}$$

6. *if* $\eta(b,a) = b - a$, $\lambda = \frac{5}{6}$ *and* $\mu = \frac{1}{6}$, *we have*

$$\begin{aligned}
&\left|\frac{1}{6}\left[f(a) + f(b) + 4f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a}\int_a^b f(x)\mathrm{d}x\right| \\
&\le \frac{(b-a)}{(s+1)^{\frac{1}{q}}(p+1)^{\frac{1}{p}}}\left(\frac{1}{2^{s+1}}\right)^{\frac{1}{q}}\left[\left(\frac{1}{3}\right)^{p+1} + \left(\frac{1}{6}\right)^{p+1}\right]^{\frac{1}{p}} \\
&\times\left\{\left[\left(2^{s+1}-1\right)\left|f'(a)\right|^q + m\left|f'(\frac{b}{m})\right|^q\right]^{\frac{1}{q}}\right. \\
&\left.+\left[\left|f'(a)\right|^q + m\left(2^{s+1}-1\right)\left|f'(\frac{b}{m})\right|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned} \tag{3.17}$$

Remark 3.4. In inequality (3.16), if we let $q = \frac{p}{p-1}$ and exchange a and b, then we get inequality (1.9). Furthermore, when applying $s = m = 1$ to inequality (3.17) then we get inequality (1.3). Thus, Theorem 3.3 and its consequences generalize some main results in [31],[28].

Acknowledgments. This work was supported by the National Natural Science foundation of China under Grant 11301296, Hubei Province Key Laboratory of Systems Science in Metallurgical Process of China under Grant Z201402, and the Natural Science Foundation of Hubei Province, China under Grants 2013CFA131. Finally we thank the referees for their time and comments.

References

[1] Antczak, T., *Mean value in invexity analysis*, Nonlinear Anal., 60 (8) (2005), 1471-1484.

[2] Barani, A., Ghazanfari, A.G., Dragomir, S.S., *Hermite-Hadamard inequality for functions whose derivatives absolute values are preinvex*, J. Inequal. Appl., 247 (2012), 9 pp., http://dx.doi.org/10.1186/1029-242X-2012-247.

[3] Chen, F.X., Feng, Y.M., *New inequalities of Hermite-Hadamard type for functions whose first derivatives absolute values are s-convex*, Italian J. Pure Appl. Math., 32 (2014), 213-222.

[4] Dragomir, S.S., Agarwal, R.P., Cerone, P., *On Simpson's inequality and applications*, J. Inequal. Appl., 5 (6) (2000), 533-579.

[5] Eftekhari, N., *Some remarks on (s, m)-convexity in the second sense*, J. Math. Inequal., 8 (3) (2014), 489-495.

[6] Godunova, E.K., Levin, V.I., *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions*, Numer. Math. Math. Phys., 166 (1985), 138-142.

[7] Hua, J., Xi, B.Y., Qi, F., *Hermite-Hadamard type inequalities for geometric-arithmetically s-convex functions*, Commun. Korean Math. Soc., 29 (1) (2014), 51-63.

[8] Iqbal, M., Bhatti, M.I., Nazeer, K., *Generalization of inequalities analogous to Hermite-Hadamard inequality via fractional integrals*, Bull. Korean Math. Soc., 52 (3) (2015), 707-716.

[9] İşcan, İ., *Hermite-Hadamard's inequalities for preinvex function via fractional integrals and related fractional inequalities*, American J. Math. Anal., 1 (3) (2013), 33-38.

[10] Latif, M.A., *On Hermite-Hadamard type integral inequalities for n-times differentiable preinvex functions with applications*, Stud. Univ. Babeş-Bolyai Math., 58 (3) (2013), 325-343.

[11] Latif, M.A., Dragomir, S.S., *Some Hermite-Hadamard type inequalities for functions whose partial derivatives in absloute value are preinvex on the co-oordinates*, Facta Universitatis (NIŠ) Ser. Math. Inform., 28 (3) (2013), 257-270.

[12] Latif, M.A., Shoaib, M., *Hermite-Hadamard type integral inequalities for differentiable m-preinvex and (α, m)-preinvex functions*, J. Egypt. Math. Soc., 23 (2) (2015), 236-241.

[13] Latif, M.A., Dragomir, S.S., *Some weighted integral inequalities for differentiable preinvex and prequasiinvex functions with applications*, J. Inequal. Appl., 575 (2013), 19 pp, http://dx.doi.org/10.1186/1029-242X-2013-575.

[14] Latif, M.A., Dragomir, S.S., Momoniat, E., *Some weighted Hermite-Hadamard-Noor type inequalities for differentiable preinvex and quasi preinvex functions*, Punjab Univ. J. Math., 47 (1) (2015), 57-72.

[15] Li, J.Y., *On Hadamard-type inequalities for s-preinvex functions*, J. Chongqing Norm. Univ. (Natural Science) China, 27 (4) (2010), 5-8.

[16] Li, Y.J., Du, T.S., *On Simpson type inequalities for functions whose derivatives are extended (s, m)-GA-convex functions*, Pure Appl. Math. China, 31 (5) (2015), 487-497.

[17] Li, Y.J., Du, T.S., *Some Simpson type integral inequalities for functions whose third derivatives are (α, m)-GA-convex functions*, J. Egypt. Math. Soc., 24 (2) (2016), 175-180.

[18] Li, T.Y., Hu, G.H., *On the strictly G-preinvex function*, J. Inequal. Appl., 210 (2014), 9 pp., http://dx.doi.org/10.1186/1029-242X-2014-210.

[19] Matłoka, M., *On some Hadamard-type inequalities for* (h_1, h_2)*-preinvex functions on the co-ordinates*, J. Inequal. Appl., 227 (2013), 12 pp., http://dx.doi.org/10.1186/1029-242X-2013-227.

[20] Matłoka, M., *Inequalities for h-preinvex functions*, Appl. Math. Comput., 234 (2014), 52-57.

[21] Noor, M.A., *Hadamard integral inequalities for product of two preinvex functions*, Nonlinear Anal. Forum, 14 (2009), 167-173.

[22] Park, J., *Simpson-like and Hermite-Hadamard-like type integral inequalities for twice differentiable preinvex functions*, Int. J. Pure Appl. Math., 79 (4) (2012), 623-640.

[23] Park, J., *Hermite-Hadamard-like inequalities for functions whose derivatives of n-th order are preinvex and logarithmic preinvex*, Appl. Math. Sci., 7 (122) (2013), 6053-6068.

[24] Park, J., *Hermite-Hadamard-like type integral inequalities for functions whose derivatives of n-th order are preinvex*, Appl. Math. Sci., 7 (133) (2013), 6637-6650.

[25] Pini, R., *Invexity and generalized convexity*, Optimization, 22 (4) (1991), 513-525.

[26] Sarikaya, M.Z., Alp, N., Bozkurt, H., *On Hermite-Hadamard type integral inequalities for preinvex and log-preinvex functions*, Contemp. Anal. Appl. Math., 1 (2) (2013), 237-252.

[27] Sarikaya, M.Z., Set, E., Özdemir, M.E., *On new inequalities of Simpson's type for s-convex functions*, Comput. Math. Appl., 60 (8) (2010), 2191-2199.

[28] Sarikaya, M.Z., Set, E., Özdemir, M.E., *On new inequalities of Simpson's type for convex functions*, RGMIA Res. Rep. Coll., 13 (2) (2010), Article 2, http://ajmaa.org/RGMIA/papers/v13n1/sso3.pdf.

[29] Wang, S.H., Qi, F., *Hermite-Hadamard type inequalities for n-times differentiable and preinvex functions*, J. Inequal. Appl., 49 (2014), 9 pp., http://dx.doi.org/10.1186/1029-242X-2014-49.

[30] Wang, Y., Wang, S.H., Qi, F., *Simpson type integral inequalities in which the power of the absolute value of the first derivative of the integrand is s-preinvex*, Facta Univ., Ser. Math. Inform., 28 (2) (2013), 151-159.

[31] Wang, Y., Xi, B.Y., Qi, F., *Hermite-Hadamard type integral inequalities when the power of the absolute value of the first derivative of the integrand is preinvex*, Le Mat., 69 (1) (2014), 89-96.

[32] Wang, Y., Zheng, M.M., Qi, F., *Integral inequalities of Hermite-Hadamard type for functions whose derivatives are α-preinvex*, J. Inequal. Appl., 97 (2014), 10 pp., http://dx.doi.org/10.1186/1029-242X-2014-97.

[33] Xi, B.Y., Qi, F., *Inequalities of Hermite-Hadamard type for extended s-convex functions and applications to means*, J. Nonlinear Convex Anal., 16 (5) (2015), 873-890.

[34] Xi, B.Y., Wang, Y., Qi, F., *Some integral inequalities of Hermite-Hadamard type for extended (s, m)-convex functions*, Transylv. J. Math. Mech., 5 (1) (2013), 69-84.

[35] Yang, X.M., Yang, X.Q., Teo, K.L., *Generalized invexity and generalized invariant monotonicity*, J. Optim. Theory Appl., 117 (3) (2003), 607-625.

[36] Yang, Z.Q., Li, Y.J., Du, T.S., *A generalization of Simpson type inequality via differentiable functions using (s, m)-convex functions*, Italian J. Pure Appl. Math., 35 (2015), 327-338.

Accepted: 15.02.2016