DIFFERENTIAL SANDWICH THEOREMS FOR $p$-VALENT FUNCTIONS ASSOCIATED WITH A CERTAIN GENERALIZED DIFFERENTIAL OPERATOR AND INTEGRAL OPERATOR

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Abstract. The purpose of this paper is to derive some subordination and superordination results for functions of the form $f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k$ which are $p$-valent in the open unit disk $\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\}$ by using certain differential operator $A_{\alpha, \beta, \mu}\lambda^p f(z)$ and integral operator $F_{\mu}(\rho, \vartheta) f(z)$. Some special cases are also considered.

Keywords: analytic functions, differential subordination, differential superordination, sandwich theorems.

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1. Introduction

Let $\mathcal{H}$ denote the class of functions which are analytic in the open unit disk

$$\mathbb{U} = \{z \in \mathbb{C}: |z| < 1\},$$

and let $\mathcal{H}[a, p]$ denote the subclass of the functions $f \in \mathcal{H}$ of the form:

$$(1.1) \quad f(z) = a + a_p z^p + a_{p+1} z^{p+1} + \cdots \quad (a \in \mathbb{C}; \ p \in \mathbb{N} := \{1, 2, \ldots\}).$$
For simplicity, \( \mathcal{H}[a] = \mathcal{H}[a, 1] \). Also, let \( \mathcal{A}(p) \) denote the subclass of \( \mathcal{H} \) consisting of functions of the form:

\[
(1.2) \quad f(z) = z^p + \sum_{k=p+1}^{\infty} a_k z^k, \quad (a_k \geq 0; p \in \mathbb{N} := \{1, 2, \ldots\}),
\]

which are \( p \)-valent in \( U \).

Given two functions \( f, g \in \mathcal{H} \). The function \( f(z) \) is said to be subordination to \( g(z) \) in \( U \), written \( f(z) \prec g(z) \), if there exists a function \( h(z) \), analytic in \( U \), with \( h(0) = 0 \) and \( |h(z)| < 1 \) for all \( z \in U \), such that \( f(z) = g(h(z)) \) for all \( z \in U \). Furthermore, if the function \( g \) is univalent in \( U \), then we have the following equivalence (see [7] and [9]):

\[
f(z) \prec g(z) \iff f(0) = g(0) \text{ and } f(U) \subset g(U).
\]

Let \( \phi : \mathbb{C}^3 \times U \to \mathbb{C} \) and \( \psi(z) \) be univalent in \( U \). If \( p(z) \) is analytic in \( U \) and satisfies the second-order differential subordination:

\[
(1.3) \quad \phi \left( p(z), zp'(z), z^2 p''(z); z \right) \prec \psi(z),
\]

then \( p(z) \) is a solution of the differential subordination \( (1.3) \). The univalent function \( q(z) \) is called a dominant of the solutions of the differential subordination \( (1.3) \) if \( p(z) \prec q(z) \) for all \( p(z) \) satisfying \( (1.3) \). A univalent dominant \( \tilde{q} \) that satisfies \( \tilde{q} \prec q \) for all dominants of \( (1.3) \) is called the best dominant. If \( p(z) \) is univalent in \( U \) and satisfies the second-order differential superordination:

\[
(1.4) \quad \psi(z) \prec \phi \left( p(z), zp'(z), z^2 p''(z); z \right),
\]

then \( p(z) \) is a solution of the differential superordination \( (1.4) \). A univalent function \( q(z) \) is called a subordinant of the solutions of the differential superordination \( (1.4) \) if \( q(z) \prec p(z) \) for all \( p(z) \) satisfying \( (1.4) \). A univalent subordinant \( \tilde{q} \) that satisfies \( q \prec \tilde{q} \) for all subordinants of \( (1.4) \) is called the best subordinant. Using the results of Miller and Mocanu [10], Al-Hawary, Amourah and Darus [1], Aljarah and Darus [3], Bulboaca [6] considered certain classes of first-order differential superordinations as well as superordination-preserving integral operators [7]. Ali et al. [2] have used the results of Bulboaca [6] to obtain sufficient conditions for normalized analytic functions \( f \in \mathcal{A}(1) \) to satisfy:

\[
q_1(z) \prec \frac{zf'(z)}{f(z)} \prec q_2(z),
\]

where \( q_1 \) and \( q_2 \) are given univalent normalized functions in \( U \) with \( q_1(0) = q_2(0) = 1 \).

Also, Tuneski [14] obtained a sufficient condition for starlikeness of \( f \in \mathcal{A}(1) \) in terms of the quantity \( \frac{f'(z)f(z)}{(f'(z))^2} \).

Recently, Shanmugam et al. [12, 11] and [13] obtained sufficient conditions for the normalized analytic function \( f \in \mathcal{A}(1) \) to satisfy:

\[
q_1(z) \prec \frac{f(z)}{zf'(z)} \prec q_2(z),
\]
and

\[ q_1(z) < \frac{z^2 f'(z)}{f^2(z)} < q_2(z). \]

Shanmugam et al. [11] also obtained the so-called sandwich results for certain classes of analytic functions.

For the function \( f \in \mathcal{A}(p) \), Faisal and Darus ([8]) defined the following differential operator:

\[
A_{\lambda,p}^0(\alpha, \beta, \mu) f(z) = f(z),
\]

\[
A_{\lambda,p}^1(\alpha, \beta, \mu) f(z) = \left(\frac{\alpha + \beta - p(\mu + \lambda)}{\alpha + \beta}\right) f(z) + \left(\frac{\mu + \lambda}{\alpha + \beta}\right) z f'(z),
\]

and for \( n = 2, 3, \ldots \),

\[
A_{\lambda,p}^n(\alpha, \beta, \mu) f(z) = A(A_{\lambda,p}^{n-1}(\alpha, \beta, \mu) f(z)),
\]

\[
(1.5) = z^p + \sum_{k=p+1}^{\infty} \left[ \frac{\alpha + (\mu + \lambda)(k - p) + \beta}{\alpha + \beta} \right]^n a_k z^k,
\]

for \( f \in \mathcal{A}(p) \), \( \alpha, \beta, \mu, \lambda \geq 0, \alpha + \beta \neq 0 \) and \( \mu + \lambda \neq 0, p \in \mathbb{N} \) and \( n \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \).

It is easily verified from (1.5) that

\[
\frac{\mu + \lambda}{\alpha + \beta} z(A_{\lambda,p}^n(\alpha, \beta, \mu) f(z))' = A_{\lambda,p}^{n+1}(\alpha, \beta, \mu) f(z)
\]

\[
(1.6) = A_{\lambda,p}(\alpha, \beta, \mu) f(z).
\]

Also, for the function \( f \in \mathcal{A}(p) \), Aouf et al. ([5]) defined the integral operator:

\[
F_p^m(\rho, \vartheta) f(z) = z^p + \sum_{k=p+1}^{\infty} \left[ \frac{p + \vartheta}{p + \vartheta + \rho(k - p)} \right]^m a_k z^k,
\]

for \( \rho > 0, \vartheta \geq 0, p \in \mathbb{N} \) and \( m \in \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). It is easily verified from (1.7) that

\[
\rho z(F_p^{m+1}(\rho, \vartheta) f(z))' = (\vartheta + p)(F_p^m(\rho, \vartheta) f(z))
\]

\[
(1.8) = [\vartheta + p(1 - \rho)](F_p^{m+1}(\rho, \vartheta) f(z)).
\]

In this paper, we derive several subordination results, superordination results and sandwich results involving the differential operator \( A_{\lambda,p}^n(\alpha, \beta, \mu) f(z) \) given by (1.5) and the integral operator \( F_p^m(\rho, \vartheta) f(z) \) given by (1.7).

2. Definitions and preliminaries

In order to prove our subordinations and superordinations results, we need the following definition and lemmas.
Definition 2.1 [10] Denote by $Q$, the set of all functions $f$ that are analytic and injective on $\mathbb{U}\setminus E(f)$, where

$$E(f) = \left\{ \zeta \in \partial \mathbb{U} \lim_{z \to \zeta} f(z) = \infty \right\},$$

and are such that $f'(\zeta) \neq 0$ for $\zeta \in \partial \mathbb{U}\setminus E(f)$.

Lemma 2.2 [10] Let $q(z)$ be univalent in $\mathbb{U}$ and let $\theta$ and $\varphi$ be analytic in a domain $D$ containing $q(\mathbb{U})$ with $\varphi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set

$$\psi(z) = zq'(z)\varphi(q(z)) \quad \text{and} \quad h(z) = \theta(q(z)) + \psi(z).$$

Suppose that

(i) $\psi$ is starlike univalent in $\mathbb{U}$,

(ii) $\text{Re} \left\{ \frac{zh'(z)}{\psi(z)} \right\} > 0$ for $z \in \mathbb{U}$.

If $p(z)$ is a analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset D$ and

$$\theta(p(z)) + zq'(z)\varphi(p(z)) \prec \theta(q(z)) + zq'(z)\varphi(q(z)),$$

then $p(z) \prec q(z)$ and $q(z)$ is the best dominant of (2.1).

Lemma 2.3 [6] Let $q(z)$ be convex univalent in $\mathbb{U}$ and let $\vartheta$ and $\phi$ be analytic in a domain $D$ containing $q(\mathbb{U})$. Suppose that

(i) $\text{Re} \left\{ \frac{\vartheta'(q(z))}{\phi(q(z))} \right\} > 0$ for $z \in \mathbb{U}$,

(ii) $\Psi(z) = zq'(z)\phi(q(z))$ is starlike univalent in $\mathbb{U}$.

If $p(z) \in \mathcal{H}[q(0), 1] \cap Q$, with $p(\mathbb{U}) \subset D$, and $\vartheta(p(z)) + zp'(z)\phi(p(z))$ is univalent in $\mathbb{U}$ and

$$\vartheta(q(z)) + zq'(z)\phi(q(z)) \prec \vartheta(p(z)) + zp'(z)\phi(p(z)),$$

then $q(z) \prec p(z)$ and $q(z)$ is the best subordinant of (2.2).

3. Subordinations and superordinations results for $A_{\lambda, p}^\alpha(\alpha, \beta; \mu)f(z)$

Unless otherwise mentioned, we assume throughout this paper that $\epsilon, \eta \in \mathbb{C}$. We begin with the following result involving differential subordination between analytic functions.
Theorem 3.1 Let $q(z)$ be univalent in $\mathbb{U}$ with $q(0) = 1$. Further, assume that
\begin{equation}
\text{Re}\left\{2(\delta + \kappa)q(z) + 1 + \frac{zq''(z)}{q'(z)}\right\} > 0, \quad (\delta \in \mathbb{C}^* = \mathbb{C} - \{0\}, \kappa \geq 0)
\end{equation}

If $f \in \mathcal{A}(p)$ satisfies the following subordination condition:
\begin{equation}
\zeta(n, \lambda, \mu, \beta, \alpha; z) < \delta zq'(z) + (\delta + \kappa) (q(z))^2,
\end{equation}
where
\begin{align*}
\zeta(n, \lambda, \mu, \beta, \alpha; z) &= \frac{\delta(\alpha + \beta)}{(\mu + \lambda)} \left[ (A^n_{\lambda,p}(\alpha, \beta, \mu)f(z))^{1-\epsilon} - \epsilon \frac{(A^{n+2}_{\lambda,p}(\alpha, \beta, \mu)f(z))(A^n_{\lambda,p}(\alpha, \beta, \mu)f(z))}{(A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z))^{1+\epsilon}} \right] \\
&+ \delta(\epsilon - 1) \left( \frac{\alpha + \beta}{\mu + \lambda} - p \right) \frac{A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)}{(A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z))^{1+\epsilon}} + \frac{(A^n_{\lambda,p}(\alpha, \beta, \mu)f(z))^{2\epsilon}}{(A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z))^{2\epsilon}},
\end{align*}
then
\begin{equation}
A^n_{\lambda,p}(\alpha, \beta, \mu)f(z) < q(z)
\end{equation}
and $q(z)$ is the best dominant.

Proof. Define a function $p(z)$ by
\begin{equation}
p(z) = \frac{A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)}{(A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z))}, \quad (z \in \mathbb{U}).
\end{equation}
Then the function $p(z)$ is analytic in $\mathbb{U}$ and $p(0) = 1$. Differentiating (3.3) logarithmically with respect to $z$ and using the identity (1.6) in the resulting equation, we get
\begin{align*}
p'(z) &= \frac{p'(z)}{p(z)} = \frac{A^n_{\lambda,p}(\alpha, \beta, \mu)f(z) - (1 - \frac{p}{\alpha + \beta}(\mu + \lambda))A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)}{\frac{\mu + \lambda}{\alpha + \beta}zA^n_{\lambda,p}(\alpha, \beta, \mu)f(z)} \\
&- \frac{\frac{\mu + \lambda}{\alpha + \beta}zA^{n+2}_{\lambda,p}(\alpha, \beta, \mu)f(z) - (1 - \frac{p}{\alpha + \beta}(\mu + \lambda))A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z)}{zA^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z)},
\end{align*}
\begin{align*}
p'(z) &= \left. \frac{(\alpha + \beta)}{z(\mu + \lambda)} \left[ (A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z))^{1-\epsilon} - \epsilon \frac{(A^{n+2}_{\lambda,p}(\alpha, \beta, \mu)f(z))(A^n_{\lambda,p}(\alpha, \beta, \mu)f(z))}{(A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z))^{1+\epsilon}} \right] \right) \\
&+ \left. \frac{(\epsilon - 1) \left( \frac{\alpha + \beta}{\mu + \lambda} - p \right)}{z} \frac{A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)}{(A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z))^{1+\epsilon}}. \right.
\end{align*}
\( \Rightarrow \delta z^p' \left( z \right) + (\delta + \kappa) \left( p(z) \right)^2 \)

\[
\frac{\delta(a + \beta)}{(\mu + \lambda)} \left[ \left( A_{\lambda,p}^n(a,\beta,\mu)f(z) \right)^{1-\epsilon} - \epsilon \left( A_{\lambda,p}^{n+2}(a,\beta,\mu)f(z) \right) \frac{A_{\lambda,p}^n(a,\beta,\mu)f(z)}{A_{\lambda,p}^{n+1}(a,\beta,\mu)f(z)} \right] \\
+ \delta(\epsilon - 1) \left( \frac{\alpha + \beta}{\mu + \lambda} - p \right) \frac{A_{\lambda,p}^n(a,\beta,\mu)f(z)}{A_{\lambda,p}^{n+1}(a,\beta,\mu)f(z)} + (\delta + \kappa) \frac{A_{\lambda,p}^n(a,\beta,\mu)f(z)^2}{A_{\lambda,p}^{n+1}(a,\beta,\mu)f(z)^2}.
\]

Using (3.2), we obtain

\[
\delta z^p'(z) + (\delta + \kappa) \left( p(z) \right)^2 < \delta z^q'(z) + (\delta + \kappa) \left( q(z) \right)^2.
\]

Therefore, now Theorem 3.1 follows by applying Lemma 2.2 by setting

\[
\theta(w) = (\delta + \kappa)w^2 \text{ and } \varphi(w) = \delta.
\]

Corollary 3.2

Let \( q(z) = \frac{1 + A z}{1 + B z} \) \((-1 \leq B < A \leq 1) \) in Theorem 3.1. Further, assume that (3.1) holds.

If \( f \in \mathcal{A}(p) \) satisfies the following subordination condition:

\[
\zeta(n, \lambda, p, \beta, \delta, \alpha; z) < \frac{\delta(A - B)z}{(1 + Bz)^2} + (\delta + \kappa) \left( \frac{1 + A z}{1 + B z} \right)^2,
\]

then

\[
\frac{A_{\lambda,p}^n(a,\beta,\mu)f(z)}{A_{\lambda,p}^{n+1}(a,\beta,\mu)f(z)} < \frac{1 + A z}{1 + B z},
\]

and the function \( \frac{1 + A z}{1 + B z} \) is the best dominant.

Also, let \( q(z) = \frac{1 + z}{1 - z} \). Then for \( f \in \mathcal{A}(p) \) we have

\[
\zeta(n, \lambda, p, \beta, \delta, \alpha; z) < \frac{2\delta z}{(1 - z)^2} + (\delta + \kappa) \left( \frac{1 + z}{1 - z} \right)^2,
\]

then

\[
\frac{A_{\lambda,p}^n(a,\beta,\mu)f(z)}{A_{\lambda,p}^{n+1}(a,\beta,\mu)f(z)} < \frac{1 + z}{1 - z},
\]

and the function \( \frac{1 + z}{1 - z} \) is the best dominant.

Finally, by taking \( q(z) = \left( \frac{1 + z}{1 - z} \right)^\mu \), \((0 < \mu \leq 1)\), then for \( f \in \mathcal{A}(p) \) we have,

\[
\zeta(n, \lambda, p, \beta, \delta, \alpha; z) < \frac{2\delta \mu z}{(1 - z)^2} \left( \frac{1 + z}{1 - z} \right)^{\mu - 1} + (\delta + \kappa) \left( \frac{1 + z}{1 - z} \right)^{2\mu}.
\]
then
\[ \frac{A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)}{[A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z)]^\tau} \prec \left( \frac{1 + z}{1 - z} \right)^\mu, \]
and the function \( \left( \frac{1 + z}{1 - z} \right)^\mu \) is the best dominant.

Next, by appealing to Lemma 2.3, we prove the following.

**Theorem 3.3** Let \( q(z) \) be convex univalent in \( U \) with \( q(0) = 1 \). Assume that
\[
\text{Re}\left\{ \frac{2(\delta + \kappa)q(z)q'(z)}{\delta} \right\} > 0. \tag{3.4}
\]
Let \( f \in A(p) \) such that
\[ \frac{A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)}{[A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z)]^\tau} \in H[q(0), 1] \cap Q, \]
is univalent in \( U \) and the following superordination condition
\[
(\delta + \kappa)(q(z))^2 + \delta zq'(z) \prec (\delta + \kappa)(p(z))^2 + \delta zp'(z), \tag{3.5}
\]
holds, then
\[
q(z) \prec \frac{A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)}{[A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z)]^\tau}, \tag{3.6}
\]
and \( q(z) \) is the best subordinant.

**Proof.** Let the function \( p(z) \) be defined by
\[
p(z) = \frac{A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)}{[A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z)]^\tau}.
\]
Then, from the assumption of Theorem 3.3, the function \( p(z) \) is analytic in \( U \) and (3.4) holds. Hence, the subordination (3.5) is equivalent to
\[
(\delta + \kappa)(q(z))^2 + \delta zq'(z) \prec (\delta + \kappa)(p(z))^2 + \delta zp'(z).
\]
The assertion (3.6) of Theorem 3.3 now follows by an application of Lemma 2.3. \( \blacksquare \)

**Corollary 3.4** Let \( q(z) = \frac{1 + Az}{1 + Bz} \) \((-1 \leq B < A \leq 1) \) in Theorem 3.3. Further, assume that (3.4) holds.
If \( f \in A(p) \) such that
\[ \frac{A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)}{[A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z)]^\tau} \in H[q(0), 1] \cap Q, \]
is univalent in \( U \) and the following superordination condition
\[
\frac{\delta(A - B)z}{(1 + Bz)^2} + (\delta + \kappa)\left( \frac{1 + Az}{1 + Bz} \right)^2 \prec (\delta + \kappa)\frac{1 + Az}{1 + Bz},
\]
holds, then
\[
q(z) \prec \frac{A^n_{\lambda,p}(\alpha, \beta, \mu)f(z)}{[A^{n+1}_{\lambda,p}(\alpha, \beta, \mu)f(z)]^\tau}, \tag{3.6}
\]
and \( q(z) \) is the best subordinant.
holds, then
\[ \frac{1 + A z}{1 + B z} < \frac{A^n_{\lambda, p}(\alpha, \beta, \mu) f(z)}{[A^{n+1}_{\lambda, p}(\alpha, \beta, \mu) f(z)]^\tau}, \]
and \( \frac{1 + A z}{1 + B z} \) is the best subordinant.

Also, let \( q(z) = \frac{1 + z}{1 - z} \), then for \( f \in A(p) \) we have
\[ \frac{2 \delta z}{(1 - z)^2} + (\delta + \kappa) \left( \frac{1 + z}{1 - z} \right)^2 < \zeta(n, \lambda, p, \beta, \delta, \alpha; z), \]
then
\[ \frac{1 + z}{1 - z} < \frac{A^n_{\lambda, p}(\alpha, \beta, \mu) f(z)}{[A^{n+1}_{\lambda, p}(\alpha, \beta, \mu) f(z)]^\tau}, \]
and the function \( \frac{1 + z}{1 - z} \) is the best subordinant.

Finally, by taking \( q(z) = \left( \frac{1 + z}{1 - z} \right)^\mu \), \( 0 < \mu \leq 1 \), then for \( f \in A(p) \) we have
\[ \frac{2 \delta \mu z}{(1 - z)^2} \left( \frac{1 + z}{1 - z} \right)^{\mu - 1} + (\delta + \kappa) \left( \frac{1 + z}{1 - z} \right)^{2\mu} < \zeta(n, \lambda, p, \beta, \delta, \alpha; z), \]
then
\[ \left( \frac{1 + z}{1 - z} \right)^\mu < \frac{A^n_{\lambda, p}(\alpha, \beta, \mu) f(z)}{[A^{n+1}_{\lambda, p}(\alpha, \beta, \mu) f(z)]^\tau}, \]
and the function \( \left( \frac{1 + z}{1 - z} \right)^\mu \) is the best subordinant.

Combining Theorem 3.1 and Theorem 3.3, we get the following sandwich theorem.

**Theorem 3.5** Let \( q_1 \) and \( q_2 \) be convex univalent in \( \mathbb{U} \) with \( q_1(0) = q_2(0) = 1 \) and satisfy (3.1) and (3.4) respectively.

If \( f \in A(p) \) such that \( \frac{A^n_{\lambda, p}(\alpha, \beta, \mu) f(z)}{[A^{n+1}_{\lambda, p}(\alpha, \beta, \mu) f(z)]^\tau} \in \mathcal{H}[q(0), 1] \cap Q, \zeta(n, \lambda, p, \beta, \delta, \alpha; z) \) is univalent in \( \mathbb{U} \) and
\[ (\delta + \kappa) (q_1(z))^2 + \delta z q'_1(z) < \zeta(n, \lambda, p, \beta, \delta, \alpha; z) < (\delta + \kappa) (q_2(z))^2 + \delta z q'_2(z), \]
holds, then
\[ q_1(z) < \frac{A^n_{\lambda, p}(\alpha, \beta, \mu) f(z)}{[A^{n+1}_{\lambda, p}(\alpha, \beta, \mu) f(z)]^\tau} < q_2(z), \]
and \( q_1(z) \) and \( q_2(z) \) are, respectively, the best subordinant and the best dominant.
Corollary 3.6 Let \( q_i(z) = \frac{1 + A_i z}{1 + B_i z} \) \((i = 1, 2; -1 \leq B_2 < B_1 < A_1 \leq A_2 \leq 1)\) in Theorem 3.5.

If \( f \in \mathcal{A}(p) \) such that
\[
\frac{A_p \alpha, \beta, \mu}{A_{p+1} \alpha, \beta, \mu} f(z) \in \mathcal{H}[q(0), 1] \cap Q, \zeta(n, \lambda, p, \beta, \delta, \alpha; z)
\]
is univalent in \( U \) and
\[
\frac{\delta (A_1 - B_1) z}{(1 + B_1 z)^2} + (\delta + \kappa) \left( \frac{1 + A_1 z}{1 + B_1 z} \right)^2 < \zeta(n, \lambda, p, \beta, \delta, \alpha; z)
\]
holds, then
\[
\frac{1 + A_1 z}{1 + B_1 z} < \frac{A_p \alpha, \beta, \mu}{A_{p+1} \alpha, \beta, \mu} f(z) < \frac{1 + A_2 z}{1 + B_2 z},
\]
and \( \frac{1 + A_1 z}{1 + B_1 z} \) and \( \frac{1 + A_2 z}{1 + B_2 z} \) are, respectively, the best subordinant and the best dominant.

4. Subordinations and superordinations results for \( F_p^m(\rho, \vartheta)f(z) \)

In this section, we derive subordination and superordination results for \( p \)-valent functions involving the integral operator \( F_p^m(\rho, \vartheta)f(z) \) given by (1.7).

\[
\text{Theorem 4.1} \quad \text{Let } q(z) \text{ be univalent in } U \text{ with } q(0) = 1. \text{ Further, assume that}
\]
\[
\text{Re} \left\{ \frac{2(\delta + \kappa)q(z)}{\delta} + 1 + \frac{zq'(z)}{q(z)} \right\} > 0. \quad (\delta \in \mathbb{C}^* = \mathbb{C} - \{0\}, \kappa \geq 0),
\]
\[
\text{If } f \in \mathcal{A}(p) \text{ satisfies the following subordination condition:}
\]
\[
\Lambda(p, \vartheta, \rho; z) < \delta zq'(z) + (\delta + \kappa) (q(z))^2,
\]
where
\[
\Lambda(p, \vartheta, \rho; z) = \frac{\delta(\vartheta + p)}{\rho} \left[ \frac{F_{p-1}(\rho, \vartheta)f(z)}{(F_{p+1}(\rho, \vartheta)f(z))^{\eta+1}} \left( F_{p+1}(\rho, \vartheta)f(z) \right) - \eta \left( F_{p}(\rho, \vartheta)f(z) \right)^2 \right]
\]
\[
+ \frac{\delta(\eta-1)[\vartheta+p(1-\rho)]}{\rho} \left( F_{p}(\rho, \vartheta)f(z) \right)^{\eta} + (\delta+\kappa) \left( F_{p}(\rho, \vartheta)f(z) \right)^{2\eta},
\]
then
\[
\frac{F_{p}(\rho, \vartheta)f(z)}{(F_{p+1}(\rho, \vartheta)f(z))^{\eta}} < q(z),
\]
and \( q(z) \) is the best dominant.
Proof. Define a function \( p(z) \) by
\[
(4.3) \quad p(z) = \frac{F_m^p(\rho, \vartheta) f(z)}{[F_{p+1}^m(\rho, \vartheta) f(z)]^\eta}, \quad (z \in U).
\]
Then the function \( p(z) \) is analytic in \( U \) and \( p(0) = 1 \). Differentiating (4.3) logarithmically with respect to \( z \) and using the identity (1.8) in the resulting equation, we have
\[
\frac{\delta (\vartheta + p)}{\rho} \left[ \frac{(F_m^{p-1}(\rho, \vartheta) f(z)) (F_m^{p+1}(\rho, \vartheta) f(z)) - \eta (F_m^m(\rho, \vartheta) f(z))^2}{(F_m^{p+1}(\rho, \vartheta) f(z))^{\eta+1}} \right] + \frac{\delta (\eta - 1)}{\rho} \frac{[\vartheta + p(1 - \rho)]}{(F_m^{p+1}(\rho, \vartheta) f(z))^\eta} + \delta + \kappa \frac{(F_m^m(\rho, \vartheta) f(z))^2}{(F_m^{p+1}(\rho, \vartheta) f(z))^{2\eta}}
\]

\[
= \delta z p'(z) + (\delta + \kappa) (p(z))^2,
\]
that is,
\[
\delta z p'(z) + (\delta + \kappa) (p(z))^2 < \delta z q'(z) + (\delta + \kappa) (q(z))^2.
\]
Therefore, Theorem 4.1 now follows by applying Lemma 2.2 by setting
\[
\theta(w) = (\delta + \kappa) w^2 \text{ and } \varphi(w) = \delta.
\]

Corollary 4.2 Let \( q(z) = \frac{1 + A z}{1 + B z} \) \((-1 \leq B < A \leq 1\) in Theorem 4.1. Further, assume that (4.1) holds.

If \( f \in A(p) \) satisfies the following subordination condition:
\[
\Lambda(p, \vartheta, \rho; z) \prec \frac{\delta(A - B) z}{(1 + B z)^2} + (\delta + \kappa) \left( \frac{1 + A z}{1 + B z} \right)^2,
\]
then
\[
\frac{F_m^p(\rho, \vartheta) f(z)}{[F_{p+1}^m(\rho, \vartheta) f(z)]^\eta} \prec \frac{1 + A z}{1 + B z},
\]
and the function \( \frac{1 + A z}{1 + B z} \) is the best dominant.

Also, let \( q(z) = \frac{1 + z}{1 - z} \). Then, for \( f \in A(p) \), we have
\[
\Lambda(p, \vartheta, \rho; z) \prec \frac{2 \delta z}{(1 - z)^2} + (\delta + \kappa) \left( \frac{1 + z}{1 - z} \right)^2,
\]
then
\[
\frac{F_m^p(\rho, \vartheta) f(z)}{[F_{p+1}^m(\rho, \vartheta) f(z)]^\eta} \prec \frac{1 + z}{1 - z}.
\]
and the function \( \frac{1+z}{1-z} \) is the best dominant.

Finally, by taking \( q(z) = \left( \frac{1+z}{1-z} \right)^\mu \), \((0 < \mu \leq 1)\), then for \( f \in \mathcal{A}(p) \) we have,

\[
\Lambda(p, \vartheta, \rho; z) \prec \frac{2\delta \mu z}{(1-z)^2} \left( \frac{1+z}{1-z} \right)^{\mu-1} + \left( \delta + \kappa \right) \left( \frac{1+z}{1-z} \right)^{2\mu},
\]

then

\[
\frac{F_m^p(\rho, \vartheta)f(z)}{[F_{m+1}^p(\rho, \vartheta)f(z)]^\eta} \prec \left( \frac{1+z}{1-z} \right)^\mu,
\]

and the function \( \left( \frac{1+z}{1-z} \right)^\mu \) is the best dominant.

Next, by appealing to Lemma 2.3 we prove the following.

**Theorem 4.3** Let \( q(z) \) be convex univalent in \( \mathbb{U} \) with \( q(0) = 1 \). Assume that

\[
\text{Re} \left\{ \frac{2(\delta + \kappa)q(z)q'(z)}{\delta} \right\} > 0. \tag{4.4}
\]

Let \( f \in \mathcal{A}(p) \) such that \( \frac{F_m^p(\rho, \vartheta)f(z)}{[F_{m+1}^p(\rho, \vartheta)f(z)]^\eta} \in \mathcal{H}[q(0), 1] \cap \mathcal{Q}, \) \( \Lambda(p, \vartheta, \rho; z) \) is univalent in \( \mathbb{U} \) and the following superordination condition

\[
(\delta + \kappa) (q(z))^2 + \delta z q'(z) \prec \Lambda(p, \vartheta, \rho; z), \tag{4.5}
\]

holds, then

\[
q(z) \prec \frac{F_m^p(\rho, \vartheta)f(z)}{[F_{m+1}^p(\rho, \vartheta)f(z)]^\eta}, \tag{4.6}
\]

and \( q(z) \) is the best subordinant.

**Proof.** Let the function \( p(z) \) be defined by

\[
p(z) = \frac{F_m^p(\rho, \vartheta)f(z)}{[F_{m+1}^p(\rho, \vartheta)f(z)]^\eta}.
\]

Then from the assumption of Theorem 4.3, the function \( p(z) \) is analytic in \( \mathbb{U} \) and (4.4) holds. Hence, the subordination (4.5) is equivalent to

\[
(\delta + \kappa) (q(z))^2 + \delta z q'(z) \prec (\delta + \kappa) (p(z))^2 + \delta z p'(z),
\]

The assertion (4.6) of Theorem 4.3 now follows by an application of Lemma 2.3. ■
Corollary 4.4 Let $q(z) = \frac{1 + Az}{1 + Bz}$ ($-1 \leq B < A \leq 1$) in Theorem 4.3. Further, assume that (4.4) holds.

If $f \in A(p)$ such that $\frac{F^m_p(\rho, \vartheta)f(z)}{[F^{m+1}_p(\rho, \vartheta)f(z)]^\eta} \in \mathcal{H}[q(0), 1] \cap Q$, $\Lambda(p, \vartheta, \rho; z)$ is univalent in $U$ and the following superordination condition

$$\frac{\delta(A - B)z}{(1 + Bz)^2} + (\delta + \kappa) \left(\frac{1 + Az}{1 + Bz}\right)^2 \prec \Lambda(p, \vartheta, \rho; z),$$

holds, then

$$\frac{1 + Az}{1 + Bz} \prec \frac{F^m_p(\rho, \vartheta)f(z)}{[F^{m+1}_p(\rho, \vartheta)f(z)]^\eta},$$

and $\frac{1 + Az}{1 + Bz}$ is the best subordinant.

Also, let $q(z) = \frac{1 + z}{1 - z}$, then for $f \in A(p)$ we have

$$\frac{2\delta z}{(1 - z)^2} + (\delta + \kappa) \left(\frac{1 + z}{1 - z}\right)^2 \prec \Lambda(p, \vartheta, \rho; z),$$

then

$$\frac{1 + z}{1 - z} \prec \frac{F^m_p(\rho, \vartheta)f(z)}{[F^{m+1}_p(\rho, \vartheta)f(z)]^\eta},$$

and the function $\frac{1 + z}{1 - z}$ is the best subordinant.

Finally, by taking $q(z) = \left(\frac{1 + z}{1 - z}\right)^\mu$, $(0 < \mu \leq 1)$, then for $f \in A(p)$ we have

$$\frac{2\delta \mu z}{(1 - z)^2} \left(\frac{1 + z}{1 - z}\right)^{\mu - 1} + (\delta + \kappa) \left(\frac{1 + z}{1 - z}\right)^{2\mu} \prec \Lambda(p, \vartheta, \rho; z),$$

then

$$\left(\frac{1 + z}{1 - z}\right)^\mu \prec \frac{F^m_p(\rho, \vartheta)f(z)}{[F^{m+1}_p(\rho, \vartheta)f(z)]^\eta},$$

and the function $\left(\frac{1 + z}{1 - z}\right)^\mu$ is the best subordinant.

Combining Theorem 4.1 and Theorem 4.3, we get the following sandwich theorem.

Theorem 4.5 Let $q_1$ and $q_2$ be convex univalent in $U$ with $q_1(0) = q_2(0) = 1$ and satisfy (4.1) and (4.4) respectively.

If $f \in A(p)$ such that $\frac{F^m_p(\rho, \vartheta)f(z)}{[F^{m+1}_p(\rho, \vartheta)f(z)]^\eta} \in \mathcal{H}[q(0), 1] \cap Q$, $\Lambda(p, \vartheta, \rho; z)$ is
univalent in \( U \) and
\[
(\delta + \kappa) (q_1(z))^2 + \delta z q'_1(z) \prec \Lambda(p, \vartheta, \rho; z) \prec (\delta + \kappa) (q_2(z))^2 + \delta z q'_2(z),
\]
holds, then
\[
q_1(z) \prec \frac{F^m_p(\rho, \vartheta)f(z)}{[F^{m+1}_p(\rho, \vartheta)f(z)]^\eta} \prec q_2(z),
\]
and \( q_1(z) \) and \( q_2(z) \) are, respectively, the best subordinant and the best dominant.

**Corollary 4.6** Let \( q_i(z) = \frac{1 + A_i z}{1 + B_i z} \) \((i = 1, 2; -1 \leq B_2 < B_1 < A_1 \leq A_2 \leq 1)\) in Theorem 4.5.

If \( f \in A(p) \) such that \( \frac{F^m_p(\rho, \vartheta)f(z)}{[F^{m+1}_p(\rho, \vartheta)f(z)]^\eta} \in \mathcal{H}(q(0), 1] \cap Q, \Lambda(p, \vartheta, \rho; z) \) is univalent in \( U \) and
\[
\frac{\delta(A_1 - B_1)z}{(1 + B_1 z)^2} + (\delta + \kappa) \left( \frac{1 + A_1 z}{1 + B_1 z} \right)^2 \prec \Lambda(p, \vartheta, \rho; z)
\]
\[
\prec \frac{\delta(A_2 - B_2)z}{(1 + B_2 z)^2} + (\delta + \kappa) \left( \frac{1 + A_2 z}{1 + B_2 z} \right)^2,
\]
holds, then
\[
1 + A_1 z
\]
\[
1 + B_1 z
\]
holds, then
\[
\frac{F^m_p(\rho, \vartheta)f(z)}{[F^{m+1}_p(\rho, \vartheta)f(z)]^\eta} \prec \frac{1 + A_2 z}{1 + B_2 z},
\]
and \( \frac{1 + A_1 z}{1 + B_1 z} \) and \( \frac{1 + A_2 z}{1 + B_2 z} \) are, respectively, the best subordinant and the best dominant.

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**References**


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