PRICING MATURITY GUARANTEE WITH DYNAMIC WITHDRAWAL BENEFIT UNDER VASICEK INTEREST RATE MODEL

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Abstract. Motivated by Ko et al. (2010), who propose a new equity-linked product called "maturity guarantee with dynamic withdrawal benefit" (MGDWB), we consider the pricing of this product under a Vasicek stochastic interest rates framework. The explicit pricing formulas for the dynamic withdrawal benefit (DWB) payment stream and the maturity guarantee can be obtained when the DWB payment level is set to be a function of zero-coupon bond.

Keywords: maturity guarantee; dynamic withdrawal benefit; forward neutral measure.

Mathematics Subject Classification: 60G51, 60J25.

1. Introduction

Both individual and institutional investors have a basic need for protection against downside risk. The dynamic guaranteed fund has been one of the most popular investment funds in the insurance industry, recently. The fund, introduced by Gerber and his co-workers, provides a dynamic guarantee for an equity index linked portfolio to its investors with a necessary payment so that the upgraded fund unit value does not fall below a guaranteed level during the protection period. See Gerber and Pafumi (2000), Gerber and Shiu (1998, 2003), Tiong (2000) and Imai and Boyle (2001).

Researches on the dynamic guaranteed fund mainly focus on the modelling of the price of underlying assets and the design of the guarantees. See, for example, Gerber et al. (2013) investigate the valuation of variable annuities with guaranteed minimum death benefits under a jump-diffusion model. Gerber et al. (2015) consider the problem of valuing guaranteed minimum death benefits under a discrete model. Inspired by dynamic guaranteed fund and motivated by the importance of "withdrawal benefits" in enhancing sales of variable annuities, Ko et al. (2010) propose a new equity-linked product, which is called "maturity guarantee with dynamic withdrawal benefit" (MGDWB).

This new equity-linked product has two payoff features. Consider a customer paying a single premium for investing in a mutual fund or stock index. As the customer's account value attains a barrier, amounts over the barrier are automatically withdrawn from the account and paid to the customer. This payoff feature is called dynamic withdrawal benefit (DWB), which is a function of market conditions, so that it can well protect the insurers issuing variable annuities with DWB. Furthermore, the customer is provided with a guarantee that at the contract maturity date T, he will have at least a predetermined amount K. When the DWB payment level is set to be a constant or an exponential function of time and the interest-free rate is assumed to be constant, Ko et al. (2010) price such MGDWB contracts.

However, since equity-indexed annuities are usually of long maturity, the assumption of constant interest rate may not be quite adequate. In this paper, we will price a MGDWB contract in the presence of a stochastic term structure of interest rates. Assume the customer holds the contract to maturity T. Denote the time-t account value without and with DWB payments by $F(t)$ and $\hat{F}(t)$, respectively. In Ko et al. (2010), the relation between the two processes is given by

$$\hat{F}(t) = F(t) \min\left\{1, \min_{0\leq s\leq t} \frac{B}{F(s)}\right\}, \tag{1.1}$$

where $B \geq K$ is a constant. Obviously, the barrier B is not necessarily chosen to be a constant. See, for example, Gerber and Pafumi (2000) consider a stronger protection, where the guaranteed value of a fund unit at time t is $Ke^{\gamma t}$, for some $\gamma > 0$. In fact, γ can be explained as a guaranteed rate of return. In this paper, we extend the protection level to a stochastic level and consider the price of a MGDWB contract under a stochastic interest rate environment. However, in general, it is not easy to obtain the explicit formula for the price of the MGDWB contract under a stochastic interest rate environment. In this paper, we aim at providing a model in which the closed form formula for the price of the MGDWB contract can be obtained under a stochastic interest rate environment.

Bernard et al. (2006) propose a stochastic level with the guaranteed rate matching the return of a government bond and derive the fair value of a participating life insur- ance contract under a Vasicek interest rate environment. Therefore, following Bernard et al. (2006), we consider the relation between the two processes as follows:

$$\hat{F}(t) = F(t) \min\left\{1, \min_{0\leq s\leq t} \frac{BP(s,T)/P(0,T)}{F(s)}\right\}, \tag{1.2}$$

where $P(s,T)$ is the zero-coupon bond price with expiry date T at s. Assume $P(0,T) < 1$ so that $B/P(0,T) \geq K$. In fact, in (1.2), the barrier is chosen as a stochastic process given by $\frac{BP(s,T)}{P(0,T)}$. As explained in Bernard et al. (2006), the protection is equivalent to buying $B/P(0,T)$ government zero-coupon bonds having an initial value equal to $P(0,T)$ at time 0, since the guaranteed value of a fund unit at time t is equal to the value at time t of $K/P(0,T)$ zero-coupon bonds maturing at time T. Therefore, comparing with the protection $Be^{\gamma t}$, which can be regarded as having a fixed guaranteed rate of return, the protection $BP(s,T)/P(0,T)$ can be regarded as having a guaranteed rate matching the return of a zero-coupon bond.

Let $\{\Omega, \Im, \{\Im_t\}_{0\le t\le T}, Q\}$ be a filtered complete probability space, where Q is the risk neutral measure and $\{\Im_t\}_{0\le t\le T}$ is a filtration satisfying the usual conditions of right continuity and completeness. Throughout the paper, it is assumed that all random variables and stochastic processes are well defined on this probability space and $\Im_T-$measurable. We assume that the fund process $\{F(t)\}$ follows a geometric Brownian motion under the risk-neutral probability measure Q:

$$\frac{dF(t)}{F(t)} = r_t dt + \sigma dW^Q(t). \tag{1.3}$$

It is assumed the short-term interest rate dynamics obey the relationship under Q:

$$dr_t = (a\theta - ar_t)dt + \gamma dZ_1^Q(t), a > 0, \gamma > 0.$$

Vasicek (1977) has showed the zero-coupon bond price with expiry date T at a fixed time t, $P(t,T)$, is a function of T and can be expressed as

$$P(t,T) = E^Q[e^{-\int_t^T r_s ds}|\Im_t] = e^{A(t,T)-B(t,T)r_t}, \tag{1.4}$$

where

$$\begin{cases} A(t,T) = \dfrac{(B(t,T)-T+t)}{a^2}\left(a^2\theta - \dfrac{\gamma^2}{2}\right) - \dfrac{B^2(t,T)\gamma^2}{4a}, \\ B(t,T) = \dfrac{1-e^{-a(T-t)}}{a}. \end{cases} \tag{1.5}$$

The zero-coupon bond price with expiry date T, $P(t,T)$, follows the stochastic diffusions

$$\frac{dP(t,T)}{P(t,T)} = r_t dt - \sigma_P(t,T)dZ_1^Q(t),$$

where $\sigma_P(t,T) = \frac{\gamma}{a}(1-e^{-a(T-t)})$. $Z_1^Q(t)$ and $W^Q(t)$ are two $Q-$standard Brownian motions with a correlation coefficient equal to ρ.

Consider a Brownian motion Z_2^Q independent from Z_1^Q, therefore, the Brownian motion W^Q can be expressed as

$$dW^Q(t) = \rho dZ_1^Q(t) + \sqrt{1-\rho^2}dZ_2^Q(t).$$

Then the dynamics of the fund in equation (A.8) can be rewritten as:

$$\frac{dF(t)}{F(t)} = r_t dt + \sigma\left(\rho dZ_1^Q(t) + \sqrt{1-\rho^2}dZ_2^Q(t)\right).$$

Denote by Q_T the $T-$forward-neutral measure. It is defined through its Radon-Nikodym derivative

$$\frac{dQ_T}{dQ} = e^{-\int_0^T \sigma_P(s,T)dZ_1^Q(s) - \frac{1}{2}\int_0^T \sigma_P^2(s,T)ds}.$$

From Girsanov theorem the process Z_1^Q defined by $dZ_1^{Q_T} = dZ_1^Q + \sigma_P(t,T)dt$ is a Q_T-Brownian motion.

Therefore, under the $T-$forward-neutral measure Q_T, the short-term interest rate dynamics obey the relationship

$$dr_t = a(\theta_t - r_t)dt + \gamma dZ_1^{Q_T}(t),$$

where $\theta_t = \theta - \gamma^2/a^2(1 - e^{-a(T-t)})$.

We also define $Z_2^{Q_T}$ such that $Z_1^{Q_T}$ and $Z_2^{Q_T}$ are noncorrelated Q_T-Brownian motions. Under the $T-$forward-neutral measure Q_T, the dynamics of $F(t)$ and $P(t,T)$ follow the stochastic differential equations

$$\frac{dF(t)}{F(t)} = (r_t - \rho\sigma\sigma_P(t,T))dt + \sigma\left(\rho dZ_1^{Q_T}(t) + \sqrt{1-\rho^2}dZ_2^{Q_T}(t)\right)$$

and

$$\frac{dP(t,T)}{P(t,T)} = (r_t + \sigma_P^2(t,T))dt - \sigma_P(t,T)dZ_1^{Q_T}(t).$$

By some calculations, we have

$$\begin{aligned}\frac{BP(s,T)}{F(s)P(0,T)} = \frac{B}{F_0}\exp\Bigg\{&-\int_0^s(\sigma_P(u,T)+\rho\sigma)dZ_1^{Q_T}(u) - \int_0^s\sigma\sqrt{1-\rho^2}dZ_2^{Q_T}(u) \\ &+\frac{1}{2}\int_0^s((\sigma_P(u,T)+\rho\sigma)^2 + \sigma^2(1-\rho^2))du\Bigg\}.\end{aligned} \tag{1.6}$$

Define the martingale H as

$$H_s = \int_0^s(\sigma_P(u,T)+\rho\sigma)dZ_1^{Q_T}(u) + \int_0^s\sigma\sqrt{1-\rho^2}dZ_2^{Q_T}(u).$$

Note that the quadratic variation of H is

$$\begin{aligned}\xi(s) = \int_0^s((\sigma_P(u,T)+\rho\sigma)^2 + \sigma^2(1-\rho^2))du &= \left(\sigma^2 + \frac{\gamma^2}{a^2} + \frac{2\rho\sigma\gamma}{a}\right)s \\ -\left(\frac{2\rho\sigma\gamma}{a^2} + \frac{2\gamma^2}{a^3}\right)(e^{-a(T-s)} - e^{-aT}) &+ \frac{\gamma^2}{2a^3}(e^{-2a(T-s)} - e^{-2aT}).\end{aligned} \tag{1.7}$$

Due to Dubins-Schwarz theorem, there exists a Q_T-Brownian motion B such that

$$H_s = B_{\xi(s)}.$$

Therefore, (1.6) becomes

$$\frac{BP(s,T)}{F(s)P(0,T)} = \frac{B}{F_0} e^{-B_{\xi(s)}+\frac{1}{2}\xi(s)}, \tag{1.8}$$

from (1.8), $F(T)$ can be expressed as

$$F(T) = \frac{F_0}{P(0,T)} e^{B_{\xi(T)}-\frac{1}{2}\xi(T)}.$$

2. Contract valuation

Denote $VW(B,T)$ by the value of the DWB payment stream between time 0 and T at time 0, denote $VP(K,B,T)$ by the value of the European put option with payoff $(K-\hat{F}(T))^+$ at time 0. For simplicity, we assume the cost of the put option is to be paid at the beginning of the contract, and $0 < K \leq F_0 \leq B$. Because the customer pays $F_0 + VP(K,B,T)$, so by the fundamental theorem of asset pricing, we have

$$F_0 = VW(B,T) + E^Q[e^{-\int_0^T r_s ds}\hat{F}(T)], \tag{2.1}$$

$$VP(K,B,T) = E^Q[e^{-\int_0^T r_s ds}(K-\hat{F}(T))^+]. \tag{2.2}$$

Moving toward forward-neutral world, we obtain

$$E^Q[e^{-\int_0^T r_s ds}\hat{F}(T)] = P(0,T)E^{Q_T}[\hat{F}(T)], \tag{2.3}$$

$$E^Q[e^{-\int_0^T r_s ds}(K-\hat{F}(T))^+] = P(0,T)E^{Q_T}[(K-\hat{F}(T))^+]. \tag{2.4}$$

From (1.8), one obtains

$$\begin{aligned} \hat{F}(T) &= F(T)\min\left\{1, \min_{0\leq s\leq T}\frac{B}{F_0}e^{-B_{\xi(s)}+\frac{1}{2}\xi(s)}\right\} \\ &= F(T)\min\left\{1, \frac{B}{F_0}e^{-\max_{0\leq s\leq \xi(T)}(B_s-\frac{1}{2}s)}\right\} \\ &= \frac{F_0}{P(0,T)}e^{B_{\xi(T)}-1/2\xi(T)}\min\left\{1, \frac{B}{F_0}e^{-\max_{0\leq s\leq \xi(T)}(B_s-\frac{1}{2}s)}\right\}. \end{aligned} \tag{2.5}$$

For simplicity, we write $X = B_{\xi(T)} - \frac{1}{2}\xi(T), M = \max_{0\leq s\leq \xi(T)}(B_s - \frac{1}{2}s)$. Then, from (2.1), (2.3) and (2.5), we obtain

$$\begin{aligned} VW(B,T) &= F_0 - P(0,T)E^{Q_T}[\hat{F}(T)] \\ &= F_0 - F_0E^{Q_T}[e^X 1\{M < b\}] - BE^{Q_T}[e^X e^{-M}1\{M > b\}], \end{aligned} \tag{2.6}$$

where $b = \ln\frac{B}{F_0} > 0$, $1\{C\}$ is the indicator of a set C. To simplify the above expression, we define a new probability $\overline{Q}$ through Radon-Nikodym derivative

$$\frac{d\overline{Q}}{dQ_T} = e^X = e^{\int_0^T(\sigma_P(u,T)+\rho\sigma)dZ_1^{Q_T}(u)+\int_0^T\sigma\sqrt{1-\rho^2}dZ_2^{Q_T}-\frac{1}{2}\xi(T)}.$$

From Girsanov theorem, we can construct two $\overline{Q}-$ standard Brownian motions $Z_1^{\overline{Q}}$ and $Z_2^{\overline{Q}}$ defined by

$$\begin{aligned} dZ_1^{\overline{Q}} &= dZ_1^{Q_T} - \int_0^s (\sigma_P(u,T) + \rho\sigma)du, \\ dZ_2^{\overline{Q}} &= dZ_2^{Q_T} - \int_0^s \sigma\sqrt{1-\rho^2}du. \end{aligned}$$

Then, under $\overline{Q}$, $F(T)$ is expressed as

$$F(T) = \frac{F_0}{P(0,T)} e^{\int_0^T (\sigma_P(s,T)+\rho\sigma)dZ_1^{\overline{Q}}(u) + \int_0^T \sigma\sqrt{1-\rho^2}dZ_2^{\overline{Q}}(u) + \frac{1}{2}\xi(T)}. \tag{2.7}$$

Similar to deriving Eq. (1.8), under $\overline{Q}$ Eq. (2.7) can be expressed as

$$F(T) = \frac{F_0}{P(0,T)} e^{\overline{B}_{\xi(T)} + \frac{1}{2}\xi(T)},$$

where $\overline{B}$ is a $\overline{Q}-$standard Brownian motion. Define $\overline{X} = \overline{B}_{\xi(T)} + \frac{1}{2}\xi(T)$ and $\overline{M} = \max\limits_{0\le s\le \xi(T)} \left(\overline{B}_s + \frac{1}{2}s\right)$. Note $E^{Q_T}[e^X] = 1$, then

$$\begin{aligned} E^{Q_T}[e^X 1\{M < b\}] &= E^{Q_T}[e^X]\overline{Q}(\overline{M} < b) = \overline{Q}(\overline{M} < b) \\ &= \Phi\left(\frac{b - 1/2\xi(T)}{\sqrt{\xi(T)}}\right) - \frac{B}{F_0}\Phi\left(\frac{-b - 1/2\xi(T)}{\sqrt{\xi(T)}}\right) \end{aligned} \tag{2.8}$$

and

$$E^{Q_T}[e^{X-M}1\{M > b\}] = E^{Q_T}[e^X]E^{\overline{Q}}[e^{-\overline{M}}1\{\overline{M} > b\}] = E^{\overline{Q}}[e^{-\overline{M}}1\{\overline{M} > b\}]. \tag{2.9}$$

Note the expectation $E^{\overline{Q}}[e^{a\overline{M}}1\{\overline{M} > b\}]$, for $a \neq -1$, has been given by Huang and Shiu (2001) and Lee (2003). To evaluate the expectation $E^{\overline{Q}}[e^{-\overline{M}}1\{\overline{M} > b\}]$, first we give the following lemma.

Lemma 2.1 *Let* $X(t) = \mu t + \sigma W(t)$, $M(t) = \max\limits_{0\le s\le t} X(s)$, *where* $W(t)$ *is a* $Q-$*standard Brownian motion. Then, we have*

$$Q(X(t) < k, M(t) > d)) = e^{\frac{2\mu d}{\sigma^2}} Q(X(t) \le k - 2d), b \ge 0, k \le b. \tag{2.10}$$

Proof. The proof of (2.10) can be found in Gerber and Shiu (2000) and Huang and Shiu (2001). From (2.10), we obtain

$$\begin{aligned} \overline{Q}(\overline{M} \le m) &= \overline{Q}(\overline{X} \le m) - \overline{Q}(\overline{X} \le m, \overline{M} \ge m) \\ &= \overline{Q}(\overline{X} \le m) - e^m \overline{Q}(\overline{X} \le -m), \end{aligned} \tag{2.11}$$

Using (2.11), we have

$$
\begin{aligned}
E^{\overline{Q}}[e^{-\overline{M}}1\{\overline{M} > b\}] &= \int_b^\infty e^{-m} d\overline{Q}(\overline{X} \le m) - \int_b^\infty e^{-m} d(e^m \overline{Q}(\overline{X} \le -m)) \\
&= \int_b^\infty \frac{2}{\sqrt{2\pi\xi(T)}} e^{-\frac{(m+1/2\xi(T))^2}{2\xi(T)}} dm - \int_b^\infty \overline{Q}(\overline{X} \le -m) dm \\
&= 2\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - \int_b^\infty \overline{Q}(\overline{X} \le -m) dm.
\end{aligned} \tag{2.12}
$$

Integrating the integral in (2.12) by parts, we obtain

$$
\begin{aligned}
&\int_b^\infty \overline{Q}(\overline{X} \le -m) dm \\
&= - b\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \int_b^\infty \frac{m}{\sqrt{2\pi\xi(T)}} e^{-\frac{(m+1/2\xi(T))^2}{2\xi(T)}} dm \\
&= - (b+1/2\xi(T))\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \sqrt{\frac{\xi(T)}{2\pi}} e^{-\frac{(b+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned} \tag{2.13}
$$

Substituting (2.13) into (2.12), we have

$$
\begin{aligned}
E^{\overline{Q}}[e^{-\overline{M}}1\{\overline{M} > b\}] = (2+b+1/2\xi(T))\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
- \sqrt{\frac{\xi(T)}{2\pi}} e^{-\frac{(b+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned} \tag{2.14}
$$

Substituting (2.8)-(2.14) into (2.6), we give the explicit expression for $VW(B,T)$

$$
\begin{aligned}
&VW(B,T) = F_0\Phi\left(-\frac{b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&- B\left(1+\ln\frac{B}{F_0}+\frac{\xi(T)}{2}\right)\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) + B\sqrt{\frac{\xi(T)}{2\pi}} e^{-\frac{(b+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned} \tag{2.15}
$$

Remark 2.1.

(1) From (1.7), we can easily obtain $\xi(T) \to +\infty$, when $T \to +\infty$, hence $VW(B,\infty) = F_0$.

(2) When B is infinity, then no withdrawal benefit payments will ever be made. Thus, $VW(\infty,T) = 0$ and it is easy to check it from (2.6) and (2.8)-(2.14).

Now, we present the evaluation of the put option price $VP(B,K,T)$. We can rewrite Eq. (2.2) by using the $T-$forward-neutral equivalent martingale measure Q_T according to:

$$
VP(B,K,T) = P(0,T)E^{Q_T}[(K-\hat{F}(T))_+]. \tag{2.16}
$$

With $c = \ln \frac{KP(0,T)}{F_0} < 0$, we divide the expectation $E^{Q_T}[(K - \hat{F}(T))_+]$ into two parts:

$$
\begin{aligned}
&E^{Q_T}[(K - \hat{F}(T))_+] \\
&\quad = E^{Q_T}[(K - \hat{F}(T))_+ 1\{M < b\}] + E^{Q_T}[(K - \hat{F}(T))_+ 1\{M > b\}] \\
&\quad = E^{Q_T}\left[\left(K - \frac{F_0}{P(0,T)} e^X\right) 1\{M < b, X < c\}\right] \\
&\quad + E^{Q_T}\left[\left(K - \frac{B}{P(0,T)} e^{X-M}\right) 1\{M > b, M - X > b - c\}\right].
\end{aligned} \tag{2.17}
$$

The first term on the right hand side of (2.17) can be calculated as follows:

$$
\begin{aligned}
&E^{Q_T}\left[\left(K - \frac{F_0}{P(0,T)} e^X\right) 1\{M < b, X < c\}\right] \\
&\quad = KQ_T(M < b, X < c) - \frac{F_0}{P(0,T)} E^{Q_T}[e^X] \overline{Q}(\overline{M} < b, \overline{X} < c) \\
&\quad = K[Q_T(X < c) - Q_T(M \geq b, X < c)] \\
&\qquad - \frac{F_0}{P(0,T)} [\overline{Q}(\overline{X} < c) - \overline{Q}(\overline{M} \geq b, \overline{X} < c)] \\
&\quad = K\left[\Phi\left(\frac{c + 1/2\xi(T)}{\sqrt{\xi(T)}}\right) - \frac{F_0}{B}\Phi\left(\frac{c - 2b + 1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right] \\
&\qquad - \frac{F_0}{P(0,T)}\left[\Phi\left(\frac{c - 1/2\xi(T)}{\sqrt{\xi(T)}}\right) - \frac{B}{F_0}\Phi\left(\frac{c - 2b - 1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right],
\end{aligned} \tag{2.18}
$$

where the last equality follows from (2.10). The second term on the right hand side of (2.17) is

$$
\begin{aligned}
&E^{Q_T}\left[\left(K - \frac{B}{P(0,T)} e^{X-M}\right) 1\{M > b, M - X > b - c\}\right] \\
&\quad = KQ_T(M > b, M - X > b - c) \\
&\quad - \frac{B}{P(0,T)} E^{Q_T}\left[e^{X-M} 1\{M > b, M - X > b - c\}\right] \\
&\quad = \frac{B}{P(0,T)}\left[\Phi\left(\frac{c - 2b - 1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \frac{F_0 K}{B}\Phi\left(\frac{c - 2b + 1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right] \\
&\quad - \frac{B}{P(0,T)}\left[(2 + 2b - c + 1/2\xi(T))\Phi\left(\frac{c - 2b - 1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right. \\
&\quad \left. - \sqrt{\frac{\xi(T)}{2\pi}} e^{-\frac{(2b - c + 1/2\xi(T))^2}{2\xi(T)}}\right],
\end{aligned} \tag{2.19}
$$

where the proof of the expressions for the probability $Q_T(M > b, M - X > b - c)$ and the expectation $E^{Q_T}[e^{X-M} 1\{M > b, M - X > b - c\}]$ are given by (A.6) and (A.8) in the Appendix.

Substituting (2.17)-(2.19) into (2.16) yields that

$$
\begin{aligned}
VP(B,K,T) =& KP(0,T)\Phi\left(\frac{c+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - F_0\Phi\left(\frac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&+ B\left(\ln\frac{KP(0,T)F_0}{B^2} - \frac{\xi(T)}{2}\right)\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \qquad (2.20)\\
&+ B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(2b-c+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned}
$$

Remark 2.2.

(1) It follows from (1.4), (1.5) that, if $a^2\theta > \frac{\gamma^2}{2}$, then $P(0,T)\to 0$ when $T\to\infty$, so we have $VP(B,K,\infty)=0$.

(2) When $K=0$, it can be directly obtained from (2.16) $VP(B,0,T)=0$.

(3) When B is infinity, then no withdrawal benefit payments will ever be made. Thus, $VW(\infty,K,T)$ must be the Black-Scholes price of the European $K-$strike put option on $F(t)$ exercisable at T and it can be expressed as

$$VP(\infty,K,T) = P(0,T)K\Phi\left(\frac{c+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - F_0\Phi\left(\frac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right).$$

In fact, $VP(\infty,K,T)$ is exactly given by the first two terms on the right hand side of (2.20). Hence, (2.20) can also be written as

$$
\begin{aligned}
&VP(B,K,T) - VP(\infty,K,T) \\
&\quad = B\left(\ln\frac{KP(0,T)F_0}{B^2} - \frac{\xi(T)}{2}\right)\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&\quad + B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(2b-c+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned}
$$

Since the left hand side of (2.20) is positive, its right hand side is also positive, which is not immediately obvious.

(4) Differentiating (2.15) with respect to F_0 and simplifying, we obtain the hedge-ratio formula,

$$\frac{\partial}{\partial F_0}VW(B,T) = \Phi\left(-\frac{b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \frac{B}{F_0}\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right).$$

Differentiating (2.20) with respect to F_0 and simplifying, we have

$$\frac{\partial}{\partial F_0}VP(B,K,T) = -\Phi\left(\frac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \frac{B}{F_0}\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right).$$

It is interesting to note that both derivatives can be quickly obtained if we pretend that b, c are not functions of F_0 when performing the differentiation. (Similarly, differentiating the Black-Scholes formula with respect to the stock price, while pretending that d_1 and d_2 are constant, immediately yields the formula for the option's delta.)

3. Numerical analysis

In this section, we make some numerical analysis on the value of the DWB payment stream $VW(B,T)$ and the value of the European put option $VP(K,B,T)$. The price of the zero-coupon bond $P(0,T)$ is considered as a function of T and hence it is can be calculated from equation (1.6). Let $F_0 = 100$, $r_0 = 0.05$, $a = 0.4$, $\sigma = 0.1$, $\gamma = 0.008$, $\rho = 0.2$, $\theta = 0.04$.

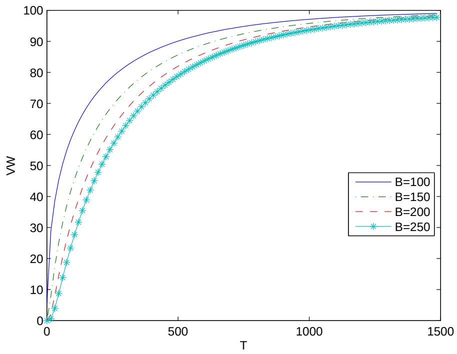

Figure 1: $VW(B,T)$ as a function of T for different B

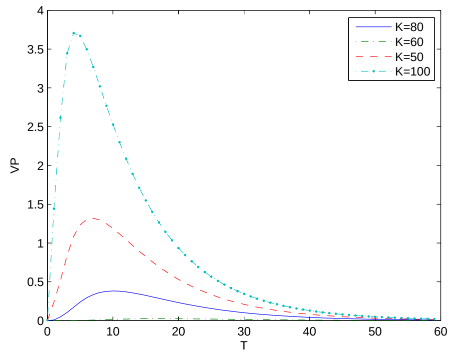

Figure 2: $VP(K,B,T)$ as a function of T for different K, $B = 120$.

Figure 1 plots $VW(B,T)$ as a function of T for different B. From Figure 1 we see, $VW(B,T)$ is a decreasing function of B. That is to say, the withdrawal benefit payments will decrease with B increasing. Figure 2 plots $VP(B,K,T)$ as a function of T for different K given $B = 120$. It is easy to see from Figure 2, $VP(B,K,T)$ is an increasing function of K. That is because $VP(B,K,T)$ is the price of put option with strike K.

4. Reflected Brownian motion

In this section, we derive the pricing formulas (2.15) and (2.20) by using the theory of Reflected Brownian motion.

$$\ln \frac{BP(t,T)}{P(0,T)\hat{F}(t)} = -\left(B_{\xi(t)} - \frac{1}{2}\xi(t)\right) + \max\left\{b, \max_{0\le s\le t}\left\{B_{\xi(s)} - \frac{1}{2}\xi(s)\right\}\right\}. \tag{4.1}$$

The process $\{\ln \frac{BP(t,T)}{P(0,T)\hat{F}(t)}\}$ is a reflected Brownian motion starting at $\ln \frac{B}{F(0)} = b$, with reflecting barrier at 0. Let $Y = \ln \frac{B}{P(0,T)\hat{F}(T)}$, then from Formula (91), Section

5.7 in Cox and Miller (1965) it follows that the probability density function of the random variable Y under Q_T is

$$\begin{aligned} p(x;b,T) &= n\left(x;b+\frac{1}{2}\xi(T),\xi(T)\right)+e^{-b}n\left(x;-b+\frac{1}{2}\xi(T),\xi(T)\right) \\ &- e^{x}\Phi\left(-\frac{x+b+1/2\xi(T)}{\sqrt{\xi(T)}}\right), x>0, \end{aligned} \tag{4.2}$$

where $n(x;\mu,\sigma^2)$ is the probability density function of the normal random variable with mean μ and variance σ^2, and $\Phi(x)$ is the standard normal distribution. This result corresponds to formula (2.5) in Ko et al. (2010). To obtain the pricing formulas (2.15) and (2.20), we also need the following formulas given in Gerber and Pafumi (2000) and Ko et al. (2010),

$$\int_a^\infty e^{cx}n(x;\mu,\sigma^2)dx = e^{\mu c+1/2\sigma^2c^2}\Phi\left(\frac{-a+\mu+c\sigma^2}{\sigma}\right), \tag{4.3}$$

where a and c are arbitrary real numbers and

$$\begin{aligned} \int_a^\infty e^{cx}\Phi\left(-\frac{x-\mu}{\sigma}\right)dx &= \frac{1}{c}e^{\mu c+1/2\sigma^2c^2}\Phi\left(\frac{-a+\mu+c\sigma^2}{\sigma}\right) \\ &- \frac{1}{c}e^{ac}\Phi\left(\frac{-a+\mu}{\sigma}\right), \end{aligned} \tag{4.4}$$

where c is an arbitrary number other than zero. By using (4.2)-(4.4), we obtain

$$\begin{aligned} VW(B,T) &= F_0 - P(0,T)E^{Q_T}[\hat{F}(T)] = F_0 - BE^{Q_T}[e^{-Y}] \\ &= F_0 - B\left[e^{-b}\Phi\left(\frac{b-1/2\xi(T)}{\sqrt{\xi(T)}}\right)+\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right. \\ &\left. - \int_0^\infty \Phi\left(\frac{-y-b-1/2\xi(T)}{\sqrt{\xi(T)}}\right)dy\right] \\ &= F_0\Phi\left(\frac{-b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - B\Phi\left(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\ &+B\int_0^\infty \Phi\left(\frac{-y-b-1/2\xi(T)}{\sqrt{\xi(T)}}\right)dy. \end{aligned} \tag{4.5}$$

Integrating the integral in (4.5) by parts, we obtain

$$\begin{aligned} \int_0^\infty \Phi(\frac{-y-b-1/2\xi(T)}{\sqrt{\xi(T)}})dy &= \int_0^\infty \frac{y}{\sqrt{2\pi\xi(T)}}e^{-\frac{(y+b+1/2\xi(T))^2}{2\xi(T)}}dy \\ &= -(\ln\frac{B}{F_0}+1/2\xi(T))\Phi(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}})+B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(b+1/2\xi(T))^2}{2\xi(T)}}. \end{aligned} \tag{4.6}$$

Substituting (4.6) into (4.5), we have

$$\begin{aligned} VW(B,T) &= F_0\Phi(-\frac{b-1/2\xi(T)}{\sqrt{\xi(T)}}) + B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(b+1/2\xi(T))^2}{2\xi(T)}} \\ &- B(1+\ln\frac{B}{F_0}+\frac{\xi(T)}{2})\Phi(-\frac{b+1/2\xi(T)}{\sqrt{\xi(T)}}), \end{aligned} \tag{4.7}$$

which is the same as equation (2.15).

Similarly, we can obtain the explicit formula for pricing the put option.

$$\begin{aligned} VP(B,K,T) &= P(0,T)E^{Q_T}[(K-\hat{F}(T))^+] \\ &= P(0,T)E^{Q_T}\left[\left(K-\frac{B}{P(0,T)}e^{-Y}\right)^+\right] \\ &= P(0,T)E^{Q_T}\left[\left(K-\frac{B}{P(0,T)}e^{-Y}\right)1\{Y\geq b-c\}\right] \\ &= P(0,T)KQ_T(Y\geq b-c) - BE^{Q_T}[e^{-Y}1\{Y\geq b-c\}] \\ &= KP(0,T)\Phi\left(\frac{c+1/2\xi(T)}{\sqrt{\xi(T)}}\right) + B\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\ &-B\left[e^{-b}\Phi\left(\frac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right) + \Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right)\right. \\ &\qquad\left. - \int_{b-c}^{\infty}\Phi\left(-\frac{y+b+1/2\xi(T)}{\sqrt{\xi(T)}}\right)dy\right] \\ &= KP(0,T)\Phi\left(\frac{c+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - F_0\Phi\left(\frac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\ &+B\int_{b-c}^{\infty}\Phi\left(-\frac{y+b+1/2\xi(T)}{\sqrt{\xi(T)}}\right)dy. \end{aligned} \tag{4.8}$$

Integrating the integral in (4.8) by parts, we obtain

$$\begin{aligned} \int_{b-c}^{\infty}\Phi\left(-\frac{y+b+1/2\xi(T)}{\sqrt{\xi(T)}}\right)dy &= -(b-c)\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\ &+ \int_{b-c}^{\infty}\frac{y}{\sqrt{2\pi\xi(T)}}e^{-\frac{(y+b+1/2\xi(T))^2}{2\xi(T)}}dy \\ &= -(2b+1/2\xi(T)-c)\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\ &+\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(2b-c+1/2\xi(T))^2}{2\xi(T)}}. \end{aligned} \tag{4.9}$$

Substituting (4.9) into (4.8) gives

$$
\begin{aligned}
VP(B,K,T) &= KP(0,T)\Phi\left(\tfrac{c+1/2\xi(T)}{\sqrt{\xi(T)}}\right) - F_0\Phi\left(\tfrac{c-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&\quad +B\left(\ln\frac{KP(0,T)F_0}{B^2}-\frac{\xi(T)}{2}\right)\Phi\left(\frac{c-2b-1/2\xi(T)}{\sqrt{\xi(T)}}\right) \\
&\quad +B\sqrt{\frac{\xi(T)}{2\pi}}e^{-\frac{(2b-c+1/2\xi(T))^2}{2\xi(T)}}.
\end{aligned}
\tag{4.10}
$$

Obviously, equation (4.10) is the same with (2.20).

5. Conclusions

This article considers the pricing of the maturity guarantee with dynamic withdrawal benefit contract under a Vasicek stochastic interest rates framework. Because the DWB payment barrier is chosen to be a function of zero-coupon bond, so the closed-form formulas for the value of the DWB payment stream and the value of the European put option can be obtained by a probabilistic method or by using the theory of reflected Brownian motion.

Appendix

From (2.10), we have

$$
Q_T(M>y, X<x) = e^{-y}\Phi\left(\frac{x-2y+1/2s}{\sqrt{s}}\right), \tag{A.1}
$$

where $x \le y$, and $s = \xi(T)$. Differentiating (A.1), we obtain the joint density function of (M, X)

$$
Q_T(M\in dy, X\in dx) = \frac{1}{\sqrt{2\pi s}}e^{\frac{(x+1/2s)^2+4y^2-4xy}{2s}}\frac{4y-2x}{s}dxdy, x\le y. \tag{A.2}
$$

From equation (A.2), we have

$$
\begin{aligned}
&E^{Q_T}[1\{M>b, M-X>b-c\}] \\
&= \int_b^\infty\int_{-\infty}^{y-b+c}\frac{1}{\sqrt{2\pi s}}e^{-\frac{(x+1/2s)^2+4y^2-4xy}{2s}}\frac{4y-2x}{s}dxdy \\
&= \int_b^\infty\int_{-\infty}^{y-b+c}\frac{1}{\sqrt{2\pi s}}e^{-\frac{(x+1/2s-2y)^2}{2s}}e^{-y}\left(\frac{4y-2x-s}{s}+1\right)dxdy \\
&= \int_b^\infty\int_{-\infty}^{y-b+c}\frac{4y-2x-s}{s\sqrt{2\pi s}}e^{-y}e^{-\frac{(x+1/2s-2y)^2}{2s}}dxdy \\
&\quad +\int_b^\infty\Phi\left(-\frac{y+b-c-1/2s}{\sqrt{s}}\right)dy \\
&= \int_b^\infty e^{-y}I_1dy + \int_b^\infty e^{-y}\Phi\left(-\frac{y+b-c-1/2s}{\sqrt{s}}\right)dy.
\end{aligned}
\tag{A.3}
$$

Integrating I_1 by parts yields

$$I_1 = \frac{2}{\sqrt{2\pi s}} e^{-\frac{(y+b-c-1/2s)^2}{2s}}. \tag{A.4}$$

Substituting (A.4) into (A.3), we have

$$\begin{aligned} &E^{Q_T}[1\{M > b, M - X > b - c\}] \\ &= \int_b^\infty \frac{2e^{-y}}{\sqrt{2\pi s}} e^{-\frac{(y+b-c-1/2s)^2}{2s}} dy + \int_b^\infty e^{-y}\Phi\left(-\frac{y+b-c-1/2s}{\sqrt{s}}\right) dy, \end{aligned} \tag{A.5}$$

Integrating the second integral in (A.5) by parts gives

$$\begin{aligned} &E^{Q_T}[1\{M > b, M - X > b - c\}] \\ &= \int_b^\infty \frac{1}{\sqrt{2\pi s}} e^{b-c} e^{-\frac{(y+b-c+1/2s)^2}{2s}} dy + e^{-b}\Phi\left(-\frac{2b-c-1/2s}{\sqrt{s}}\right) \\ &= e^{b-c}\Phi\left(\frac{c-2b-1/2s}{\sqrt{s}}\right) + e^{-b}\Phi\left(\frac{c-2b+1/2s}{\sqrt{s}}\right). \end{aligned} \tag{A.6}$$

Similarly,

$$\begin{aligned} &E^{Q_T}[e^{X-M}1\{M > b, M - X > b - c\}] \\ &= \int_b^\infty \int_{-\infty}^{y-b+c} e^{x-y} \frac{1}{\sqrt{2\pi s}}\ e^{-\frac{(x+1/2s)^2+4y^2-4xy}{2s}}\ \frac{4y-2x}{s}\ dxdy \\ &= \int_b^\infty \int_{-\infty}^{y-b+c} \frac{4y-2x+s}{s\sqrt{2\pi s}}\ e^{-\frac{(x-1/2s-2y)^2}{2s}} dxdy \\ &- \int_b^\infty \Phi\left(-\frac{y+b-c+1/2s}{\sqrt{s}}\right) dy \end{aligned} \tag{A.7}$$

We integrate the above two integrals in (A.7) by parts

$$\begin{aligned} &E^{Q_T}[e^{X-M}1\{M > b, M - X > b - c\}] \\ &= \int_b^\infty \frac{2}{\sqrt{2\pi s}}\ e^{-\frac{(y+b-c+1/2s)^2}{2s}} dy + b\Phi\left(\frac{c-2b-1/2s}{\sqrt{s}}\right) \\ &- \int_b^\infty \frac{y}{\sqrt{2\pi s}}\ e^{-\frac{(y+b-c+1/2s)^2}{2s}}\ dy \\ &= (2+2b-c+1/2s)\Phi\left(\frac{c-2b-1/2s}{\sqrt{s}}\right) \\ &- \int_b^\infty \frac{(y+b-c+1/2s)}{\sqrt{2\pi s}}\ e^{-\frac{(y+b-c+1/2s)^2}{2s}}\ dy \\ &= (2+2b-c+1/2s)\Phi\left(\frac{c-2b-1/2s}{\sqrt{s}}\right) - \sqrt{\frac{s}{2\pi}}\ e^{-\frac{(2b-c+1/2s)^2}{2s}}, \end{aligned} \tag{A.8}$$

where the last equality is obtained by integrating by parts.

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