# TRIGONOMETRIC FUNCTIONS – ONE POSSIBLE DEFINITION

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**Abstract.** The paper deals with trigonometric functions, presents several possibilities of their introducing and offers one definition based on geometrical ideas and using analytical means. The basic properties of such functions are derived, too.

Trigonometric functions play very important role in geometry, analysis and other branches of mathematics. But definitions of these functions can be more subtle than one might suspect.

## 1. Some historical remarks

Trigonometry was studied in ancient Egypt, Babylon, Chaldea and Greece. For instance Hipparchos of Nicaea (180–125 BC) tabulated values for length of the chord of some angles (called chord function). It is directly proportional to the function sine (see Fig. 1), because

$$\text{chord}\, A = 2r \sin \frac{\alpha}{2}\,\cdot$$

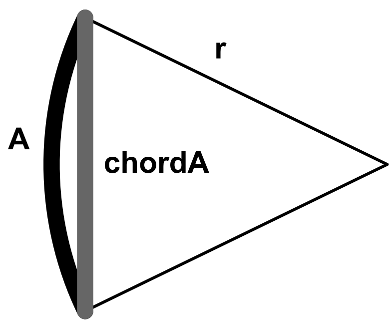


Figure 1: Chord function

Also Claudius Ptolemy (90–165 AD) solved astronomical problems linked with trigonometry.

The concepts closest to our trigonometric functions were introduced by Indian scholars at the period of Gupta dynasty, i.e. approximately from 320 to 550. In the treatise Surya Siddhanta, there are described notions $jy\bar{a}$, $koti - jy\bar{a}$ (and $utkrama - jy\bar{a}$) which correspond to the sine and cosine (see Fig. 2).

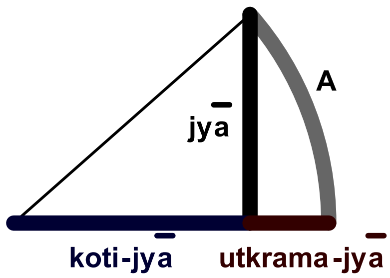


Figure 2: Trigonometric functions in Surya Siddhanta

Aryabhatiya (476–550) put together the tables of half chords which means sine. Bhaskara I. (600–680) invented the formula for approximate calculation of sine

$$\sin A = \frac{16A(\pi - A)}{5\pi^2 - 4A(\pi - A)}$$

(the formula is adjusted for radians).

Another Indian mathematician and astronomer Madhava of Sangamagrama (1340–1425) performed something like expansion of trigonometric functions into series and composed the accurate tables of sine for multiples of angle $3°45'$ (which is $1/24$ of the right angle).

For more remarks about Indian ancient mathematicians see [6].

Indian mathematics was mediated to European culture usually by Arabian scholars (e.g. al-Khwarizmí) in period from 9th to 15th century. In rapid development of sciences, the trigonometric functions became the basic concept in works of many European mathematicians, let us name Leonhard Euler (1707–1783), for all, and his treatment "Introductio in analysis infinitorum". Signa sin, cos and tan themselves were introduced by Albert Girard (1595–1632).

## 2. Usual ways of definitions

Trigonometric functions are defined in several ways.

On elementary level, these functions are introduced geometrically by use of a rectangle triangle or a unit circle. Let us remind also one non frequented possibility – from general triangle ABC by cosine theorem

$$\cos\gamma = \frac{a^2 + b^2 - c^2}{2ab} \,.$$

In mathematical analysis, trigonometric functions are described usually by series, integrals or differential equations, see [1], [2] and especially [8]. The most often way of the definition uses the power series

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!}, \qquad \cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} \,.$$

Trigonometric functions can be defined by means of differential equations. The sine is a solution of Cauchy's problem

$$y'' + y = 0 \quad \text{with conditions} \quad y(0) = 0,\ y'(0) = 1 \ .$$

Or both sine and cosine are solutions of differential equation system

$$y_1' = y_2,\ y_2' = -y_1 \quad \text{with conditions} \quad y_1(0) = 0,\ y_2(0) = 1 \ .$$

Another possibility how to introduce trigonometric functions is to define some of the inverse function to them by integral. We can define

$$\arctan x = \int \frac{1}{1+x^2}\,dx \quad \text{or} \quad \arcsin x = \int \frac{1}{\sqrt{1-x^2}}\,dx$$

and tangent or sine as the corresponding inverse functions which can be appropriately extended.

Similar situation is with the exponential function. It can be introduced by power series

$$\mathrm{e}^x = \sum_{n=0}^{\infty} \frac{x^n}{n} \ ,$$

as a solution of differential equation

$$y' - y = 0 \quad \text{with condition} \quad y(0) = 0$$

or as an inverse function to logarithm

$$\ln x = \int \frac{1}{x}\,dx \quad \text{for } x > 0 \ .$$

One interesting definition of trigonometric functions using area of circular section is published in [7].

Each of these definitions requires quite complicated mathematical apparatus. But we often use trigonometric (or exponential) functions and their properties during studies of differential or integral calculus, including chapters on power series, differential equations or integrals. That is why we looked for some simple and sufficiently correct way of definition of trigonometric functions.

## 3. One simple definition of arcsine

The aim of this paper is to offer quite simple introduction of the function arcsine and then sine from it as an extension of inverse function to arcsine. This definition is based on geometrical ideas but it uses analytical means.

At first we shall calculate the longitude of curve described by (continuous and smooth) function $f$ on $\langle a; b\rangle$ by means of so called variation. This idea comes from Pythagoras theorem.

For any partition

$$D : a = t_0 < t_1 < \cdots < t_n = b$$

of the interval $\langle a; b\rangle$ let us consider the sum

$$s(D) = \sum_{k=1}^{n} \sqrt{(t_k - t_{k-1})^2 + (f(t_k) - f(t_{k-1}))^2}\ .$$

If there exists finite supremum $S$ of such sums $s(D)$ for all partitions $D$ of interval $\langle a; b\rangle$, we call this number $S$ a longitude of curve $f$ on $\langle a; b\rangle$, i.e.,

$$S = \sup\{s(D); D \text{ is a partition of interval } \langle a; b\rangle\}\ .$$

Now, we choose function describing a unit circle in the first quadrant

$$f(t) = \sqrt{1 - t^2}$$

and denote by $AS(x)$ the longitude of this function on the interval $\langle 0; x\rangle$ for any $x \in \langle 0, 1\rangle$. So, for any partition $D:\ 0 = t_0 < t_1 < \cdots < t_n = x$ of interval $\langle 0; x\rangle$, we have

$$\begin{aligned} AS(x) &= \sup_D \sum_{k=1}^{n} \sqrt{(t_k - t_{k-1})^2 + (f(t_k) - f(t_{k-1}))^2} = \\ &= \sup_D \sum_{k=1}^{n} \sqrt{(t_k - t_{k-1})^2 + \left(\sqrt{1 - t_k^2} - \sqrt{1 - t_{k-1}^2}\right)^2} = \\ &= \sqrt{2} \sup_D \sum_{k=1}^{n} \sqrt{1 - t_k t_{k-1} - \sqrt{1 - t_k^2 - t_{k-1}^2 + t_k^2 t_{k-1}^2}}\ . \end{aligned}$$

The function introduced in this way for $0 \le x \le 1$ is well known arcussine $\arcsin x = AS(x)$.

Geometrically, it is illustrated on Fig. 3.

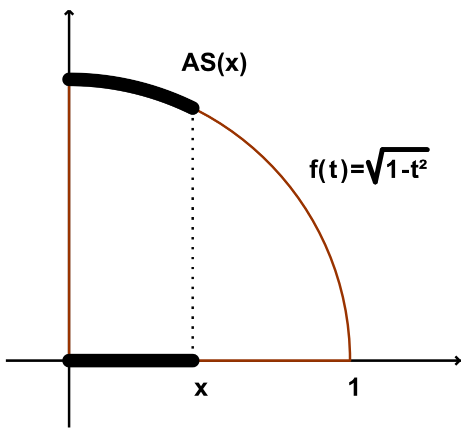


Figure 3: Definition of arcsine

For negative $-1 \leq x < 0$, arcussine can be defined by the formula

$$\arcsin x = -AS(-x).$$

Then arcussine is an odd function.

## 4. Properties of arcsine

Let us investigate some of arcsine properties:

1. The function $AS$ is increasing.

It is obvious from definition.

2. We define $\pi = 2 \cdot AS(1)$.

Relation $AS(1) = \frac{1}{2}\pi$ corresponds with our geometrical ideas about Ludolf number $\pi$.

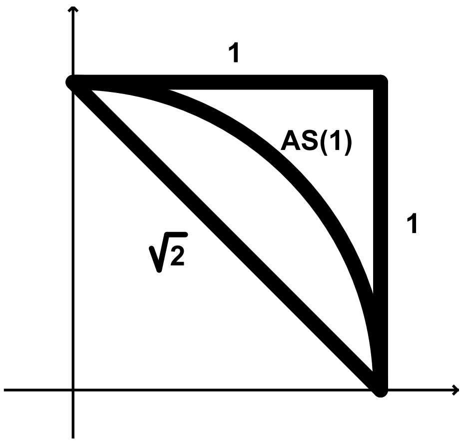


Figure 4: Second property

From Fig. 4 we can see the inequality $\sqrt{2} \leq AS(1) \leq 2$.

3. $AS(x) + AS(\sqrt{1-x^2}) = \frac{1}{2}\pi$.

This equality is obvious from Fig. 5.

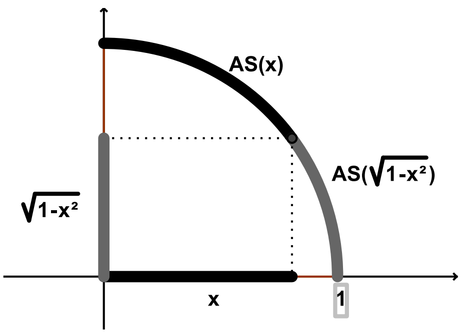


Figure 5: Third property

4. $\lim\limits_{x\to 0^+} \dfrac{AS(x)}{x} = 1.$

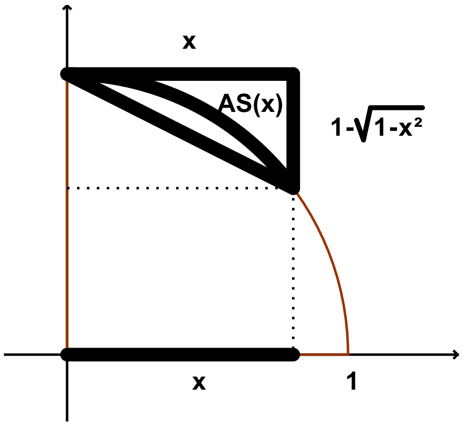


Figure 6: Fourth property

From Fig. 6 we can see the inequality

$$x + 1 - \sqrt{1-x^2} \geq AS(x) \geq \sqrt{x^2 + (1-\sqrt{1-x^2})^2} = \sqrt{2}\sqrt{1-\sqrt{1-x^2}}$$

and from that

$$\sqrt{2}\frac{\sqrt{1-\sqrt{1-x^2}}}{x} \leq \frac{AS(x)}{x} \leq 1 + \frac{1-\sqrt{1-x^2}}{x} .$$

Because of the limits $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x^2} = \dfrac{1}{2}$ and $\lim\limits_{x\to 0} \dfrac{1-\sqrt{1-x^2}}{x} = 0$ , it holds

$$\frac{AS(x)}{x} \to 1.$$

5. $\lim\limits_{x\to 1^-} \dfrac{AS(x) - AS(1)}{x-1} = \infty.$

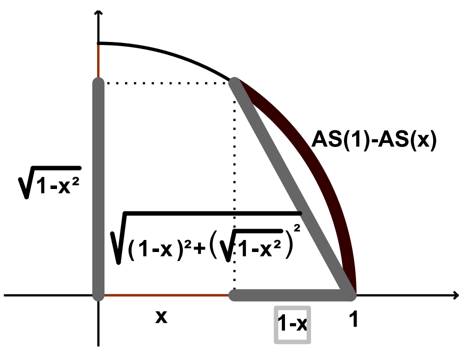


Figure 7: Fifth property

Pythagoras theorem used on triangle on Fig. 7 yields the inequality

$$AS(1) - AS(x) \geq \sqrt{(1-x)^2 + (\sqrt{1-x^2})^2} = \sqrt{2}\sqrt{1-x}$$

and from here we get

$$\frac{AS(1) - AS(x)}{1-x} \geq \frac{\sqrt{2}}{\sqrt{1-x}}.$$

Because $\lim\limits_{x \to 1^-} \dfrac{\sqrt{2}}{\sqrt{1-x}} = \infty$, our limit is also $\infty$.

This property is important to the possibility of a smooth extension of the inverse function.

6. $AS(y) - AS(x) = AS(x\sqrt{1-y^2} - y\sqrt{1-x^2})$.

The lengths of arc of function $f(t) = \sqrt{1-t^2}$ on interval $\langle 0; x\rangle$ and $\langle 0; y\rangle$ are presented by $AS(x)$ and $AS(y)$ respectively. Their difference $AS(y) - AS(x)$ presents the length of arc $PQ$ of unit circle, the situation is described on Fig. 8 for $x < y$.

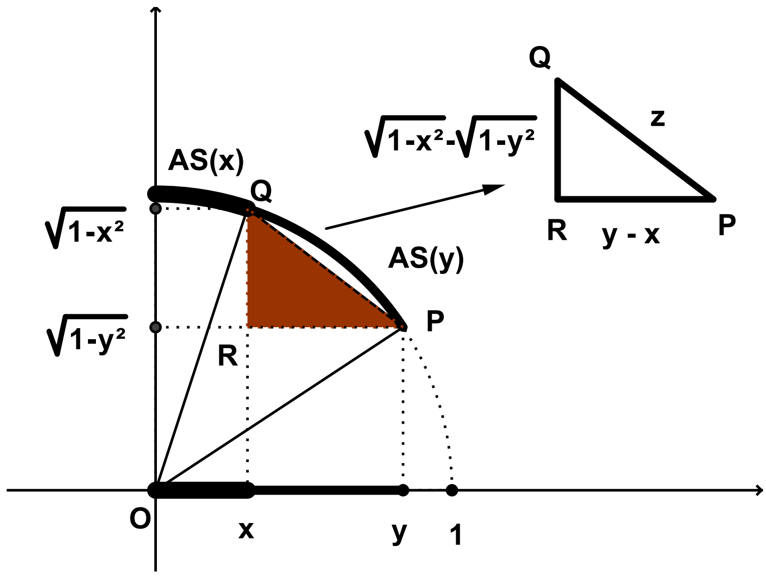


Figure 8: Sixth property

First we would like to work out the length $z = PQ$ of hypothenuse in triangle $PQR$. We use Pythagoras theorem and we get

$$z^2 = 2 - 2xy - 2\sqrt{1-x^2}\sqrt{1-y^2}.$$

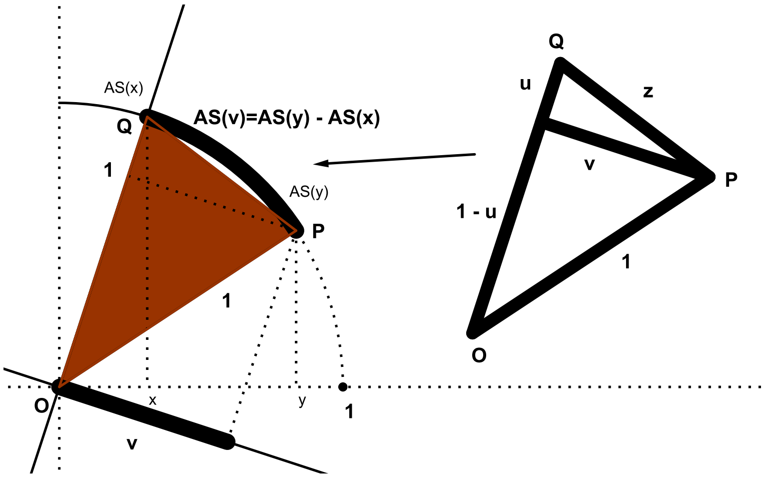


Figure 9: Sixth property

Let us consider different system of axes rotated in order that the second axis goes trough the points $O$ and $Q$, it is described on Fig. 9. From this situation we see that the length of arc $PQ$ of the unit circle is presented by $AS(v)$ for a certain $v$ described on Fig. 9. It holds

$$AS(v) = AS(y) - AS(x)\ .$$

For deriving the required equality it remains only to express the longitude of $v$ as a function of $z$ and consequently as a function of $x$ and $y$. We shall start from Fig. 9 and we use again Pythagoras theorem for both rectangular triangles in triangle $OPQ$

$$u^2 + v^2 = z^2 \quad \text{and} \quad (1-u)^2 + v^2 = 1\ .$$

We get from these equations

$$u = \frac{z^2}{2} \quad \text{and} \quad v = \frac{z}{2}\sqrt{4 - z^2}\ .$$

Finally we calculate

$$\begin{aligned} v &= \frac{1}{2}\sqrt{2 - 2xy - 2\sqrt{1-x^2}\sqrt{1-y^2}} \cdot \sqrt{2 + 2xy + 2\sqrt{1-x^2}\sqrt{1-y^2}} \\ &= \sqrt{x^2 + y^2 - 2x^2y^2 - 2xy\sqrt{1-x^2}\sqrt{1-y^2}} \\ &= \sqrt{x^2(1-y^2) - 2xy\sqrt{1-x^2}\sqrt{1-y^2} + y^2(1-x^2)} \\ &= \sqrt{\left(x\sqrt{1-y^2} - y\sqrt{1-x^2}\right)^2} \\ &= x\sqrt{1-y^2} - y\sqrt{1-x^2} \end{aligned}$$

and the equality of the sixth property is proved.

## 5. Definition and properties of sine and cosine

Now, we shall introduce the sin step by step for the intervals $\langle 0; \frac{\pi}{2}\rangle$, $\langle 0; \pi\rangle$, $\langle -\pi; \pi\rangle$ and $(-\infty; \infty)$.

At first we define the function $S_1 : \langle 0; \frac{\pi}{2}\rangle \to \langle 0; 1\rangle$ as the inverse function of $AS : \langle 0; 1\rangle \to \langle 0; \frac{\pi}{2}\rangle$. We obtain

$$S_1\left(\frac{\pi}{2}\right) = 1 \ , \qquad \lim_{x\to 0^+} \frac{S_1(x)}{x} = 1$$

and

(i)
$$\lim_{x\to \frac{\pi}{2}^-} \frac{1 - S_1(x)}{\frac{\pi}{2} - x} = 0$$

from the properties 2, 4 and 5 of function $AS$.

To derive the summation formula for $x < y$, we put $u = S_1(x)$ and $v = S_1(y)$ into property 6

$$AS(v) - AS(u) = AS(u\sqrt{1-v^2} - v\sqrt{1-u^2}) \ .$$

Functions $S_1$ and $AS$ are mutually inverse, hence $x = AS(u)$, $y = AS(v)$ and

$$S_1(AS(v) - AS(u)) = u\sqrt{1-v^2} - v\sqrt{1-u^2} \ .$$

From here, we obtain the formula

(ii)
$$S_1(y - x) = S_1(x)\sqrt{1 - S_1^2(y)} - S_1(y)\sqrt{1 - S_1^2(x)} \ .$$

Then, we define the function $S_2$ on $\langle 0; \pi\rangle$ by means of $S_1$

$$S_2(x) = \begin{cases} S_1(x) & \text{for } x \in \langle 0; \frac{\pi}{2}\rangle, \\ S_1(\pi - x) & \text{for } x \in (\frac{\pi}{2}; \pi\rangle. \end{cases}$$

Formula (i) shows that the derivatives of $S_2$ at $\frac{\pi}{2}$ from the left and from the right are the same and equal to 0, hence function $S_2$ is smooth here.

Then, the function can be defined on $\langle -\pi; \pi\rangle$ symmetrically around the coordinates origin ($S_3(x) = -S_2(-x)$ for $-\pi \le x \le 0$) and at the end on $(\infty; \infty)$ periodically ($S_4(x) = S_3(x - 2k\pi)$ for $(2k-1)\pi \le x \le (2k+1)\pi$, $k$ integers). This process again preserves the smoothness of the functions $S_3$ and $S_4$ in points $(2k+1)\pi$ because of property 4.

The function $S_4$ is called sine and denoted sin. From symmetry (and periodicity), we have that it is an odd function, i.e., $\sin(-x) = -\sin x$. The second of trigonometric functions cosine is only shifted sine

$$\cos x = \sin\left(\tfrac{\pi}{2} - x\right) \ .$$

It is an even function, so $\cos(-x) = \cos x$ and its graph is symmetrical about the axis $y$.

From here it follows that $\cos x = \sqrt{1 - \sin^2 x}$ on the interval $\langle 0; \frac{1}{2}\pi \rangle$, so we can rewrite summation formula (ii) as

$$\sin(y - x) = \sin y \cos x - \sin x \cos y$$

on this interval. Properties of introduced functions imply this relation even for all $x \in \mathbb{R}$.

Let us summarize that we have two functions defined on real numbers sin, cos and a number $\pi$ for which the following properties hold:

(a) sin is increasing on $\langle 0; \frac{\pi}{2} \rangle$, $\sin(0) = 0$ and $\sin(\frac{\pi}{2}) = 1$,

(b) $\lim\limits_{x \to 0} \frac{\sin x}{x} = 1$,

(c) $\sin(y - x) = \sin y \cos x - \sin x \cos y$ for all $x, y \in \mathbb{R}$.

All other properties of these functions can be derived from here.

Let us notice that the similar way can be used for alternative introduction of generalized trigonometric functions $\sin_{p,q}$ and $\cos_{p,q}$ and number $\pi_{p,q}$, only the function

$$f_{p,q} = \sqrt[p]{1 - x^q}$$

is considered (see [3], [4], [5]).

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