COMMUTATIVE FUNDAMENTAL RELATION IN FUZZY HYPERSEMIGROUPS

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Abstract. We introduce and study the construction of commutative fundamental relation in fuzzy hypersemigroups and investigate its basic properties. We will determine some necessary and sufficient conditions so that this relation is transitive in fuzzy hypersemigroups. Also we show that this relation is transitive in every fuzzy hypergroup.

Keywords and phrases: fuzzy hypersemigroups, fuzzy regular relation, commutative fundamental relation.

1. Introduction

The study of fuzzy hyperstructures is an interesting research topic for fuzzy sets. There are many works on the connections between fuzzy sets and hyperstructures. This can be considered into three groups. A *first* group of papers studies *crisp* hyperoperations defined through fuzzy sets. This study was initiated by Corsini in [2], [3] and then continuated by other researchers. A *second* group of papers concerns the *fuzzy hyperalgebras*. This is a direct extension of the concept of fuzzy algebras. This was intiated by Zahedi in [12]. A *third* group was introduced by Corsini and Tofan in [5]. The basic idea in this group of papers is the following: a multioperation assigns to every pair of elements of S a nonempty subset of S, while a fuzzy multioperation assigns to every pair of elements of S a nonzero fuzzy set on S. This idea was continuated by Sen, Ameri and Chowdhury in [11] where fuzzy semihypergroups are introduced. The fundamental relations are one of the most important and interesting concepts in fuzzy hyperstructures that ordinary algebraic structures are derived from fuzzy hyperstructures by them. Fundamental relation α^* on fuzzy hypersemigroups is studied in [1]. In this paper, continuing our previous work [1], we show that the relation γ^* is the smallest strongly regular equivalence relation on a fuzzy hypersemigroup S such that the quotient $S/\gamma^*$is a commutative semigroup. The structure of this paper is follows:

After an introduction in section 2, we recall some basic notions and results on fuzzy hypersemigroups. In section 3, we introduce the relation γ^* in fuzzy hypersemigroup S and prove that S/γ^* is a commutative semigroup. In section 4, the notion of complete part of a fuzzy hypersemigroup is introduced and we determine some necessary and sufficient conditions so that γ is transitive in fuzzy hypersemigroups, then we will prove that this relation is transitive in every fuzzy hypergroup.

2. Preliminaries

Recall that for a non-empty set S, a fuzzy subset μ of S is a function from S into the real unite interval $[0, 1]$. We denote the set of all nonzero fuzzy subsets of S by $F^*(S)$. Also for fuzzy subsets μ_1 and μ_2 of S, then μ_1 is *smaller* than μ_2 and write $\mu_1 \leq \mu_2$ iff for all $x \in S$, we have $\mu_1(x) \leq \mu_2(x)$. Define $\mu_1 \vee \mu_2$ and $\mu_1 \wedge \mu_2$ as follows: $\forall x \in S, \quad (\mu_1 \vee \mu_2)(x) = max\{\mu_1(x), \mu_2(x)\}$ and $(\mu_1 \wedge \mu_2)(x) = min\{\mu_1(x), \mu_2(x)\}$.

A *fuzzy hyperoperation* on S is a mapping $\circ : S \times S \mapsto F^*(S)$ written as $(a, b) \mapsto a \circ b = ab$. The couple $(S, \circ)$ is called a *fuzzy hypergropoid.*

Definition 2.1 A fuzzy hypergropoid $(S, \circ)$ is called a *fuzzy hypersemigroup* if for all $a, b, c \in S$, $(a \circ b) \circ c = a \circ (b \circ c)$, where for any fuzzy subset μ of S

$$(a \circ \mu)(r) = \begin{cases} \bigvee_{t \in S} ((a \circ t)(r) \wedge \mu(t)), & \mu \neq 0 \\ 0, & \mu = 0 \end{cases}$$

$$(\mu \circ a)(r) = \begin{cases} \bigvee_{t \in S} (\mu(t) \wedge (t \circ a)(r)), & \mu \neq 0 \\ 0, & \mu = 0, \end{cases}$$

for all $r \in S$.

Definition 2.2 Let μ, ν be two fuzzy subsets of a fuzzy hypergropoid $(S, \circ)$. Then we define $\mu \circ \nu$ by $(\mu \circ \nu)(t) = \bigvee\limits_{p,q \in S} (\mu(p) \wedge (p \circ q)(t) \wedge \nu(q))$, for all $t \in S$.

Definition 2.3 A fuzzy hypersemigroup $(S, \circ)$ is called *fuzzy hypergroup* if $x \circ S = S \circ x = \chi_S$, for all $x \in S$, where χ_S is characteristic function of S.

Example 2.4 Consider a fuzzy hyperoperation $\circ$ on a non-empty set S by $a \circ b = \chi_{\{a,b\}}$, for all $a, b \in S$. Then, $(S, \circ)$ is a fuzzy hypersemigroup and fuzzy hypergroup as well.

Theorem 2.5 *Let $(S, \circ)$ be a fuzzy hypersemigroup. Then, $\chi_a \circ \chi_b = a \circ b$, for all $a, b \in S$.*

Definition 2.6 Let $(S, \star)$ and $(T, \circ)$ be two fuzzy hypersemigroups. A mapping $\phi : S \mapsto T$ is said to be a

(i) *fuzzy homomorphism* if $\phi(a \star b) \leq \phi(a) \circ \phi(b),\ (a, b \in S)$.

(ii) *fuzzy good homomorphism* if $\phi(a \star b) = \phi(a) \circ \phi(b),\ (a, b \in S)$.

Definition 2.7 Let ρ be an equivalence relation on a fuzzy hypersemigroup $(S, \circ)$, we define two relations $\overline{\rho}$ and $\overline{\overline{\rho}}$ on $F^*(S)$ as follows: for $\mu, \nu \in F^*(S)$; $\mu\overline{\rho}\nu$ if $\mu(a) > 0$ then there exists $b \in S$ such that $\nu(b) > 0$ and $a\rho b$, also if $\nu(x) > 0$ then there exists $y \in S$, such that $\mu(y) > 0$ and $x\rho y$. $\mu\overline{\overline{\rho}}\nu$ if for all $x \in S$ such that $\mu(x) > 0$ and for all $y \in S$ such that $\nu(y) > 0$, $x\rho y$.

Definition 2.8 An equivalence relation ρ on a fuzzy hypersemigroup $(S, \circ)$ is said to be *(strongly) fuzzy regular* if $a\rho b, a'\rho b'$ implies $a \circ a'\ \overline{\rho}\ b \circ b'(a \circ a'\ \overline{\overline{\rho}}\ b \circ b')$.

3. Commutative fundamental relation

Let $(S, \circ)$ be a fuzzy hypersemigroup. For every map $f : S \to S'$ where S' is a semigroup, we define $f(a \circ b) = \{f(t) : (a \circ b)(t) > 0\}$.
If ρ is a equivalence relation on a fuzzy hypersemigroup $(S, \circ)$, then we consider the following hyperoperation on the quotient set S/ρ as follows:
for every $a\rho, b\rho \in S/\rho$,

$$a\rho \oplus b\rho = \{c\rho : (a' \circ b')(c) > 0, a\rho a', b\rho b'\}$$

Theorem 3.1 *Let $(S, \circ)$ be a fuzzy hypersemigroup and ρ be an equivalence relation on S. Then*

(i) *the relation ρ is fuzzy regular on $(S, \circ)$ iff $(S/\rho, \oplus)$ is a hypersemigroup.*

(ii) *the relation ρ is strongly fuzzy regular on $(S, \circ)$ iff $(S/\rho, \oplus)$ is a semigroup.*

(iii) *if the map $f : S \to S'$, where S' is a semigroup, satisfies in the following condition:*

$$f(a \circ b) = f(a) \circ f(b) \qquad \forall a, b \in S.$$

Then the equivalence relation associated with f is strongly fuzzy regular.

Proof. Straightforward. ∎

Definition 3.2 Let $(S, \circ)$ be a fuzzy hypersemigroup. A *commutative fundamental relation* on $(S, \circ)$ is the smallest equivalence relation ρ on S such that the quotient structure $(S/\rho, \oplus)$ is a commutative semigroup.

Let $(S, \circ)$ be a fuzzy hypersemigroup. We define the relation γ on S in the following way, $\gamma = \bigcup\limits_{n \geq 1} \gamma_n$ where $\gamma_1 = \{(s, s) : s \in S\}$ and for every $n \geq 2$, $a\gamma_n b$ if

$\exists x_1, ..., x_n \in H(n \in \mathbb{N}), \exists \sigma \in S_n : (x_1 \circ \ldots \circ x_n)(a) > 0$ and $(x_{\sigma_1} \circ \ldots \circ x_{\sigma_n})(b) > 0$.

It is clear that γ is symetric and reflexive. We take γ^* to be the transitive closure of γ. Then γ^* is an equivalence relation on S.

Theorem 3.3 *The relation γ^* is a strongly fuzzy regular relation.*

Proof. First we have to show that for each $x, y, a \in S$

$$x\gamma y \Rightarrow xa\overline{\overline{\gamma}}ya, \quad ax\overline{\overline{\gamma}}ay. \tag{$*$}$$

If $x\gamma y$, then there exist $n \in \mathbb{N}$, $(z_1, \ldots, z_n) \in S^n$ and $\sigma \in S_n$ such that $\prod_{i=1}^{n} z_i(x) > 0$, $\prod_{i=1}^{n} z_{\sigma(i)}(y) > 0$. For every $a \in S$, set $a = z_{n+1}$ and suppose that τ be the permutation of S_n such that

$$\tau(i) = \begin{cases} \sigma(i), & \forall i \in \{1, 2, \ldots, n\} \\ n+1, & i = n+1. \end{cases}$$

For every $v, w \in S$ such that $(xa)(v) > 0, (ya)(w) > 0$, we can write

$$\left(\prod_{i=1}^{n+1} z_i\right)(v) = \left[\left(\prod_{i=1}^{n} z_i\right) a\right](v) = \bigvee_{t \in S}\left[\left(\prod_{i=1}^{n} z_i\right)(t) \wedge (ta)(v)\right]$$

For $t = x$, $\left(\prod_{i=1}^{n} z_i\right)(x) > 0$ and $(xa)(v) > 0$, so $\left(\prod_{i=1}^{n+1} z_i\right)(v) > 0$, and

$$\begin{aligned}\left(\prod_{i=1}^{n+1} z_{\tau(i)}\right)(w) &= \left[\left(\prod_{i=1}^{n} z_{\sigma(i)}\right) z_{n+1}\right](w) = \left[\left(\prod_{i=1}^{n} z_{\sigma(i)}\right) a\right](w) \\ &= \bigvee_{t \in S}\left[\left(\prod_{i=1}^{n} z_{\sigma(i)}\right)(t) \wedge (ta)(w)\right].\end{aligned}$$

For $t = y$, $\prod_{i=1}^{n} z_{\sigma(i)}(y) > 0$ and $(ya)(w) > 0$, hence

$$\left(\prod_{i=1}^{n+1} z_{\tau(i)}\right)(w) > 0.$$

Therefore, $v\gamma w$. Thus, $xa\overline{\overline{\gamma}}ya$. In the same way, we see $x\gamma y$ implies that $ax\overline{\overline{\gamma}}ay$.

Now, if $x\gamma^* y$, there exist $m \in \mathbb{N}$ and $w_0 = x, w_1, \ldots, w_m = y \in S$ such that $w_0 = x\gamma w_1 \gamma w_2 \gamma \ldots \gamma w_m = y$, by $(*)$ we have $w_0 a = xa\overline{\overline{\gamma}}w_1 a\overline{\overline{\gamma}}w_2 a\overline{\overline{\gamma}} \ldots \overline{\overline{\gamma}}w_m a = ya$. For each $v, w \in S$ such that $(xa)(v) = (w_0 a)(v) > 0$ and $(ya)(w) = (w_m a)(w) > 0$, by taking $(z_1, z_2, \ldots, z_{m-1}) \in S^{m-1}$ such that $(w_1 a)(z_1) > 0, (w_2 a)(z_2) > 0, \ldots, (w_{m-1}a)(z_{m-1}) > 0$, we have $v\gamma z_1 \gamma z_2 \gamma \ldots \gamma z_{m-1} \gamma w$, and so $v\gamma^* w$. Therefore,

$$x\gamma^* y \Rightarrow xa\overline{\overline{\gamma^*}}ya.$$

Similarly, we can prove $x\gamma^* y$ implies that $ax\overline{\overline{\gamma^*}}ay$, hence γ^* is strongly fuzzy regular. ∎

Corollary 3.4 *Let S be a fuzzy hypersemigroup. Then the quotient S/γ^* is a commutative semigroup.*

Proof. Since γ^* is a strongly fuzzy regular relation, S/γ^* is a semigroup under the following operation

$$\gamma^*(x_1) \oplus \gamma^*(x_2) = \gamma^*(z), \forall z \in S \ s.t \ \ (x_1x_2)(z) > 0.$$

Consider $\sigma \in S_2$, such that $\sigma(1) = 2$. Then for every $z, w \in S$ such that $(x_1x_2)(z) > 0$ and $(x_{\sigma(1)}x_{\sigma(2)})(w) > 0$, we have $z\gamma w$, thus $z\gamma^* w$ and we obtain

$$\gamma^*(x_1) \oplus \gamma^*(x_2) = \gamma^*(z) = \gamma^*(w) = \gamma^*(x_2) \oplus \gamma^*(x_1).$$

So S/γ^* is a commutative semigroup. ∎

Theorem 3.5 *The relation γ^* is the commutative fundamental relation on fuzzy hypersemigroup $(S, \circ)$.*

Proof. Let ρ be a strongly fuzzy regular equivalence relation on $(S, \circ)$ such that S/ρ is a commutative semigroup. The canonical map $\phi : S \to S/\rho$ satisfies in following condition

$$\phi(xy) = \phi(x) \oplus \phi(y) \qquad \forall x, y \in S$$

Now if $x\gamma y$, then there exist $n \in \mathbb{N}$, $(z_1, \ldots, z_n) \in S^n$ and $\sigma \in S_n$ such that

$$\left(\prod_{i=1}^{n} z_i\right)(x) > 0 \ \text{ and } \ \left(\prod_{i=1}^{n} z_{\sigma(i)}\right)(y) > 0.$$

S/ρ is a semigroup, thus

$$\phi(x) = \prod_{i=1}^{n} \phi(z_i) \ \text{ and } \ \phi(y) = \prod_{i=1}^{n} \phi(z_{\sigma(i)}).$$

By commutativity of S/ρ, we have $\phi(x) = \phi(y)$ and $x\rho y$. Thus $\gamma \subseteq \rho$. Now, if $x\gamma^* y$, there exist $m \in \mathbb{N}$ and $(u_0 = x, u_1, \ldots, u_m = y) \in S^{m+1}$ such that, $x\gamma u_1 \gamma u_2 \gamma \ldots \gamma u_{m-1} \gamma y$. Therefore, $x\rho u_1 \rho u_2 \rho \ldots \rho u_{m-1} \rho y$. Since ρ is transitive, we conclude $x\rho y$. So $\gamma^* \subseteq \rho$. ∎

4. Complete parts and transitivity of γ

In the following, we will determine some necessary and sufficient conditions so that the relation γ is transitive.

Definition 4.1 Let S be a fuzzy hypersemigroup and M be a non-empty subset of S. We say that M is a complete part of S, if it satisfies in this condition: for every $n \in \mathbb{N}$, $(z_1, \ldots, z_n) \in S^n$, when there exists $z \in M$ such that $\left(\prod_{i=1}^{n} z_i\right)(z) > 0$, then for every $w \in S - M$ and $\sigma \in S_n$, we have $\left(\prod_{i=1}^{n} z_{\sigma(i)}\right)(w) = 0$.

Lemma 4.2 *Let S be a fuzzy hypersemigroup and M be a non-empty subset of S. The following conditions are equivalent:*

(i) *M is complete part of S;*

(ii) *if $x \in M$ and $x\gamma y$, then $y \in M$;*

(iii) *if $x \in M$ and $x\gamma^* y$, then $y \in M$.*

Proof. (i) $\to$ (ii) Let $x, y \in S$, $x \in M$ and $x\gamma y$. Then, there exist $n \in \mathbb{N}$, $(z_1, ..., z_n) \in S^n$, and $\sigma \in S_n$ such that

$$\left(\prod_{i=1}^{n} z_i\right)(x) > 0 \quad \text{and} \quad (\prod_{i=1}^{n} z_{\sigma(i)})(y) > 0.$$

Since M is complete part, $\left(\prod_{i=1}^{n} z_{\sigma(i)}\right)(y) > 0$ implies that $y \in M$.

(ii) $\to$ (iii) Let $x, y \in S$, $x \in M$ and $x\gamma^* y$. Then there exist $m \in \mathbb{N}$ and $(u_0 = x, u_1, ..., u_m = y) \in S^{m+1}$ such that $x\gamma u_1 \gamma u_2 \gamma \ldots \gamma u_{m-1} \gamma y$. Since $x \in M$, by applying (ii) m times, we see $y \in M$.

(iii) $\to$ (i) Let $x \in M$ and $(z_1, ..., z_n) \in S^n$ such that $\left(\prod_{i=1}^{n} z_i\right)(x) > 0$. Then, if there exist $\sigma \in S_n$ and $y \in S$ such that $\left(\prod_{i=1}^{n} z_{\sigma(i)}\right)(y) > 0$, we have $x\gamma y$, therefore, $x\gamma^* y$ and $x \in M$. By (iii) we obtain $y \in M$, whence M is a complete part of S. ∎

Now, we are going to introduce new notations:

Let S be a fuzzy hypersemigroup and $x \in S$, we set;

$$\begin{aligned}
T_n(x) &= \left\{(x_1, x_2, ..., x_n) \in S^n \mid \left(\prod_{i=1}^{n} x_i\right)(x) > 0\right\} \\
P_n(x) &= \bigcup \left\{\prod_{i=1}^{n} x_{\sigma(i)} \mid \sigma \in S_n, (x_1, x_2, ..., x_n) \in T_n(x)\right\} \\
P(x) &= \bigcup_{n \geq 1} P_n(x).
\end{aligned}$$

By the above notations, we have the following results:

Lemma 4.3 *For every $x, y \in S$,*

$$x\gamma y \Leftrightarrow P(x)(y) > 0.$$

Proof.

$$\begin{aligned} x\gamma y &\Leftrightarrow \exists (n \in \mathbb{N}, (x_1, ..., x_n) \in S^n, \sigma \in S_n) : \left(\prod_{i=1}^{n} x_i\right)(x) > 0, \left(\prod_{i=1}^{n} x_{\sigma(i)}\right)(y) > 0 \\ &\Leftrightarrow \exists n \in \mathbb{N}, P_n(x)(y) > 0 \\ &\Leftrightarrow P(x)(y) > 0. \end{aligned}$$

Theorem 4.4 *If S is a fuzzy hypersemigroup, then the following conditions are equivalent:*

(i) *γ is transitive,*

(ii) *for every $x \in S, \gamma^*(x) = \{y \in S : P(x)(y) > 0\}$*

(iii) *for every $x \in S$, $\gamma(x)$ is a complete part.*

Proof. (i) $\to$ (ii) For every $(x, y) \in S^2$, by Lemma 4.3 we have

$$y \in \gamma^*(x) \Leftrightarrow x\gamma^* y \Leftrightarrow x\gamma y \Leftrightarrow P(x)(y) > 0.$$

(ii) $\to$ (iii) By Lemma 4.2, if $\varnothing \neq M \subseteq S$, then M is a complete part of S if and only if M is the union of equivalence classes modulo γ^*. In particular, $\gamma^*(x)$ is a $\gamma-$ part.

(iii) $\to$ (i) If $x\gamma y$ and $y\gamma z$, then there exist $n, m \in \mathbb{N}$, $(x_1, \ldots, x_n) \in T_n(x)$, $(y_1, \ldots, y_m) \in T_m(y)$, $\sigma \in S_n$ and $\tau \in S_m$ such that $\left(\prod_{i=1}^{n} x_{\sigma(i)}\right)(y) > 0$ and $\left(\prod_{i=1}^{m} y_{\tau(i)}\right)(z) > 0$. Since $\gamma(x)$ is a complete part, $\left(\prod_{i=1}^{n} x_{\sigma(i)}\right)(y) > 0$ implies that $y \in \gamma(x)$. We have $\left(\prod_{i=1}^{m} y_i\right)(y) > 0$ and $\gamma(x)$ is a complete part, therefore $\left(\prod_{i=1}^{m} y_{\tau(i)}\right)(z) > 0$ implies that $z \in \gamma(x)$. This means $z\gamma x$. ∎

Remark 1. If S is a commutative fuzzy hypersemigroup, then the relation γ is equal to α in [1]. We proved in Ref [1], the relation α is transitive in every fuzzy hypergroup, hence γ is transitive in every commutative fuzzy hypergroup.

Remark 2. If S is a fuzzy hypergroup, then S/γ^* is a group. We define $\omega_S = \phi^{-1}(1_{S/\gamma^*})$, in which $\phi : S \to S/\gamma^*$ is the canonical projection.

Lemma 4.5 *If S is a fuzzy hypergroup and M is a non-empty subset of S, then*

(i) $\phi^{-1}(\phi(M)) = \{x \in S : (\omega_S M)(x) > 0\} = \{x \in S : (M\omega_S)(x) > 0\}$

(ii) *If M is a complete part of S, then $\phi^{-1}(\phi(M)) = M$.*

Proof. (i) Let $x \in S$ and $(t, y) \in \omega_S \times M$ such that $(ty)(x) > 0$, so $\phi(x) = \phi(t) \oplus \phi(y) = 1_{S/\gamma^*} \oplus \phi(y) = \phi(y)$, therefore $x \in \phi^{-1}(\phi(y)) \subset \phi^{-1}(\phi(M))$. Conversely, for every $x \in \phi^{-1}(\phi(M))$, there exists $b \in M$ such that $\phi(x) = \phi(b)$. By reproducibility, $a \in S$ exists such that $(ab)(x) > 0$, so $\phi(b) = \phi(x) = \phi(a) \oplus \phi(b)$. This implies $\phi(a) = 1_{S/\gamma^*}$ and $a \in \phi^{-1}(1_{S/\gamma^*}) = \omega_S$. Therefore $(\omega_S M)(x) > 0$. In the same way, we can prove that $\phi^{-1}(\phi(M)) = \{x \in S : (M\omega_S)(x) > 0\}$.

(ii) We know $M \subseteq \phi^{-1}(\phi(M))$. If $x \in \phi^{-1}(\phi(M))$, then there exists $b \in M$ such that $\phi(x) = \phi(b)$. Therefore $x \in \gamma^*(x) = \gamma^*(b)$. Since M is a complete part of S and $b \in M$, by Lemma 4.2 we conclude $\gamma^*(b) \subseteq M$ and $x \in M$. ∎

The next theorem shows that the relation γ is transitive in every fuzzy hypergroup.

Theorem 4.6 *If S is a fuzzy hypergroup, then the relation γ is transitive.*

Proof. We know if S is a fuzzy hypergroup, then S/γ^* is a group, let $z \in \omega_S$. Now, if $x\gamma^* y$, then $x \in \phi^{-1}(\phi(y)) = \{u \in S \mid (y\omega_S)(u) > 0\}$, thus there exists $t \in \omega_S$ such that $(yt)(x) > 0$. Since $t \in \omega_S$, there exist $n \in \mathbb{N}$, $(x_1, ..., x_n) \in S^n$ and $\sigma \in S_n$ such that $\left(\prod_{i=1}^{n} x_i\right)(z) > 0$ and $\left(\prod_{i=1}^{n} x_{\sigma(i)}\right)(t) > 0$.

Now, by the reproducibility of S , there is $v \in S$ such that $(vz)(y) > 0$, and since $1_{S/\gamma^*} \oplus 1_{S/\gamma^*} = 1_{S/\gamma^*}$, there is $w \in \omega_S$ such that $(wz)(z) > 0$. Moreover, since $\{z, w\} \subset \omega_S$ and the relation α is transitive in every fuzzy hypergroup (see [1]), there is an m- tuple $(y_1, ..., y_m) \in S^n$ such that

$$\left(\prod_{i=1}^{m} y_i\right)(z) > 0 \quad \text{and} \quad \left(\prod_{i=1}^{m} y_i\right)(w) > 0.$$

Now,

$$\left(\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(z) = \bigvee_{p,q \in S} \left[\left(\prod_{i=1}^{m} y_i\right)(p) \wedge \left(\prod_{i=1}^{n} x_i\right)(q) \wedge (pq)(z)\right].$$

Considering $p = w$ and $q = z$, we obtain $\left(\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(z) > 0$. From this, and by setting $k = z$ in the following, we have

$$\left(w \prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(z) = \bigvee_{k \in S} \left[(wk)(z) \wedge \left(\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(k)\right] > 0.$$

Again, by last statement, and setting $g = z$, we obtain

$$\left(vw\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(y) = \bigvee_{g\in S}\left[(vg)(y) \wedge \left(w\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_i\right)(g)\right] > 0. \tag{1}$$

On the other hand, if we set $p = y$ and $q = t$ in

$$\left(vw\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_{\sigma(i)}\right)(x) = \bigvee_{p,q\in S}\left[\left(vw\prod_{i=1}^{m} y_i\right)(p) \wedge \left(\prod_{i=1}^{n} x_{\sigma(i)}\right)(q) \wedge (p,q)(x)\right],$$

then we obtain

$$\left(vw\prod_{i=1}^{m} y_i\right)(y) = \bigvee_{k\in S}\left[(vk)(y) \wedge \left(w\prod_{i=1}^{m} y_i\right)(k)\right]$$

by setting $k = z$ in the above, and $f = z$ in the following, we have

$$\left(w\prod_{i=1}^{m} y_i\right)(z) = \bigvee_{f\in S}\left[(wf)(z) \wedge \left(\prod_{i=1}^{m} y_i\right)(f)\right] > 0.$$

Therefore,

$$\left(vw\prod_{i=1}^{m} y_i \prod_{i=1}^{n} x_{\sigma(i)}\right)(x) > 0. \tag{2}$$

From (1) and (2), we conclude that $x\gamma y$, thus γ is transitive. ■

Conclusions.

We introduced the concepts of commutative fundamental relation and complete parts of a fuzzy hypersemigroup. We determined some conditions so that this relation is transitive in fuzzy hypersemigroups and we proved that this relation is transitive in every fuzzy hypergroup. We will study about fundamental relation in fuzzy hyperrings.

Acknowledgements. The author is grateful to the referee(s) for reading the paper.

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Accepted: 18.01.2016