COLORING OF BIFUZZY GRAPHS

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Abstract. In this paper, we introduce coloring function of a bifuzzy edge graph $G = (V, B)$ using (α, β)-cuts of G and determine the chromatic number of G. We compute k-coloring of bifuzzy graph $\tilde{G} = (V, A, B)$ based on the family of bifuzzy sets on V. We establish strong coloring of a bifuzzy graph and compute strong chromatic number. We also present a new channel assignment problem by using a bifuzzy graph.

Keywords: bifuzzy sets, bifuzzy graphs, chromatic number, bifuzzy number.
Mathematics Subject Classification 2000: 03E72, 05C72, 05C78, 05C99.

1. Introduction

In 1983, Atanassov [7] introduced the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets [28]. Atanassov added a new component(which determines the degree of non-membership) in the definition of fuzzy set. The fuzzy sets give the degree of membership of an element in a given set (and the non-membership degree equals one minus the degree of membership), while intuitionistic fuzzy sets give both a degree of membership and a degree of non-membership which are more-or-less independent from each other, the only requirement is that the sum of these two degrees is not greater than 1. Gerstenkorn and Mańko [14] renamed the intuitionistic fuzzy sets as bifuzzy sets. The concept of bifuzzy set can be viewed as an alternative approach in case where available information is not sufficient to define the impreciseness by the conventional fuzzy set. Intuitionistic fuzzy sets are being studied and used in different fields of science including computer science, mathematics, medical and engineering [12], [15], [19]. The concept of fuzzy numbers and its arithmetic operations were first introduced and investigated by Chang and Zadeh [11]. Burillo et al. [10] proposed the definition of intuitionistic fuzzy number. Mahapatra and Mahapatra [21] defined trapezoidal intuitionistic fuzzy number and arithmetic operations of trapezoidal intuitionistic fuzzy number based on (α, β)-cut method. Based on Zadeh's fuzzy relations [29] Kaufmann defined in

[17] a fuzzy graph. Rosenfeld [26] described the structure of fuzzy graphs obtaining analogs of several graph theoretical concepts. Bhattacharya [9] gave some remarks on operations on fuzzy graphs introduced by Mordeson and Nair in [23]. Kóczy [19] discussed fuzzy graphs by considering fuzzy edge set with crisp vertex set, fuzzy vertex set with crisp edge set, fuzzy vertex and edge set or crisp vertices and edges with fuzzy connectivity and crisp graph with fuzzy weights. The concept of chromatic number of fuzzy graph was introduced by Muñoz et. al. [24]. The authors considered fuzzy graphs with crisp vertex set, i.e., fuzzy graphs $G = (\sigma, \mu)$ for which $\sigma(v) = 1 \ \forall \ v \in V$, and edges with membership degree in $[0, 1]$. Eslahchi and Onagh [13] introduced coloring of fuzzy graphs $G = (\sigma, \mu)$, that fuzzy coloring partitions the vertex set into different color classes and the number of distinct color classes is the fuzzy chromatic number. Shannon and Atanassov [27] introduced the concept of intuitionistic fuzzy graphs and investigated some of their properties. Parvathi et al. defined operations on intuitionistic fuzzy graphs in [25]. Akram et al. [1]–[4] introduced many new concepts, including strong intuitionistic fuzzy graphs, intuitionistic fuzzy trees, intuitionistic fuzzy hypergraphs, and intuitionistic fuzzy digraphs in decision support systems. In this paper, we introduce coloring function of a bifuzzy edge graph $G = (V, B)$ using (α, β)-cuts of G and determine the chromatic number of G. We compute k-coloring of bifuzzy graph $\tilde{G} = (V, A, B)$ based on the family of bifuzzy sets on V. We establish strong coloring of a bifuzzy graph and compute strong chromatic number. We also present a new channel assignment problem by using a bifuzzy graph.

2. Prelimilaries

In this section, we review some basic definitions which will be used in our next sections.

Definition 2.1 [28],[29] A *fuzzy subset* μ on a set X is a mapping $\mu : X \to [0, 1]$. A mapping $\nu : X \times X \to [0, 1]$ is called a *fuzzy relation* on X if $\nu(x, y) \leq \min\{\mu(x), \mu(y)\}$ for all $x, y \in X$.

Definition 2.2 [24] If $G = (V, \mu)$ is a fuzzy graph, where $V = \{1, 2, ..., n\}$ and μ is a fuzzy relation on the set of all subsets of $V \times V$. Assume $I = \alpha \cup \{0\}$, where $\alpha = \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\}$ is the fundamental set (level set) of G. For each $\alpha \in I$, G_α denotes the crisp graph $G_\alpha = (V, E_\alpha)$, where $E_\alpha = \{ij \mid 1 \leq i < j \leq n, \mu(i, j) \geq \alpha\}$ and $\chi_\alpha = \chi(G_\alpha)$ denotes the chromatic number of crisp graph G_α. The chromatic number of fuzzy graph G is the fuzzy number $\chi(G) = \{(x, \nu(x)) \mid x \in X\}$, where $X = \{1, 2, ..., |V|\}$, $\nu(x) = \sup\{\alpha \in I \mid x \in A_\alpha\}$, $\forall x \in X$ and $A_\alpha = \{1, 2, ..., \chi_\alpha\}$, $\forall \alpha \in I$.

Definition 2.3 [13] A family $\Gamma = \{\gamma_1, \gamma_2, ..., \gamma_k\}$ of fuzzy sets on V is called a *k-fuzzy coloring* of $G = (V, \sigma, \mu)$ if

(a) $\bigvee \Gamma = \sigma$,

(b) $\gamma_i \bigwedge \gamma_j = 0$,

(c) for every strong edge xy of G, $\min\{\gamma_i(x), \gamma_i(y)\} = 0$ $(0 \le i \le k)$.
The least value of k for which G has a *k-fuzzy coloring*, denoted by $\chi^f(G)$, is called the *fuzzy chromatic number* of G.

Gerstenkorn and Mańko [14] re-named the intuitionistic fuzzy sets as bifuzzy sets.

Definition 2.4 [7] A mapping $A = (\mu_A, \nu_A) : X \to [0,1] \times [0,1]$ is called a *bifuzzy set* in X if $\mu_A(x) + \nu_A(x) \le 1$ $\forall x \in X$, where the mappings $\mu_A : X \to [0,1]$ and $\nu_A : X \to [0,1]$ denote the degree of membership (namely $\mu_A(x)$) and the degree of non-membership (namely $\nu_A(x)$) of each element $x \in X$ to A, respectively.

Definition 2.5 [10] A bifuzzy set $A = \{(x, \mu_A(x)), \nu_A(x) | x \in X\}$ is called a bifuzzy number (IFN) if the following holds:

(i) A is a bifuzzy subset of real line R,

(ii) normal, i.e., there is any $x_0 \in R$ such that $\mu_A(x_0) = 1$, $\nu_A(x_0) = 0$ (x_0 is called the mean value of A),

(iii) A is convex set for the membership function $\mu_A(x)$, i.e., $\mu_A(\lambda x_1 + (1-\lambda)x_2) \ge \min\{\mu_A(x_1), \mu_A(x_2)\}$, $\forall\, x_1, x_2 \in R$, $\forall\, \lambda \in [0,1]$,

(iv) A is concave set for non-membership function $\nu_A(x)$, i.e., $\nu_A(\lambda x_1 + (1-\lambda)x_2) \le \max\{\nu_A(x_1), \nu_A(x_2)\}$, $\forall\, x_1, x_2 \in R, \forall\, \lambda \in [0,1]$.

Definition 2.6 [22] Let A=(V, μ_A, ν_A) be a bifuzzy set and $\alpha, \beta \in [0,1]$, $\alpha + \beta \le 1$. Then we call $A^{(\alpha,\beta)} = \{x \in V |\ \mu_A(x) \ge \alpha,\ \nu_A(x) \le \beta\}$ the $(\alpha, \beta)-$*cut set* of A.

Definition 2.7 [8] Let $A = (\mu_A, \nu_A)$ be a bifuzzy set on V. The *support* of A is defined as $\mathrm{supp}(A) = \{x \in V |\ \mu_A(x) \ne 0, \nu_A(x) \ne 0\}$.

3. Coloring of bifuzzy graph with crisp nodes

In this section, we present coloring function of bifuzzy graphs with crisp nodes and bifuzzy edges.

Definition 3.8 A *bifuzzy graph* is defined as a pair $G = (V, B)$ such that

(i) V is the crisp set of nodes,

(ii) the functions $\mu_B : E \subseteq V \times V \to [0,1]$ and $\nu_B : E \subseteq V \times V \to [0,1]$ are defined by $\mu_B(x,y) \le \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_B(x,y) \ge \max\{\nu_A(x), \nu_A(y)\}$ such that $0 \le \mu_B(x,y) + \nu_B(x,y) \le 1$ $\forall$ $\{x,y\} \in E$.

We call $B = (\mu_B, \nu_B)$ the bifuzzy edge set.

We consider bifuzzy graphs with crisp vertex set, i.e., bifuzzy graphs $G = (A, B)$ for which $\mu_A(x) = 1$, $\nu_A(x) = 0$ $\forall$ $x \in V$, and edges with membership and non-membership degrees in $[0,1]$.

Example 3.1 Consider a simple graph $G^* = (V, E)$ such that $V = \{u_1, u_2, u_3, u_4\}$ and $E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}$. Let B be a bifuzzy set on E defined by

$$B = \{(u_1u_2, 0.3, 0.1), (u_2u_3, 0.4, 0.1), (u_3u_4, 0.2, 0.1), (u_4u_1, 0.3, 0.2)\}.$$

$G = (V, B)$ is a bifuzzy graph of $G^* = (V, E)$ with crisp set of vertices and bifuzzy set of edges as shown in Figure 1.

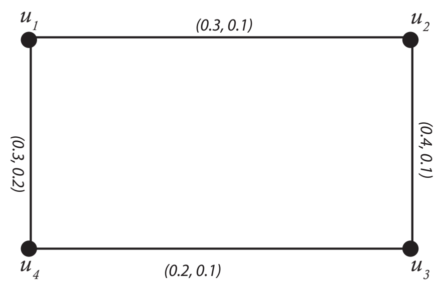

Figure 1: Bifuzzy graph $G = (V, B)$

Definition 3.9 For any $\alpha_i, \beta_i \in I$ such that $\alpha_i + \beta_i \leq 1$ and $I = \alpha \cup \beta \cup \{0\} \cup \{1\}$, where $\alpha = \{\alpha_1 < \alpha_2 < \cdots < \alpha_k\}$ and $\beta = \{\beta_1 < \beta_2 < \cdots < \beta_k\}$ gives fundamental level set of bifuzzy(IF) graph $G = (V, B)$. For each $(\alpha_i, \beta_i) \in I$, $G_{(\alpha_i,\beta_i)} = (V, B^{(\alpha_i,\beta_i)})$ denotes the crisp graph where $B^{(\alpha_i,\beta_i)} = \{(u, v) \in V \times V \mid \mu_B(u, v) \geq \alpha_i, \nu_B(u, v) \leq \beta_i\}$.

Let $\{G_{(\alpha_i,\beta_i)} = (V, B^{(\alpha_i,\beta_i)}) \mid (\alpha_i, \beta_i) \in I\}$ be the family of (α_i, β_i)-cuts sets of bifuzzy graph G. Any (crisp) $k-$coloring $C^k_{(\alpha_i,\beta_i)}$ can be defined on $G_{(\alpha_i,\beta_i)}$. The k-coloring function of bifuzzy graph is defined through these (α_i, β_i)-cuts.

Example 3.2 Consider a simple graph $G^* = (V, E)$ such that $V = \{v_1, v_2, v_3, v_4\}$ and $E = \{v_1v_2, v_2v_3, v_3v_4, v_4v_2\}$. For bifuzzy subgraph G of G^* with crisp set of nodes and bifuzzy set of edges as shown in Figure 2.

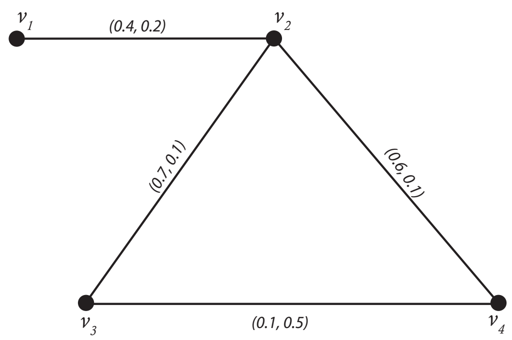

Figure 2: Bifuzzy graph $G = (V, B)$

The $(0.6, 0.1)$-cut set of G is a crisp graph $G_{(0.6,0.1)} = (V, B^{(0.6,0.1)})$, where $B^{(0.6,0.1)} = \{v_2v_3, v_4v_2\}$ as shown in Figure 3.

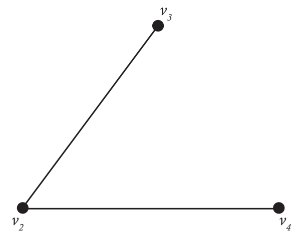

Figure 3: $G_{(0.6,0.1)}$

Definition 3.10 For a bifuzzy graph $G = (V, B)$, its *chromatic number* is a bifuzzy number

$$\chi(G) = \{(x, \nu(x), \omega(x)) \mid x \in X\},$$

where $X = \{1, 2, ..., |V|\}$, $\nu(x) = \sup\{\alpha_i \in I \mid x \in A_{\alpha_i,\beta_i}\}$, $\omega(x) = \{\beta_i \in I \mid \nu(x) = \alpha_i,\ x \in A_{\alpha_i,\beta_i}\}$ $\forall x \in X$ and $A_{\alpha_i,\beta_i} = \{1, 2, ..., \chi_{(\alpha_i,\beta_i)}\}$, $\forall(\alpha_i, \beta_i) \in I$.

The chromatic number of a bifuzzy graph is a normalized bifuzzy number whose normalized value is associated with the graph having no edge. The bifuzzy graph coloring problem consists of determining the chromatic number of a bifuzzy graph and an associated coloring function. In our method, for any level (α_i, β_i), the minimum number of colors needed to color the crisp graph $G_{(\alpha_i,\beta_i)}$ will be computed. In this way, the bifuzzy chromatic number $\chi(G)$ will be defined as a bifuzzy number through (α_i, β_i)-cuts.

4. Coloring of bifuzzy graphs with bifuzzy nodes

In Section 3, we introduced the coloring function and chromatic number of bifuzzy graphs $G = (V, B)$, V is crisp set of vertices ($\mu_A(v) = 1, \nu_A(v) = 0\ \forall\ v \in V$), B is bifuzzy set of edges. Two vertices u and v in a graph G are adjacent then the edge uv is called strong edge and weak otherwise. In the following definition, the chromatic number of bifuzzy graph is a number not a bifuzzy number.

Definition 4.11 [3] A *bifuzzy graph* with underlying set V is defined to be a pair $\tilde{G} = (V, A, B)$ such that

(i) $\mu_A : V \to [0, 1]$ and $\nu_A : V \to [0, 1]$ denote the degree of membership and non-membership of the element $x \in V$, respectively such that

$$\mu_A(x) + \nu_A(x) \leq 1,\ \forall\ x \in V,$$

(ii) the functions $\mu_B : E \subseteq V \times V \to [0, 1]$ and $\nu_B : E \subseteq V \times V \to [0, 1]$ are defined by $\mu_B(x, y) \leq \min\{\mu_A(x), \mu_A(y)\}$ and $\nu_B(x, y) \geq \max\{\nu_A(x), \nu_A(y)\}$ such that

$$0 \leq \mu_B(x, y) + \nu_B(x, y) \leq 1,\ \forall xy \in E.$$

We call A the bifuzzy vertex set of V, B the bifuzzy edge set of E, respectively. $\tilde{G} = (V, A, B)$ is a *bifuzzy graph* of $G^* = (V, E)$.

The level set of bifuzzy set A is defined as

$$L_A = \{(\alpha, \beta) \mid \mu_A(u) = (\alpha, \beta) \text{for some } u \in V\},$$

and the level set of bifuzzy set B is defined as

$$L_B = \{(\alpha, \beta) \mid \mu_B(u, v) = (\alpha, \beta) \text{ for some } (u, v) \in E\}.$$

Definition 4.12 The fundamental set of bifuzzy graph $\tilde{G} = (V, A, B)$ is defined as $L = L_A \cup L_B$, for each $(\alpha, \beta) \in L$, $\tilde{G}_{\alpha,\beta}$ denotes crisp graph $\tilde{G}_{\alpha,\beta} = (A^{\alpha,\beta}, B^{\alpha,\beta})$ where

$$A^{\alpha,\beta} = \{v \in V \mid \mu_A(v) \geq \alpha, \lambda_A(v) \leq \beta\},$$
$$B^{\alpha,\beta} = \{(u, v) \in V \times V \mid \mu_B(u, v) \geq \alpha, \lambda_B(u, v) \leq \beta\}.$$

Example 4.3 Consider a simple graph $G^* = (V, E)$ such that

$$V = \{u_1, u_2, u_3, u_4\} \text{ and } E = \{u_1u_2, u_2u_3, u_3u_4, u_4u_1\}.$$

For bifuzzy graph $\tilde{G}$ of G^*, as shown in Figure 4, $A^{(0.2,0.7)} = \{u_1, u_2, u_3, u_4\}$ and $B^{(0.2,0.7)} = \{u_1u_2, u_2u_3\}$.

So, $\tilde{G}_{(0.2,0.7)} = (A^{(0.2,0.7)}, B^{(0.2,0.7)})$ is $(0.2, 0.7)$-cut set of $\tilde{G}$.

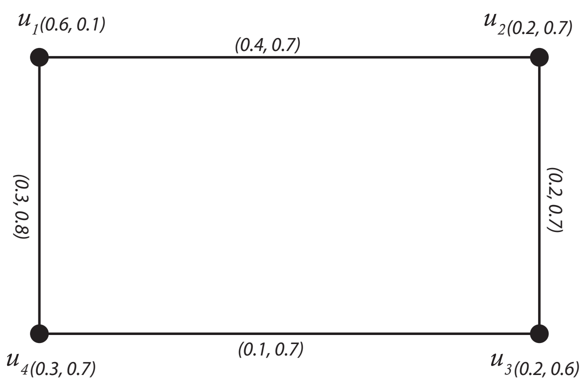

Figure 4: Bifuzzy graph $\tilde{G} = (V, A, B)$

Definition 4.13 A family $\Gamma = \{a_1, a_2, \cdots, a_k\}$ of bifuzzy sets on V is called a *k-coloring* of bifuzzy graph $\tilde{G} = (V, A, B)$ if

(a) $\bigvee \Gamma = A$,

(b) $a_i \bigwedge a_j = 0$,

(c) for every strong edge xy of $\tilde{G}$, $\min\{a_i(x), a_i(y)\} = 0$ $(0 \leq i \leq k)$.

The least value of k for which $\tilde{G}$ has a k-coloring of $\tilde{G} = (V, A, B)$, denoted by $\chi^{if}(\tilde{G})$, is called the bifuzzy chromatic number of $\tilde{G}$.

In Definition 4.3, bifuzzy coloring means that one of the vertices is having color 0 in particular color class and support set for each bifuzzy set a_i, $1 \leq i \leq k$ is an independent set of bifuzzy graph $\tilde{G}$. As in crisp coloring bifuzzy coloring partitioned the set of vertices into independent sets of vertices.

Example 4.4 Consider the following bifuzzy graph $\tilde{G} = (V, A, B)$ as shown in Figure 5.

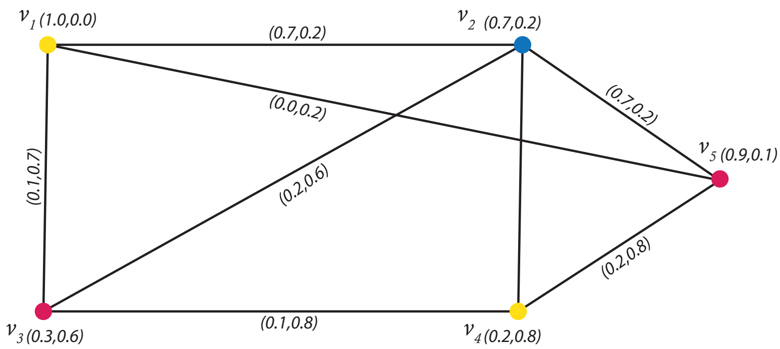

Figure 5: Intuitionistic fuzzy graph $\tilde{G} = (V, A, B)$

The bifuzzy coloring is $\Gamma = \{a_1, a_2, a_3\}$,

$$a_1(v_i) = \begin{cases} (1.0, 0.0) & \text{i=1;} \\ (0.2, 0.8) & \text{i=4;} \\ 0 & \text{otherwise,} \end{cases} \qquad a_2(v_i) = \begin{cases} (0.3, 0.6) & \text{i=3;} \\ (0.9, 0.1) & \text{i=5;} \\ 0 & \text{otherwise,} \end{cases}$$

$$a_3(v_i) = \begin{cases} (0.7, 0.2) & \text{i=2;} \\ 0 & \text{otherwise.} \end{cases} \quad \text{Hence } \chi^{if}(\tilde{G}) = 3.$$

Remark 4.1 If a bifuzzy graph $G = (V, B)$ with crisp set of vertices and bifuzzy set of edges, then there exist $(\alpha, \beta) \in I$ such that $\chi^{if}(G) = \chi_{\alpha,\beta}$. For bifuzzy graph $\tilde{G} = (V, A, B)$, it is possible that there does not exist $(\alpha, \beta) \in I$ such that $\chi^{if}(\tilde{G}) \neq \chi_{\alpha,\beta}$.

Example 4.5 Consider a bifuzzy graph $\tilde{G} = (V, A, B)$, as shown in Figure 6, such that

$$A = \{(v_1, 0.5, 0.0), (v_2, 0.7, 0.2), (v_3, 0.7, 0.0), (v_4, 0.5, 0.2)\},$$
$$B = \{(v_1v_2, 0.4, 0.2),\ (v_1v_4, 0.5, 0.2), (v_1v_3, 0.3, 0.1), (v_3v_4, 0.5, 0.2)\}.$$

We see that

$$\chi_{0.5,0.0} = \chi_{0.7,0.0} = \chi_{0.7,0.2} = 1, \chi_{0.3,0.1} = \chi_{0.5,0.2} = \chi_{0.4,0.2} = 2,$$

but $\chi^{if}(\tilde{G}) = 3$.

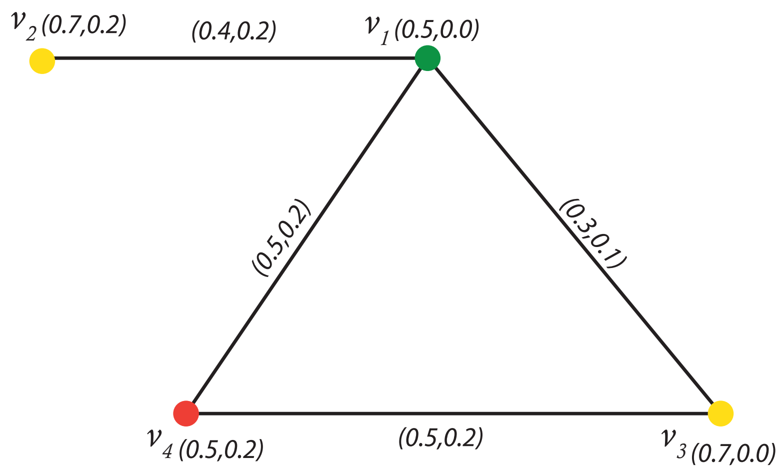

Figure 6: Bifuzzy graph $\tilde{G} = (V, A, B)$

When a fundamental set of bifuzzy set $\tilde{G}$ have some $(\alpha, \beta) \in L$ such that $\chi_{\alpha,\beta} = \chi^{if}(\tilde{G})$, then the following definition and theorem must hold.

Definition 4.14 The *chromatic number* of bifuzzy graph $\tilde{G} = (V, A, B)$ is defined as $\chi(\tilde{G}) = \max\{\chi_{\alpha,\beta} | \ (\alpha, \beta) \in L\}$, where $\chi_{\alpha,\beta} = \chi(\tilde{G}_{\alpha,\beta})$.

Theorem 4.1 *For a bifuzzy graph* $\tilde{G} = (V, A, B)$, $\chi(\tilde{G}) = \chi^{if}(\tilde{G})$.

Proof. Let $\tilde{G} = (V, A, B)$ be a bifuzzy graph on n vertices $V = \{v_1, v_2, ..., v_n\}$. Let $\chi^{if}(\tilde{G}) = k \Leftrightarrow \Gamma = \{a_1, a_2, ..., a_k\}$ is a k-coloring of bifuzzy graph $\tilde{G}$ and C_j color assigned to vertices lies in a_j^*, $j = 1, 2, ..., k \Leftrightarrow \{a_1, a_2, ..., a_k\}$ is a family of bifuzzy sets where $a_j = \{(v_j, \mu_{a_j}(v_j), \nu_{a_j}(v_j))\} \cup \{(v_i, \mu_{a_j}(v_i), \nu_{a_j}(v_i)) \mid v_j v_i \ \notin E, i \neq j\}$ which follows from (a) and (c) of Definition 4.3. It follows from (b) of Definition 4.3 $\bigcup_{j=1}^{k} a_j^* = V$ and $a_i^* \cap a_j^* = \emptyset, i \neq j \Leftrightarrow a_j^*$ is an independent set of vertices that is no two vertices in a_j^* are adjacent for each $j = 1, 2, ..., k \Leftrightarrow \chi(G^*) = k$, G^* is the underlying crisp graph of $\tilde{G}$.

Now, chromatic number of G^* is equal to chromatic number of some $\tilde{G}_{\alpha,\beta}$, i.e.,

$$\chi(G^*) = \chi(\tilde{G}_{\alpha,\beta}) = \chi_{\alpha,\beta} = \max\{\chi_{\alpha,\beta} \mid (\alpha, \beta) \in L\}. \qquad \blacksquare$$

Example 4.6 Consider bifuzzy graph $\tilde{G}$ defined on the vertex set $\{u_1, u_2, u_3, u_4\}$ and the crisp graphs $\tilde{G}_{(0.2,0.7)}$, $\tilde{G}_{(0.1,0.7)}$, $\tilde{G}_{(0.2,0.1)}$, $\tilde{G}_{(0.2,0.8)}$ as shown in Figure 7 corresponding to the level set of the fuzzy graph,

$$L = \{(0.2, 0.7), (0.1, 0.6), (0.2, 0.1), (0.2, 0.8)\}.$$

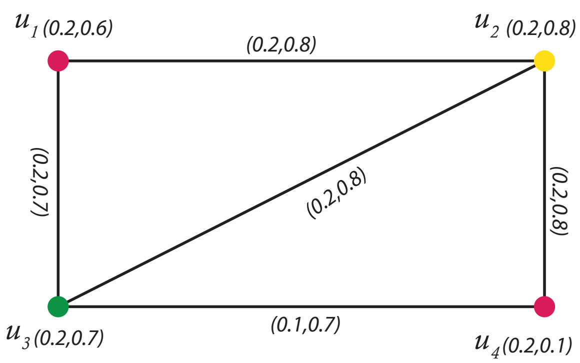

Figure 7: Bifuzzy graph $\tilde{G} = (V, A, B)$

The bifuzzy coloring is $\Gamma = \{a_1, a_2, a_3\}$,

$$a_1(v_i) = \begin{cases} (0.2, 0.6) & \text{i=1;} \\ (0.2, 0.1) & \text{i=4;} \\ 0 & \text{otherwise,} \end{cases} \qquad a_2(v_i) = \begin{cases} (0.2, 0.7) & \text{i=2;} \\ 0 & \text{otherwise,} \end{cases}$$

$$a_3(v_i) = \begin{cases} (0.2, 0.7) & \text{i=3;} \\ 0 & \text{otherwise.} \end{cases}$$

So, $\chi^{if}(\tilde{G}) = 3$ and the coloring of crisp graphs yields

$$\chi_{0.2,0.7} = \chi_{0.1,0.7} = 2, \chi_{0.2,0.1} = 1 \text{ and } \chi_{0.2,0.8} = 3.$$

Hence,

$$\chi^{if}(\tilde{G}) = \max\{\chi_{\alpha,\beta} \mid (\alpha, \beta) \in L\} = \chi_{0.2,0.8} = 3.$$

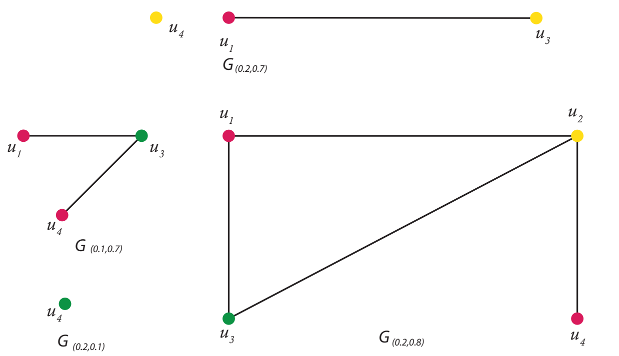

Figure 8: Crisp graphs $\tilde{G}_{(0.2,0.7)}$, $\tilde{G}_{(0.1,0.7)}$, $\tilde{G}_{(0.2,0.1)}$, $\tilde{G}_{(0.2,0.1)}$.

5. Coloring of bifuzzy graphs based on strong arcs

Coloring of bifuzzy graph will be used to solve many real life problems. Now, we define the coloring in a bifuzzy graph based on strong arcs. Moreover, the strong chromatic number of bifuzzy graph, complete bifuzzy graph and bifuzzy tree are discussed.

Definition 5.15 [16] If $u, w \in V$, the μ-strength of connectedness between u and w is

$$CONN_{\mu(B)}(u,w) = \max\{\mu_B(u,v_1) \wedge \mu_B(v_1,v_2) \wedge ... \wedge \mu_B(v_{k-1},w) \mid u, v_1, v_2, ..., v_{k-1}, w \in V,\ k = 1,2,...,n\}.$$

The ν-strength of connectedness between u and w is

$$CONN_{\nu(B)}(u,w) = \min\{\mu_B(u,v_1) \vee \mu_B(v_1,v_2) \vee ... \vee \mu_B(v_{k-1},w) \mid u, v_1, v_2, ..., v_{k-1}, w \in V,\ k = 1,2,...,n\}.$$

The μ-strength and ν-strength of connectedness between u and w in $\tilde{G}$ are denoted by $CONN_{\mu(\tilde{G})}(u,w)$ and $CONN_{\nu(\tilde{G})}(u,w)$, respectively.

Definition 5.16 Let $\tilde{G} = (V, A, B)$ be a bifuzzy graph. The *strong coloring* of a bifuzzy graph $\tilde{G}$ is a mapping $C : V(\tilde{G}) \to N$ such that $C(u) \neq C(v)$ if (u,v) is a strong arc (α-strong and β-strong) in $\tilde{G}$ is called strong coloring.

A bifuzzy graph $\tilde{G}$ is said to be $k-$strong colorable if there exist a strong coloring of G from a set of k-colors. The minimum number of colors k for which $\tilde{G}$ is $k-$strong colorable is called strong chromatic number of bifuzzy graph $\tilde{G}$ denoted by $\chi^{is}(\tilde{G})$. The end nodes of a $\delta-$arc can be assigned the same color in strong coloring of bifuzzy graph.

Example 5.7 Consider the bifuzzy graph $\tilde{G}$ as shown in Figure 9.

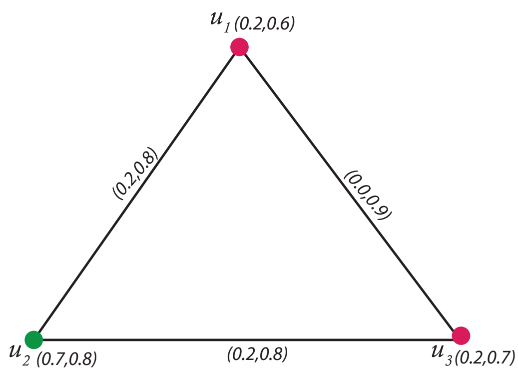

Figure 9: Strong coloring of bifuzzy graph $\tilde{G}$

We find that arc (u_1, u_2) is $\alpha-$strong arc as

$\mu_B(u_1, u_2) = 0.2 > CONN_{\mu(\tilde{G})-(u_1,u_2)}(u_1, u_2) = 0.0$ and

$\nu_B(u_1, u_2) = 0.8 < CONN_{\nu(\tilde{G})-(u_1,u_2)}(u_1, u_2) = 0.9$.

Similarly, arc (u_2, u_3) is $\alpha-$strong. But (u_1, u_3) is $\delta-$ arc as

$\mu_B(u_1, u_3) = 0.0 < CONN_{\mu(\tilde{G})-(u_1,u_3)}(u_1, u_3) = 0.2$ and

$\nu_B(u_1, u_3) = 0.9 > CONN_{\nu(\tilde{G})-(u_1,u_3)}(u_1, u_3) = 0.8$.

Hence strong coloring of bifuzzy graph gives $\chi^{is}(\tilde{G}) = 2$.

Proposition 5.1 *For a bifuzzy graph $\tilde{G} = (V, A, B)$, if $\tilde{G}$ is a complete bifuzzy graph or bifuzzy cycle then $\chi^{is}(\tilde{G}) = \chi^{if}(\tilde{G})$.*

Proof. Let $\tilde{G}$ be a complete bifuzzy graph of the complete graph G^* on n vertices, since a complete bifuzzy graph has no δ-arcs [16] and all the arcs are strong arcs, then $\chi^{is}(G) = \chi^{if}(\tilde{G}) = n$. $\tilde{G}$ is a bifuzzy cycle on n vertices of the cycle graph G^* then all the arcs in $\tilde{G}$ are strong arcs($\tilde{G}$ has at least two β-arcs) and $\tilde{G}$ has no δ-arcs in bifuzzy cycle. Thus, $\chi^{is}(\tilde{G}) = \chi^{if}(\tilde{G}) = 2$, when n is even and $\chi^{is}(\tilde{G}) = \chi^{if}(\tilde{G}) = 3$, when n is odd. ∎

Remark 5.2 The converse of the above proposition is not hold.

Example 5.8 Consider the bifuzzy graph $\tilde{G}$ as shown in Figure 10, we find that arcs (v_1, v_2), (v_1, v_5), (v_2, v_3) (v_3, v_4), (v_3, v_5) and (v_2, v_4) are strong arcs.

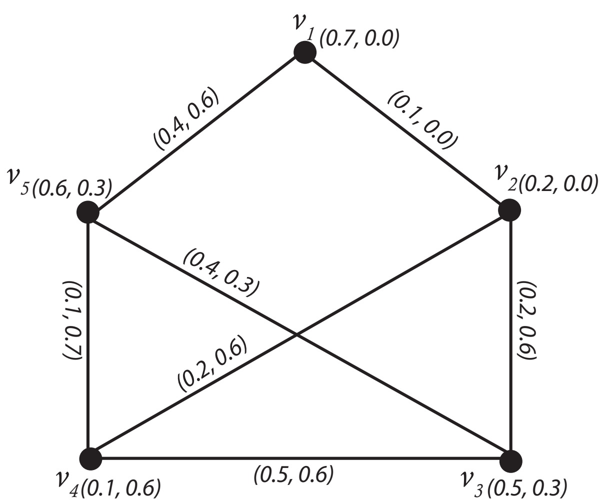

Figure 10: Bifuzzy graph $\tilde{G}$

The chromatic number and strong chromatic number of intuitionistic graph $\tilde{G}$ is 3, i.e., $\chi^{is}(\tilde{G}) = \chi^{if}(\tilde{G}) = 3$, but $\tilde{G}$ is neither complete nor is a bifuzzy cycle.

Proposition 5.2 *If $\tilde{G}$ is a bifuzzy tree, then $\chi^{is}(\tilde{G}) = \chi^{if}(T)$ where T is maximum spanning tree of $\tilde{G}$.*

Proof. Since there are no $\beta-$strong arcs in bifuzzy tree $\tilde{G}$, an arc (u, v) in a bifuzzy tree is strong if and only if (u, v) is an arc in T and the arcs of T are bifuzzy bridge. In bifuzzy graph an arc (u, v) is bifuzzy bridge if and only if (u, v) is α-strong arc [16]. Hence strong coloring of $\tilde{G}$ is exactly same as coloring of T and $\chi^{is}(\tilde{G}) = \chi^{if}(T)$. ∎

6. Application

A graph is a convenient way of representing information involving relationship between objects. The objects are represented by vertices and relations by edges. In many real world problems, we get partial information about that problem. So there is vagueness in the description of the objects or in its relationships or in both. To describe this type of relation, we need to design graph model with fission of bifuzzy set. This fission of bifuzzy set with graph is known as bifuzzy graph. For solving bifuzzy coloring problem, an algorithm which calculates the chromatic number of every crisp graph $G_{(\alpha_i,\beta_i)}$ can be used. In order to illustrate the concept of the coloring function of bifuzzy graph and its associated bifuzzy chromatic number, a bifuzzy coloring problem is presented. The problem of deciding which radio channels are best to use at different places is known as channel assignment problem. When there are n receiver and m transmitter, construct a graph in which each node represents a radio transmitter and an edge represents the joining of two nodes if the transmitters of the corresponding nodes would interfere. The interference of two nodes arises if they were assigned the same channel. An edge between two nodes represents the incompatibility of nodes. In many real life problems, such incompatibility is not crisp. The incompatibility between two nodes can be represented by different degrees. We solve this channel assignment problem with the help of bifuzzy coloring, there will be no interference between nodes.

We construct a bifuzzy graph $G = (V, B)$, as shown in Figure 11, with crisp set of vertices and bifuzzy set of edges i.e., $V = \{T_1, T_2, T_3, T_4, T_5, T_6\}$

$$\begin{aligned} B = \{&(e_1 = T_1T_2, 0.9, 0.1), (e_2 = T_1T_3, 0.7, 0.2), (e_3 = T_1T_4, 0.2, 0.8), \\ &(e_4 = T_1T_5, 0.3, 0.6), (e_5 = T_2T_3, 0.0, 0.2), (e_6 = T_2T_6, 0.1, 0.7), \\ &(e_7 = T_3T_4, 0.7, 0.1), (e_8 = T_3T_5, 0.9, 0.1), (e_9 = T_3T_6, 0.1, 0.7), \\ &(e_{10} = T_4T_5, 0.3, 0.6), (e_{11} = T_4T_6, 0.1, 0.8), (e_{12} = T_5T_6, 0.2, 0.6), \\ &(e_{13} = T_1T_6, 0.0, 0.0), (e_{14} = T_2T_4, 0.0, 0.0), (e_{15} = T_2T_5, 0.0, 0.0)\} \end{aligned}$$

In Figure 11, degree of membership and non-membership of edges represents the incompatibility and compatibility between nodes. The fundamental set of bifuzzy graph $G = (V, B)$ is defined as

$$I = \{(1, 0), (0.9, 0.1), (0.7, 0.2), (0.3, 0.6), (0.2, 0.8), (0.1, 0.7), (0.1, 0.8), (0, 1)\}.$$

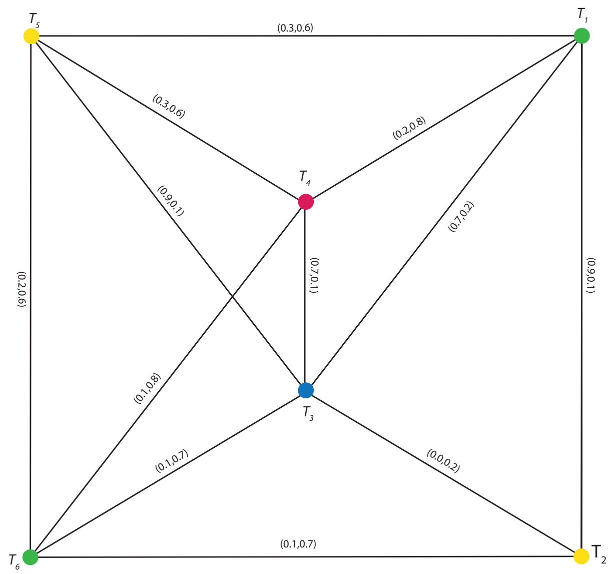

Figure 11: Bifuzzy graph $G = (V, B)$

The family of $(\alpha, \beta)-$cuts sets of G is

$$\{G_{(1,0)}, G_{(0.9,0.1)}, G_{(0.7,0.2)}, G_{(0.3,0.6)}, G_{(0.2,0.8)}, G_{(0.1,0.7)}, G_{(0.1,0.8)}, G_{(0,1)}\}.$$

For each $(\alpha, \beta) \in I$, $(\alpha, \beta)-$cuts sets of bifuzzy graph are crisp graphs, crisp coloring can be defined on each $(\alpha, \beta)-$cut, Table 1 contains the edge set $E_{\alpha,\beta}$, the chromatic number $\chi_{\alpha,\beta}$ and a $\chi_{\alpha,\beta}$-coloring $C_{\alpha,\beta}^{\chi_{\alpha,\beta}}$.

Table 1

(α, β)	$E_{\alpha,\beta}$	$\chi_{\alpha,\beta}$	$c^{\chi_{\alpha,\beta}}(T_1)$	$c^{\chi_{\alpha,\beta}}(T_2)$	$c^{\chi_{\alpha,\beta}}(T_3)$	$c^{\chi_{\alpha,\beta}}(T_4)$	$c^{\chi_{\alpha,\beta}}(T_5)$	$c^{\chi_{\alpha,\beta}}(T_6)$
$(0, 1)$	$e_1, e_2, e_3, e_4, e_{13}, e_5, e_{14}, e_{15}, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}$	6	1	2	3	4	5	6
$(0.9, 0.1)$	e_1, e_8	2	1	2	1	1	1	1
$(0.7, 0.2)$	e_1, e_2, e_7, e_8	2	1	2	1	2	2	1
$(0.3, 0.6)$	$e_1, e_2, e_4, e_7, e_8, e_{10}$	3	1	2	3	1	2	1
$(0.2, 0.8)$	$e_1, e_2, e_3, e_4, e_7 e_8, e_{10}, e_{12}$	4	1	2	2	3	4	1
$(0.1, 0.8)$	$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}$	4	1	2	3	2	4	1
$(0.1, 0.7)$	$e_1, e_2, e_3, e_4, e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11}, e_{12}$	4	1	2	3	2	4	1
$(1, 0)$	Φ	1	1	1	1	1	1	1

The chromatic number of bifuzzy graph $G = (V, B)$ is

$$\chi(G) = \{(1, 1, 0), (2, 0.9, 0.1), (3, 0.3, 0.6), (4, 0.2, 0.8), (5, 0, 1), (6, 0, 1)\}.$$

The analysis of chromatic number $\chi(G)$ of bifuzzy graph shows that there are more incompatible links between nodes for the lower value of α in $(\alpha, \beta)-$cut of G. Therefore more colors are needed to assign channels to the nodes in such a way nodes that are adjacent get different channels, then there will be no interference. On the other hand, there are less incompatible links between nodes for higher value of α in $(\alpha, \beta)-$cut of G.

7. Conclusion and future work

Fuzzy graph coloring has a variety of applications to problems involving scheduling and assignments. A bifuzzy model is a generalization of the fuzzy model. The bifuzzy models give more precision, flexibility and compatibility to the system as compared to fuzzy models. In this paper, we have defined the coloring function of bifuzzy graphs with crisp nodes and bifuzzy edges, the channel assignment problem solved with this coloring function, chromatic number can be defined according to the significance of bifuzzy edges of the bifuzzy graph. We defined coloring of bifuzzy graph with bifuzzy sets of nodes. We also defined strong chromatic number of a bifuzzy graph. We plan to extend our research of fuzzification to (1) m-polar fuzzy hypergraphs, (2) m-polar fuzzy soft hypergraphs, and (3) m-polar fuzzy rough hypergraphs.

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Accepted: 10.01.2016