THE COMPOSITION, CONVERGENCE AND TRANSITIVITY OF POWERS AND ADJOINT OF GENERALIZED FUZZY MATRICES

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Abstract. Path algebras are additively idempotent semirings and generalize Boolean algebras, fuzzy algebras, distributive lattices and inclines. Thus the Boolean matrices, the fuzzy matrices, the lattice matrices and the incline matrices are prototypical examples of matrices over path algebras. In this paper, generalized fuzzy matrices are considered as matrices over path algebras. Compositions of generalized fuzzy matrices are discussed, and a new transitive matrix is constructed from given matrices. Furthermore, the transitivity and the convergent index for powers of generalized fuzzy matrices are studied, some properties of powers are also established through adjoint matrix, and finally the invertibility of a matrix is investigated. Some results obtained here generalize and develop the corresponding ones on fuzzy matrices, lattice matrices and incline matrices shown in the references.

Keywords: generalized fuzzy matrix, compositions, transitivity, convergence, path algebra.

1. Introduction

Generalized fuzzy matrices [24] over a special type of semiring are considered. The semiring is called path algebra (see [19]) (or additively idempotent semring (see [12]). Path algebras are useful tools in diverse domains such as the design of switching circuits, automata theory, information systems, dynamic programming, decision theory and so on [2], [13]. Path algebras generalize Boolean algebra, fuzzy algebra, distributive lattice, incline, max-plus algebra and min-plus algebra. Then, the Boolean matrices, the fuzzy matrices, the lattice matrices and the incline matrices are prototypical examples of generalized fuzzy matrices over path algebras.

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The techniques of matrices over semirings play a very important role in optimization theory, models of discrete event networks, graph theory and so on [1]. The study of matrices over general semirings has a long history. In 1964, Rutherford [30] gave a proof of the Cayley-Hamilton theorem for a commutative semiring avoiding the use of determinants. Since then, a number of works on the theory of matrices over semirings were published (see e.g. [10], [27], [29]). At the same time, these issues about compositions, transitivity, convergence and invertibility of matrices over some special path algebras were discussed by many authors.

The study of compositions of matrices over some special types of semirings also has a long history. In 1985, Hashimoto [18] discussed the problem of decomposition of fuzzy rectangular matrices. In 1995, some properties of compositions of fuzzy matrices were given by Ragab, Emam and Hashimaton (see [8]). In 2002, Tan generalized some previous results related to compositions of fuzzy matrices to the matrices over distributive lattices (see [32]). In 2011, Jiang et al. [22] obtained some properties of compositions of transitive matrices over inclines. In this paper, we continue to concentrate on the compositions of generalized fuzzy matrices over path algebras.

Many results about properties of powers of matrices over some special semirings have been obtained (see e.g. [19], [17]-[33], [35], [16], [14], [34], [9]-[5]). Transitive matrix is an important type of generalized fuzzy matrices and it represents transitive relation (see e.g. [7],[11]). Transitive relation is widely used in clustering and information retrieval preference (see e.g. [28], [31], [20]). The transitivity of matrices over some special semirings has been studied in [19], [17]-[35], [14], [9], [22], [36]. In 1982, The concept of transitive binary Boolean matrices was introduced by Kim (see [23]). Hashimoto [17] presented the concept of transitive fuzzy matrices and considered the convergence of powers of transitive fuzzy matrices. Tan [37], [35] discussed the convergence of powers of transitive lattice matrices. In 2011, he [33] also described the transitive closure of a generalized fuzzy matrix with periods and obtained an expression for the transitive closure. Jiang et al. [22] discussed the transitive closure, the convergence and the composition of incline transitive matrix. In 2013, Tan [36] gave some elementary characteristics of transitive matrices and obtained some properties of the transitive closure of matrix over path algebra.

The convergence of powers of matrices over some special semirings has been considered by many authors, see e.g. [17], [6], [33], [35], [14], [38]-[22]. In 1977, Thomason [38] proved that the power sequence of a fuzzy matrix either converges in finite steps or oscillates with finite period. Since then, many authors have studied the convergence of powers of fuzzy matrices (see e.g. [26]-[21]). In [35], [4], some previous results related to the index and period of a fuzzy matrix were generalized to the matrices over distributive lattices and some results about the index and period were developed. In [22], Han and Li generalized some previous results about the index and period to the matrices over inclines. Recently, Tan [33] studied the generalized fuzzy matrices over commutative path algebras with periodic elements.

The property of adjoint matrix seems to have appeared firstly in the work of Ragab et al. [9], in [9], some properties of a square fuzzy matrix, such as reflexivity, transitivity, circularity and construction, have been proposed through the adjoint matrix. In 2004, Han et al. [15] studied the inverse of incline matrix from the adjoint matrix. Duan [6] proved that A^{n-1} was equal to the adjoint matrix of A when $A \geq I_n$. In 2013, Tan [36] discussed the transitivity of generalized fuzzy matrix with the adjoint matrix over path algebra.

In this paper, we continue to study the matrices over more general path algebras, namely over a class of commutative path algebras. In Section 3, some properties of compositions of generalized fuzzy matrices are obtained, and a new transitive matrix such as the idempotent matrix is constructed from given matrices. In Section 4, the power sequence of generalized fuzzy matrices is studied in detail. Firstly, the transitivity of generalized fuzzy matrices is discussed. As a result, we get a result $t(A) = A^s$, where A^s is a transitive matrix and the result simplifies the expression of the transitive closure $t(A)$. Secondly, the convergent index for powers of generalized fuzzy matrices is considered. Finally, some properties of powers are established with the adjoint matrix. In section 5, the invertibility of a matrix is also investigated, and some necessary and sufficient conditions for a matrix to be invertible are studied. Some results obtained in this paper generalize the corresponding results in [32], [6], [18], [16], [34], [9], [22].

2. Definitions and preliminaries

At this section, we shall give some definitions and lemmas.

Definition 2.1. [12] An algebraic system $(L, +, \cdot)$ is called a semiring if $(L, +)$ is an Abelian monoid with identity element 0 and $(L, \cdot)$ is another monoid with identity element 1 and $0 \neq 1$, connected by ring-like distributivity, and $0r = r0 = 0$ for all $r \in L$.

A semiring L is called commutative if $ab = ba$ for all $a, b \in L$; L is called an antiring if $a + b = 0$ implies that $a = b = 0$ for all $a, b \in L$. Antirings have been studied in [12] under the name of zerosumfree semirings.

Definition 2.2. [19], [2] A semiring L is called a path algebra if $a + a = a$ for all $a \in L$.

Path algebras have been studied in [12] under the name of additively idempotent semiring. It is easy to see that every path algebra is an antiring. Path algebras are abundant, for example, every Boolean algebra, the fuzzy algebra $F = ([0, 1], \vee, \top)$, where $\vee = max$ and $\top$ is a $t-$norm (for $t-$norm, refer to [25]), every bounded distributive lattice, any incline [12], the max-plus algebra $(R \cup \{-\infty\}, \vee, +)$ and the min-plus algebra $(R \cup \{+\infty\}, \wedge, +)$ [39].

In this paper, the path algebra $(L, +, \cdot)$ is always assumed to be a commutative path algebra with the least element 0. A path algebra L is commutative if $ab = ba$ for all $a, b \in L$. Boolean algebra, bounded distributive lattice, incline, max-plus algebra $(R \cup \{-\infty\}, \vee, +)$ and min-plus algebra $(R \cup \{+\infty\}, \wedge, +)$ are commutative path algebras.

In a path algebra L, the relation $\leq$ is defined by $x \leq y$ if $x + y = y$. It is easy to see that $\leq$ is a partial order relation over L and satisfies the following properties:

Proposition 2.1. *Let L be a path algebra and $a, b, c, d \in L$, then*

(1) $a \geq 0$;

(2) *if* $a \leq b$, *then* $a + c \leq b + c$, $ac \leq bc$ *and* $ca \leq cb$;

(3) *if* $a \leq b$ *and* $c \leq d$, *then* $a + c \leq b + d$ *and* $ac \leq bd$;

(4) $a \leq a + b$, *and* $a + b$ *is the least upper bound of* a *and* b, *in other words, if there is an element* c *satisfying* $a \leq c$ *and* $b \leq c$, *then* $a + b \leq c$;

(5) $a + b = 0$ *if and only if* $a = b = 0$.

Let L be a path algebra. An element a in L is said to be idempotent if $a^2 = a$. The set of all idempotent elements in L is denoted by $I(L)$, i.e., $I(L) = \{a \in L \mid a^2 = a\}$. An element a in L is said to be nilpotent if $a^k = 0$ for some positive integer k. The least positive integer k satisfying $a^k = 0$ is called the nilpotent index of a and denoted by $h(a)$. An element a in L is said to be almost periodic if there exist positive integers k and d such that $a^k = a^{k+d}$. The least such positive integers k and d are called the index and the period of a, and denoted by $k(a)$ and $d(a)$, respectively.

A matrix is called a generalized fuzzy matrix if its entries belong to a path algebra. Let $M_{m\times n}(L)$ be the set of all $m \times n$ matrices over L. Especially, put $M_n(L) = M_{n\times n}(L)$ and $V_n(L) = M_{n\times 1}(L)$. The elements in $V_n(L)$ are called the column vectors of order n over L. For any A in $M_{m\times n}(L)$, we denote by a_{ij} or A_{ij} the element of L which stands in the (i, j)-entry of A, and denote by A^T the transpose of A. We shall use Z_+ to denote the set of all positive integers. For convenience, we define:

$\underline{n} = \{1, 2, ..., n\}$;

$\underline{n}_i = \{1, 2, ..., n\}/\{i\} = \{1, 2, ..., i-1, i+1, ..., n\}$;

$a^k = \underbrace{a \cdot a \cdots a}_{k \ \ times}$.

For any $A,\ B \in M_{m\times n}(L)$ and $C \in M_{n\times l}(L)$, we define:

$A + B = (a_{ij} + b_{ij})_{m\times n}$;

$$AC = \left(\sum_{k=1}^{n} a_{ik}c_{kj}\right)_{m\times l};$$

$A^T = D$ if and only if $d_{ij} = a_{ji}$ for all $i \in \underline{n}$ and $j \in \underline{m}$;

$kA = B$ $(k \in L)$ if and only if $b_{ij} = ka_{ij}$ for all $i \in \underline{m}$ and $j \in \underline{n}$;

$A \leq B$ if and only if $a_{ij} \leq b_{ij}$ for all $i \in \underline{m}$ and $j \in \underline{n}$;

$$I_n = (t_{ij}), \text{ where } t_{ij} = \begin{cases} 1, i = j \\ 0, i \neq j \end{cases} \text{ for } i, j \in \underline{n}.$$

For any A in $M_n(L)$, the powers of A are defined as follows:
$A^0 = I_n$, $A^l = A^{l-1}A$, $l \in Z_+$.
The (i, j)-entry of A^l is denoted by $a_{ij}^{(l)}$ $(l \in Z_+)$, and obviously

$$a_{ij}^{(l)} = \sum_{1 \le i_1, i_2, \ldots, i_{l-1} \le n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{l-1} j}.$$

It is easy to see that $(M_n(L), +, \cdot)$ is a path algebra, and we have that $A + C \le B + D$ and $AC \le BD$ for all $A, B, C, D \in M_n(L)$ with $A \le B$ and $C \le D$.

Let $A \in M_n(L)$, A is said to be almost periodic if there exist positive integers k and d such that $A^k = A^{k+d}$. The least positive integers k and d are called the index and the period of A, and denoted by $k(A)$ and $d(A)$, respectively. In particular, if $d(A) = 1$ then A is said to converge in a finite number of steps. If A converges to the zero matrix in a finite number of steps, then A is said to be nilpotent. The least positive integer k satisfying $A^k = 0$ is called the nilpotent index of A, and denoted by $h(A)$.

Let $A = (a_{ij}) \in M_n(L)$. If $A^2 \le A$, then A is called transitive; if $A^2 = A$, then A is called idempotent (any idempotent matrix is transitive); if $A^T = A$, then A is called symmetric; if $A \ge I_n$, then A is called reflexive; if $a_{ii} = 0$ for all $i \in \underline{n}$, then A is called irreflexive; A is called a permutation matrix if exact one of the elements of its every row and every column is 1 and the others are 0.

Let $B \in M_n(L)$. The matrix B is called the transitive closure of A if B is transitive and $A \le B$ and if for any transitive matrix C in $M_n(L)$ satisfying $A \le C$ we have $B \le C$. The transitive closure of A is denoted by $t(A)$. It is clear that if A has a transitive closure then it is unique. However, for general path algebra L, there may be some matrices which have no transitive closure over L.

Definition 2.3. Let $A = (a_{ij}) \in M_{m\times n}(L)$ and $m \le n$. The determinant $\det(A)$ of A is defined by

$$det(A) = \sum_{\sigma \in s} \prod_{i \in \underline{m}} a_{i\sigma(i)},$$

where s is the set of all injective mappings from $\underline{m}$ to $\underline{n}$.

If $A \in M_n(L)$, then the determinant $\det(A)$ is the sum of $n!$ terms which are all possible products of n entries of A taken exactly one in each row and each column.

Definition 2.4. *Let $A = (a_{ij}) \in M_n(L)$. The adjoint matrix $adj(A) \in M_n(L)$ of A is the matrix whose (i, j)-entry is $det(A(j|i))$ for all $i, j \in \underline{n}$, where $A(j|i)$ is the matrix of order $n - 1$ formed by deleting row j and column i from A.*

3. Compositions of generalized fuzzy matrices

In this section, some properties of compositions of generalized fuzzy matrices are given.

Definition 3.1. A matrix $A = (a_{ij}) \in M_n(L)$ is said to be

(1) row diagonally dominant if $a_{ij} = a_{ii}a_{ij}$ for all $i, j \in \underline{n}$;

(2) column diagonally dominant if $a_{ij} = a_{ij}a_{jj}$ for all $i, j \in \underline{n}$;

(3) weakly diagonally dominant if for any $i \in \underline{n}$, either $a_{ii}a_{ij} = a_{ij}$ for all $j \in \underline{n}$ or $a_{ii}a_{ji} = a_{ji}$ for all $j \in \underline{n}$;

(4) strongly diagonally dominant if $a_{ij} = a_{ii}a_{ij} = a_{ij}a_{jj}$ for all $i, j \in \underline{n}$;

(5) nearly irreflexive if $a_{ii}a_{kj} = a_{ii}$ for all $i, j, k \in \underline{n}$.

Theorem 3.1. *Let $A = (a_{ij}) \in M_n(L)$. If A is transitive and row diagonally dominant, then $C = (c_{ij})$ is idempotent, where $c_{ij} = a_{ij}a_{ji}$.*

Proof. For any $i, j \in \underline{n}$,

$$c_{ij}^{(2)} = \sum_{k=1}^{n} c_{ik}c_{kj} = \sum_{k=1}^{n} a_{ik}a_{ki}a_{kj}a_{jk} = \sum_{k=1}^{n}(a_{ik}a_{kj})(a_{jk}a_{ki}) \leq \sum_{k=1}^{n}(a_{ij}a_{ji})$$

(because A is transitive) $= \sum_{k=1}^{n} c_{ij} = c_{ij}$. Thus, $C^2 \leq C$. On the other hand, since $c_{ij}^{(2)} = \sum_{k=1}^{n} c_{ik}c_{kj} \geq c_{ii}c_{ij} = a_{ii}a_{ii}a_{ij}a_{ji} = a_{ij}a_{ji}$ (because A is row diagonally dominant) $= c_{ij}$, we have $C^2 \geq C$. Therefore, $C^2 = C$. This completes the proof. ∎

Definition 3.2. Let $A = (a_{ik}) \in M_{m\times n}(L)$, $D = (d_{kj}) \in M_{n\times l}(L)$. We define $A \circ D = C$ if and only if $c_{ij} = \prod_{k=1}^{n}(a_{ik} + d_{kj})$ for any $i \in \underline{m}, k \in \underline{n}$ and $j \in \underline{l}$.

Proposition 3.1. *Let $A = (a_{ij}) \in M_n(L)$ be a symmetric and nearly irreflexive matrix. Then $A \circ A$ is symmetric and nearly irreflexive.*

Proof. Let $S = A \circ A$. Then

$$s_{ji} = \prod_{l=1}^{n}(a_{jl} + a_{li}) = \prod_{l=1}^{n}(a_{il} + a_{lj}) = s_{ij},$$

so that S is symmetric. Also,

$$\begin{aligned} s_{ii}s_{kj} &= \left[\prod_{l=1}^{n}(a_{il} + a_{li})\right]\left[\prod_{l=1}^{n}(a_{kl} + a_{lj})\right] = \prod_{l=1}^{n} a_{il}\left[\prod_{l=1}^{n}(a_{kl} + a_{lj})\right] \\ &= a_{ii}\left[\prod_{l=1}^{n}(a_{kl} + a_{lj})\right] = a_{ii} = \prod_{l=1}^{n} a_{il} = \prod_{l=1}^{n}(a_{il} + a_{li}) = s_{ii}, \end{aligned}$$

and so S is nearly irreflexive. This completes the proof. ∎

Proposition 3.1 generalizes Proposition 3.1 of Tan [32].

Proposition 3.2. *Let $A, B \in M_{m\times n}(L), C \in M_{n\times l}(L)$ and $D \in M_{p\times m}(L)$. Then*

(1) $(B \circ C)^T = C^T \circ B^T$;

(2) *If $A \leq B$, then $D \circ A \leq D \circ B$ and $A \circ C \leq B \circ C$.*

The proof is trivial.

Proposition 3.3. *Let $A = (a_{ij}) \in M_{m\times n}(L)$ be a symmetric and nearly irreflexive matrix. Then $A \circ A^T$ is symmetric and nearly irreflexive.*

Proof. Let $R = A \circ A^T$. Then

$$r_{ji} = \prod_{l=1}^{n}(a_{jl} + a_{il}) = \prod_{l=1}^{n}(a_{il} + a_{jl}) = r_{ij},$$

so that R is symmetric. Furthermore, since

$$\begin{aligned} r_{ii}r_{kj} &= \left[\prod_{l=1}^{n}(a_{il} + a_{il})\right]\left[\prod_{l=1}^{n}(a_{kl} + a_{jl})\right] = \prod_{l=1}^{n} a_{il}\left[\prod_{l=1}^{n}(a_{kl} + a_{lj})\right] \\ &= a_{ii}\left[\prod_{l=1}^{n}(a_{kl} + a_{lj})\right] \text{ (because } A \text{ is nearly irreflexive)} \\ &= a_{ii} = \prod_{l=1}^{n} a_{il} \text{ (because } A \text{ is nearly irreflexive)} \\ &= \prod_{l=1}^{n}(a_{il} + a_{il}) = r_{ii}, \end{aligned}$$

we draw the conclusion that $R = A \circ A^T$ is nearly irreflexive. This completes the proof. ■

Proposition 3.3 generalizes Theorem 3.3 of Tan [32] and Theorem 3.3 of Ragab and Emam [8].

Theorem 3.2. *Let $A = (a_{ij}) \in M_n(L)$ be a symmetric and nearly irreflexive matrix. Then the matrix $R = B \circ A$ is idempotent and constant, where*

$$B = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}, \; a_{ii} \in L, i \in \underline{n}.$$

Proof. Because the hypothesis $a_{ii}a_{kj} = a_{ii}$ for all $i, j, k \in \underline{n}$, we have $a_{11} = a_{22} = \cdots = a_{nn}$. Furthermore, based on the definitions of the matrices A, B and R, we have

$$\begin{aligned} r_{ij} &= \left(\prod_{k\neq i} a_{kj})(a_{ii} + a_{ij}\right) = \left(\prod_{k\neq i} a_{jk}\right)(a_{ii} + a_{ij}) \\ &= a_{ii} + \prod_{k\neq i} a_{jk}a_{ij} \text{(because } A \text{ is nearly irreflexive)} \\ &= a_{ii} = a_{jj}. \end{aligned}$$

For any $i, j \in \underline{n}$, $r_{ij}^{(2)} = \sum_{k=1}^{n} r_{ik} r_{kj} = \sum_{k=1}^{n} a_{ii} a_{jj} = a_{ii} = a_{jj}$. Thus, $R^2 = R = (a_{ii})_{n\times n}$. Therefore, R is idempotent and constant. This completes the proof of the theorem. ∎

Theorem 3.2 generalizes Theorem 3.5 of Tan [32] and Corollary 3.6 of Ragab and Emam [8].

Theorem 3.3. *Let $A = (a_{ij}) \in M_{m\times n}(L)$ be a symmetric and nearly irreflexive matrix. Then $I_m \circ (A \circ A^T)$ is idempotent, where $A = (a_{ij})$ with $(a_{it}+a_{kj})(a_{it}+a_{kj}) = (a_{it} + a_{kj})(i, k \in \underline{m}; t, j \in \underline{n})$.*

Proof. Let $B = A \circ A^T$ and $S = I_m \circ (A \circ A^T)$. Then

$$
\begin{aligned}
s_{ij} &= \prod_{l\neq i} b_{lj}(1 + b_{ij}) = \prod_{l\neq i} b_{lj} + b_{ij} \prod_{l\neq i} b_{lj} \\
&= \prod_{l\neq i} b_{lj} + \prod_{l=1}^{n} b_{lj} = \prod_{l\neq i} b_{lj} + b_{jj} = \begin{cases} b_{jj} + \prod_{l\neq i} b_{lj} = b_{ii} + \prod_{l\neq i} b_{li}, i = j, \\ b_{jj}, i \neq j. \end{cases}.
\end{aligned}
$$

The $(i, j)th$ entry of S^2 is $s_{ij}^{(2)} = \sum_{k=1}^{n} s_{ik} s_{kj}$.

Case 1: $i \neq j$. In this case, we have

$$
\begin{aligned}
s_{ij}^{(2)} &= \sum_{k\neq i,j} s_{ik} s_{kj} + s_{ii} s_{ij} + s_{ij} s_{jj} \\
&= \sum_{k\neq i,j} b_{kk} b_{jj} + \left(\prod_{l\neq i} b_{li} + b_{ii}\right) b_{jj} + b_{jj} \left(\prod_{l\neq j} b_{lj} + b_{jj}\right) \\
&= b_{jj} + b_{jj} + b_{jj} \text{ (by Proposition 3.3, } B \text{ is nearly irreflexive)} \\
&= b_{jj} = s_{ij},
\end{aligned}
$$

Case 2: $i = j$. In this case, we have

$$
\begin{aligned}
s_{ii}^{(2)} &= \sum_{l\neq i} s_{il} s_{li} + s_{ii} s_{ii} = \sum_{l\neq i} b_{ll} b_{ii} + \left(\prod_{l\neq i} b_{li} + b_{ii}\right)\left(\prod_{l\neq i} b_{li} + b_{ii}\right) \\
&= b_{ii} + \prod_{l\neq i} b_{li} \prod_{l\neq i} b_{li} = b_{ii} + \prod_{l\neq i} b_{li} \text{ (since } A = (a_{ij}) \\
&\text{with } (a_{it} + a_{kj})(a_{it} + a_{kj}) = (a_{it} + a_{kj})(i, k \in \underline{m}; t, j \in \underline{n})) \\
&= s_{ii}.
\end{aligned}
$$

Thus, $S^2 = S$. Therefore, $S = I_m \circ (A \circ A^T)$ is idempotent. This completes the proof of the theorem. ∎

Theorem 3.3 generalizes Corollary 3.7(1) of Tan [32].

Corollary 3.1. *Let $A = (a_{ij}) \in M_{m\times n}(L)$ be a symmetric and nearly irreflexive matrix. Then $(A \circ A^T) \circ I_m$ is idempotent, where $A = (a_{ij})$ with $(a_{it}+a_{kj})(a_{it}+a_{kj}) = (a_{it} + a_{kj})(i, k \in \underline{m}; t, j \in \underline{n})$.*

Proof. The proof of Corollary 3.1 is similar to that of Theorem 3.3. This completes the proof. ∎

Corollary 3.1 generalizes Corollary 3.8 of Tan [32].

Theorem 3.4. *Let $A = (a_{ij}) \in M_n(L)$ be a symmetric and nearly irreflexive matrix. Then $I_n \circ (A \circ A)$ is idempotent, where $A = (a_{ij})$ with $(a_{im}+a_{kj})(a_{im}+a_{kj}) = (a_{im} + a_{kj})(i, m, k, j \in \underline{n})$.*

Proof. The proof of Theorem 3.4 is similar to that of Theorem 3.3. This completes the proof. ∎

Corollary 3.2. *Let $A = (a_{ij}) \in M_n(L)$ be a symmetric and nearly irreflexive matrix. Then $(A \circ A) \circ I_n$ is idempotent, where $A = (a_{ij})$ with $(a_{im}+a_{kj})(a_{im}+a_{kj}) = (a_{im} + a_{kj})(i, m, k, j \in \underline{n})$.*

Proof. The proof of Corollary 3.2 is similar to that of Theorem 3.3. This completes the proof. ∎

Theorem 3.5. *Let $A = (a_{ij}) \in M_{m\times n}(L)$ be a symmetric and nearly irreflexive matrix. Then $I_m \circ A$ is transitive, where $A \geq A^2$.*

Proof. Let $T = I_m \circ A$. Then

$$t_{ij} = \prod_{l\neq i} a_{lj}(1 + a_{ij}) = \prod_{l\neq i} a_{lj} + a_{ij} \prod_{l\neq i} a_{lj} = \prod_{l\neq i} a_{lj} + a_{jj}$$
$$= \begin{cases} a_{jj} + \prod_{l\neq i} a_{lj} = a_{ii} + \prod_{l\neq i} a_{li}, i = j, \\ a_{jj}, i \neq j. \end{cases}.$$

The $(i, j)th$ entry of T^2 is $t_{ij}^{(2)} = \sum_{k=1}^{n} t_{ik}t_{kj}$.

Case 1: $i \neq j$. In this case, we have

$$\begin{aligned} t_{ij}^{(2)} &= \sum_{k\neq i,j} t_{ik}t_{kj} + t_{ii}t_{ij} + t_{ij}t_{jj} \\ &= \sum_{k\neq i,j} a_{kk}a_{jj} + \left(\prod_{l\neq i} a_{li} + a_{ii}\right) a_{jj} + a_{jj} \left(\prod_{l\neq j} a_{lj} + a_{jj}\right) \\ &= a_{jj} = t_{ij}, \end{aligned}$$

Case 2: $i = j$. In this case, we have

$$\begin{aligned} t_{ii}^{(2)} &= \sum_{l \neq i} t_{il} t_{li} + t_{ii} t_{ii} = \sum_{l \neq i} a_{ll} a_{ii} + \left(\prod_{l \neq i} a_{li} + a_{ii} \right) \left(\prod_{l \neq i} a_{li} + a_{ii} \right) \\ &= a_{ii} + \prod_{l \neq i} a_{li} \prod_{l \neq i} a_{li} \text{ (because } A \text{ is nearly irreflexive)} \\ &= a_{ii} + \prod_{l \neq i} a_{il} \prod_{l \neq i} a_{li} \text{ (since } A \text{ is symmetric)} \\ &\leq a_{ii} + a_{ii} \text{ (since } A \text{ is transitive)} \\ &= a_{ii} \leq a_{ii} + \prod_{l \neq i} a_{li} = t_{ij}, \end{aligned}$$

Thus, $T^2 \leq T$. Therefore, $T = I_n \circ A$ is transitive. This completes the proof of the theorem. ∎

Theorem 3.6. *Let $A = (a_{ij}) \in M_n(L)$ be an irreflexive and transitive. Then*

(1) $A \circ A^T = 0$;

(2) $A^T \circ A = 0$.

Proof. (1) Let $R = A \circ A^T$. Then

$$\begin{aligned} r_{ij} &= \prod_{k=1}^{n} (a_{ik} + a_{jk}) = (a_{i1} + a_{j1})(a_{i2} + a_{j2}) \cdots (a_{in} + a_{jn}) \\ &\leq \sum_{k \in N} a_{ij} a_{ji} a_{ik} + \sum_{k,l \in N} a_{ij} a_{ji} a_{ik} a_{jl} + \sum_{l \in N} a_{ij} a_{ji} a_{jl} \\ &\leq \sum_{k \in N} a_{ii} a_{ik} + \sum_{k,l \in N} a_{ii} a_{ik} a_{jl} + \sum_{l \in N} a_{ii} a_{jl} \text{ (since } A \text{ is transitive)} \\ &= 0. \end{aligned}$$

Thus, $R = 0$.

(2) The proof of (2) is similar to that of (1). This completes the proof of the theorem. ∎

Theorem 3.6 generalizes Proposition 3.9 of Tan [32].

Definition 3.3. A path algebra $\widetilde{L}$ is said to be a Brouwerian path algebra if for any $a, b \in L$, there exists an element $b \longrightarrow a \in \widetilde{L}$ such that $bx \leq a \Leftrightarrow x \leq b \longrightarrow a$.

Definition 3.4. $A \leftarrow D = C$ if and only if $c_{ij} = \prod_{k=1}^{n} (d_{kj} \rightarrow a_{ik})$ for any $i, j \in \underline{n}$.

Lemma 3.1. *Let $\widetilde{L}$ be a Brouwerian path algebra. Then, for any $a \in \widetilde{L}$, $1 \rightarrow a = a$.*

Proof. By $1x = x$ for any $x \in \widetilde{L}$, the proof is trivial. ∎

Theorem 3.7. *Let $A \in M_{m\times n}(\widetilde{L})$. Then $A \leftarrow A^T$ is transitive.*

Proof. Let $R = A \leftarrow A^T$. Then $r_{ij} = \prod_{k=1}^{n}(a_{jk} \to a_{ik})$ for any $i, j \in \underline{n}$. Since $a_{jk}(a_{lk} \to a_{ik})(a_{jk} \to a_{lk}) = (a_{lk} \to a_{ik})(a_{jk}(a_{jk} \to a_{lk}) \leq (a_{lk} \to a_{ik})a_{lk} \leq a_{ik}$, we have $(a_{lk} \to a_{ik})(a_{jk} \to a_{lk}) \leq a_{jk} \to a_{ik}$. Thus, $r_{il}r_{lj} = \prod_{k=1}^{n}(a_{lk} \to a_{ik})(a_{jk} \to a_{lk})$ $\leq \prod_{k=1}^{n}(a_{jk} \to a_{ik}) = r_{ij}$, and so $r_{ij}^{(2)} = \sum_{l=1}^{n} r_{il}r_{lj} \leq \sum_{l=1}^{n} r_{ij} = r_{ij}$. Therefore, $R^2 \leq R$. This proves the theorem. ∎

Theorem 3.7 generalizes Lemma 4.1 of Tan [32].

Theorem 3.8. *Let $A \in M_n(I(\widetilde{L}))$is transitive. Then $A \leq A \leftarrow A^T$.*

Proof. Let $T = A \leftarrow A^T$. Then $t_{ij} = \prod_{k=1}^{n}(a_{jk} \to a_{ik})$ for all $i, j \in \underline{n}$. Since A is transitive, we have $a_{ik} \geq a_{ij}a_{jk}$ for all $i, j, k \in \underline{n}$. Thus, $a_{ij} \leq (a_{jk} \to a_{ik})$, and so $a_{ij} \leq \prod_{k=1}^{n}(a_{jk} \to a_{ik}) = t_{ij}$ (because $A \in M_n(I(\widetilde{L}))$). Therefore, $A \leq A \leftarrow A^T$. This proves the theorem. ∎

4. Basic properties of powers of generalized fuzzy matrices

This section studies the power sequence of generalized fuzzy matrices in detail. First, the reduction of the transitive closure of a matrix with a period over L is posed. Second, the convergent index for powers of generalized fuzzy matrices is discussed. Finally, some properties of powers are established by adjoint matrix.

Lemma 4.1. [33] *Let $A \in M_n(L)$ be almost periodic, then A has transitive closure and $t(A) = \sum_{k=1}^{m} A^k$ for all $m \geq k(A) + d(A) - 1$.*

Lemma 4.2. [33] *Let $A \in M_n(L)$ be almost periodic, then there exist a positive integers s such that A^s is idempotent.*

Proof. Since A is almost periodic, we have $A^k = A^{k+d}$ for some positive integers k and d. Then $A^{(k+t)d} = A^{(k+t-1)d}$ for all $t = 1, 2, \cdots, k$. Therefore, $(A^{kd})^2 = A^{(k+k)d} = A^{(2k-1)d} = A^{(2k-2)d} = \cdots = A^{kd}$. Taking $s = kd$, we can get that A^s is idempotent. ∎

Proposition 4.1. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic. If A is row (or column) diagonally dominant, then $t(A) = A^s$, where A^s is idempotent.*

Proof. We first consider the case A is row diagonally dominant. By Lemma 4.2, A^s is idempotent, then $A^{ks} = A^s$ for any integer $k > 0$, and so $a_{ij}^{(ks)} = a_{ij}^{(s)}$,

$\forall i, j \in \underline{n}$. If A is row diagonally dominant, then $a_{ij}^{(k)} = a_{ii} \cdots a_{ii} a_{ij}^{(k)} = a_{ii}^{ks-k} a_{ij}^{(k)} \leq a_{ij}^{(ks)}$ (since $a_{ij}^{ks-k} a_{ij}^{(k)}$ is the sum of some term in $a_{ij}^{(ks)}$). Thus, $a_{ij}^{(k)} \leq a_{ij}^{(s)}$. Hence, $\sum_{k=1}^{m} a_{ij}^{(k)} \leq a_{ij}^{(s)} (\forall m \geq k(A) + d(A) - 1)$, then $t(A) \leq A^s$. On the other hand, since $t(A) = \sum_{k=1}^{m} A^k$for all $m \geq k(A) + d(A) - 1$, we have $t(A) \geq A^s$. Therefore, $t(A) = A^s$.

The proof of the case that A is column diagonally dominant is similar. This completes the proof. ∎

Corollary 4.1. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic. If A is strongly diagonally dominant, then $t(A) = A^s$, where A^s is transitive.*

Proof. Since A is strongly diagonally dominant, we have A is column (or row) diagonally dominant. Therefore, the conclusion is from Proposition 4.1. ∎

Proposition 4.2. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic. If A is weakly diagonally dominant, then $t(A) = A^s$, where A^s is idempotent.*

Proof. By Lemma 4.2, A^s is idempotent, then $A^{ks} = A^s$ for any integer $k > 0$, and so $a_{ij}^{(ks)} = a_{ij}^{(s)}$ for any $i, j \in \underline{n}$. If A is weakly diagonally dominant, then for any $i \in \underline{n}$, either $a_{ii} a_{ij} = a_{ij}$ for all $j \in \underline{n}$ or $a_{ii} a_{ji} = a_{ji}$ for all $j \in \underline{n}$.

Let $a_{ij}^{(k)} = \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j}$, $(\forall i, j \in \underline{n})$. If $a_{i_1 i_1} a_{i_1 j} = a_{i_1 j}$ for all $j \in \underline{n}$, we have $a_{i_1 i_1} a_{i_1 i_2} = a_{i_1 i_2}$. Then

$$\begin{aligned}
a_{ij}^{(k)} &= \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} = \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_1} \cdots a_{i_1 i_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\
&= \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_1}^{ks-k} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\
&\leq a_{ij}^{(ks)} (\text{because } \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_1}^{ks-k} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\
&\qquad \text{is the sum of some term in } a_{ij}^{(ks)}) \\
&= a_{ij}^{(s)}.
\end{aligned}$$

If $a_{i_1 i_1} a_{j i_1} = a_{j i_1}$, for all $j \in \underline{n}$, we have $a_{i_1 i_1} a_{i i_1} = a_{i i_1}$. Then

$$\begin{aligned}
a_{ij}^{(k)} &= \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} = \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{i_1 i_1} \cdots a_{i_1 i_1} a_{i i_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\
&= \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{i_1 i_1}^{ks-k} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{k-1} j} = \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_1}^{ks-k} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\
&\leq a_{ij}^{(ks)} \ (\text{because } \sum_{1 \leq i_1, \cdots, i_{k-1} \leq n} a_{ii_1} a_{i_1 i_1}^{ks-k} a_{i_1 i_2} \cdots a_{i_{k-1} j} \\
&\qquad \text{is the sum of some term in } a_{ij}^{(ks)}) \\
&= a_{ij}^{(s)}.
\end{aligned}$$

From above, we have $a_{ij}^{(k)} \leq a_{ij}^{(s)}$ for any $i, j \in \underline{n}$. Hence, $\sum_{k=1}^{m} a_{ij}^{(k)} \leq a_{ij}^{(s)}$ for all $m \geq k(A)+d(A)-1$, and then $t(A) \leq A^s$. On the other hand, since $t(A) = \sum_{k=1}^{m} A^k$ for all $m \geq k(A) + d(A) - 1$, we have $t(A) \geq A^s$. Therefore, $t(A) = A^s$. ∎

Proposition 4.3. *If $t(A)$ is reflexive, then $t(t(A)) = t(A)$.*

Proof. By the definition of $t(A)$, $t(A) \geq (t(A))^2$. Since $t(A)$ is reflexive, we have $t(A) \geq I_n$, and so $(t(A))^2 \geq t(A)$. Hence, $(t(A))^2 = t(A)$. Therefore, $t(t(A)) = t(A)$. This completes the proof. ∎

In [37], Tan studied the convergence for powers of a lattice matrix. In the following, the convergent index for powers of transitive generalized fuzzy matrices over a path algebra is discussed.

Proposition 4.4. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic, then*

(1) $A^m \leq \sum_{k=1}^{k(A)+d(A)-1} A^k$ *for any integer* $m \geq 1$*;*

(2) $\sum_{l=1}^{+\infty} A^l = \sum_{l=1}^{k(A)+d(A)-1} A^l.$

Proof. Since A is almost periodic, the proof is obvious.

Theorem 4.1. *Let $A = (a_{ij}) \in M_n(L)$. If $A \geq A^2$ and $a_{ii} \in I(L) (\forall i \in \underline{n})$, then*

(1) $a_{ii} = a_{ii}^{(s)}$ *for any integer* $s \geq 1$*;*

(2) *A converges to $A^{k(A)}$ with $k(A) \leq n$.*

Proof. (1) By the hypothesis $A \geq A^2$, we have $A^k \geq A^{k+1} (\forall k \in Z_+)$. Hence $A^n \geq A^{n+1}$ and $a_{ii} \geq a_{ii}^{(s)} (\forall s \in Z_+)$. Since $a_{ii} \in I(L)$, we can get $a_{ii} = a_{ii} \cdots a_{ii} = a_{ii}^s \leq a_{ii}^{(s)} (\forall s \in Z_+)$ (since a_{ii}^s is a term of $a_{ii}^{(s)}$). Therefore, $a_{ii} = a_{ii}^{(s)}$ and $a_{ii}^{(s)} \in I(L)$.

(2) Next, we shall verify $A^n \leq A^{n+1}$.

Let $a_{ij}^{(n)} = \sum_{1 \leq i_1, i_2, \cdots, i_{n-1} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-1} j} (\forall i, j \in \underline{n})$. Since the number of indices in any term $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-1} j}$ is greater than n, there must be two indices i_u and i_v such that $i_u = i_v$ for some u, v $(0 \leq u < v \leq n, i_0 = i, i_n = j)$. Then

$$
\begin{aligned}
a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-1} j} &= a_{ii_1} a_{i_1 i_2} \cdots a_{i_{u-1} i_u} a_{i_u i_{u+1}} \cdots a_{i_{v-1} i_v} a_{i_v i_{v+1}} \cdots a_{i_{n-1} j} \\
&\leq a_{ii_1} a_{i_1 i_2} \cdots a_{i_{u-1} i_u} a_{i_u i_u}^{(v-u)} a_{i_v i_{v+1}} \cdots a_{i_{n-1} j} \\
&\qquad (\text{since } a_{i_u i_{u+1}} \cdots a_{i_{v-1} i_v} \text{ is a term of } a_{i_u i_u}^{(v-u)}) \\
&= a_{ii_1} a_{i_1 i_2} \cdots a_{i_{u-1} i_u} a_{i_u i_u}^{(v-u)} a_{i_u i_u}^{(v-u)} a_{i_v i_{v+1}} \cdots a_{i_{n-1} j} \\
&\qquad (\text{since } a_{ii} = a_{ii}^{(s)} (\forall i, s \in \underline{n}) \text{ and } a_{ii} \in I(L))
\end{aligned}
$$

$$\begin{aligned}
&\leq a_{ij}^{(n+(v-u))} (v-u \geq 1) \\
&\quad (\text{since } a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_u}^{(v-u)}a_{i_ui_u}^{(v-u)}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \\
&\quad \text{is the sum of some term in } a_{ij}^{(n+(v-u))}) \\
&\leq a_{ij}^{(n+1)} \ (\text{since } A^k \geq A^{k+1} \text{ for any } k \in Z_+).
\end{aligned}$$

Thus, $a_{ij}^{(n)} \leq a_{ij}^{(n+1)}$. Therefore, $A^n \leq A^{n+1}$. From above, we can get $A^n = A^{n+1}$. This completes the proof. ∎

Proposition 4.5. *If $A \in M_n(L)$ is almost periodic and $A \geq I_n$, then*

(1) *A converges to $A^{m(A)}$ with $m(A) \leq k(A)$;*

(2) $t(A) = \sum_{l=1}^{k(A)} A^l$;

(3) *$t(A)$ converges to $A^{k(A)}$.*

The proof is trivial.

Theorem 4.2. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii} \geq a_{jk}$ and $a_{ii}a_{jk} = a_{jk}$ ($\forall i, j, k \in \underline{n}$), then*

(1) *$a_{ii} = a_{ii}^s = a_{ii}^{(s)}$ for any integer $s \geq 1$;*

(2) *A converges to $A^{k(A)}$ with $k(A) \leq n-1$.*

Proof. (1) By the hypothesis $a_{ii}a_{jk} = a_{jk}$ for all $i, j, k \in \underline{n}$, we have $a_{11} = a_{22} = \cdots = a_{nn}$ and $a_{ii} = a_{ii}^s$ ($\forall i \in \underline{n}$ and $\forall s \in Z_+$). Since a_{ii}^s is a term of $a_{ii}^{(s)}$, we can get $a_{ii} = a_{ii}^s \leq a_{ii}^{(s)}$. Let $a_{ii}^{(s)} = \sum_{1 \leq i_1, \cdots, i_{s-1} \leq n} a_{ii_1}a_{i_1i_2}\cdots a_{i_{s-1}i} \leq \sum_{1 \leq i_1, \cdots, i_{s-1} \leq n} a_{ii}a_{i_1i_1}\cdots a_{i_{s-1}i_{s-1}}$ (by the hypothesis $a_{ii} \geq a_{jk}$ for any $i, j, k \in \underline{n}$) $= \sum_{1 \leq i_1, \cdots, i_{s-1} \leq n} a_{ii}^s$ (since $a_{11} = a_{22} = \cdots = a_{nn}$) $= \sum_{1 \leq i_1, \cdots, i_{s-1} \leq n} a_{ii}$ (since $a_{ii} = a_{ii}^s$) $= a_{ii}$. Therefore, $a_{ii} = a_{ii}^s = a_{ii}^{(s)}$ for any integer $s \in Z_+$.

(2) Since $a_{ij}^{(2)} = \sum_{k=1}^n a_{ik}a_{kj}$ for any $i, j \in \underline{n}$ and the hypothesis $a_{ii}a_{jk} = a_{jk}$ for any $i, j, k \in \underline{n}$, we can get $a_{ij}^{(2)} = \sum_{k=1}^n a_{ik}a_{kj} = \sum_{k=1}^n a_{ik}a_{kk}a_{kj} \leq a_{ij}^{(3)}$ (because $\sum_{k=1}^n a_{ik}a_{kk}a_{kj}$ is the sum of some term in $a_{ij}^{(3)}$). Thus, $A^2 \leq A^3$, and so $A^{n-1} \leq A^n$.

In the following, we will show that $A^{n-1} \geq A^n$. Since (1)$a_{ii} = a_{ii}^{(s)}$ for any $i \in \underline{n}$ and $s \in Z_+$, we only need to prove that $a_{ij}^{(n-1)} \geq a_{ij}^{(n)}$ for $i \neq j$. Let $a_{ij}^{(n)} = \sum_{1 \leq i_1, \cdots, i_{n-1} \leq n} a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}$ ($\forall i, j \in \underline{n}$ and $i \neq j$). Since the number of indices in $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}(i \neq j)$ is $n+1$, there must be two indices i_u and i_v such that $i_u = i_v$ for some $u, v(u < v)$.

If $v - u = 1$, then we can get

$$\begin{aligned} &a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j} \\ &\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_{u+1}}a_{i_{u+1}i_{u+2}}\cdots a_{i_{n-1}j} \\ &\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_v}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ (since } v = u+1) \\ &\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ (since } i_u = i_v) \\ &\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ (since } a_{ii}a_{jk} = a_{jk}(\forall i,j,k \in \underline{n})) \\ &\quad \leq a_{ij}^{(n-1)} \text{ (since } a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ is a term of } a_{ij}^{(n-1)}), \end{aligned}$$

and so $a_{ij}^{(n)} \leq a_{ij}^{(n-1)}$. Thus, $A^n \leq A^{n-1}$.

If $v - u > 1$, then we can get

$$\begin{aligned} &a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j} \\ &\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_v}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \\ &\quad \leq a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_v}^{(v-u)}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ (since } a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_v} \\ &\qquad \text{is a term of } a_{i_ui_u}^{(v-u)}) \\ &\quad = a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ (since } a_{ii} = a_{ii}^{(s)}(\forall i \in \underline{n}, \forall s \in Z_+)) \\ &\quad \leq a_{ij}^{(n+1-(v-u))} \text{ (since } a_{ii_1}a_{i_1i_2}\cdots a_{i_{u-1}i_u}a_{i_ui_u}a_{i_vi_{v+1}}\cdots a_{i_{n-1}j} \text{ is a term of} \\ &\quad a_{ij}^{(n+1-(v-u))}) \\ &\quad \leq a_{ij}^{(n-1)} \text{ (because } A^{k-1} \leq A^k \text{ for any integer } k \geq 3), \end{aligned}$$

and so $a_{ij}^{(n)} \leq a_{ij}^{(n-1)}$. Thus, $A^n \leq A^{n-1}$.

From above, we have $A^{n-1} = A^n$. This proves the conclusion. ∎

Proposition 4.6. *If $A = (a_{ij}) \in M_n(L)$ is irreflexive and transitive, then A converges to $A^{k(A)} = 0$ with $k(A) \leq n$.*

Proof. We only need to show $A^n = 0$. Let $a_{ij}^{(n)} = \sum_{1\leq i_1,\ldots,i_{n-1}\leq n} a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}$ for any $i, j \in \underline{n}$. We consider any term $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}$ of $a_{ij}^{(n)}$. Since $i, i_1, ..., i_{n-1}$, $j \in \underline{n}$, there are u, v such that $i_u = i_v$ $(0 \leq u < v \leq n, i_0 = i,\ i_n = j)$, and so any term $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j}$ has the factor $a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_v}$. Since $a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_v} = a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_u}$ (since $i_u = i_v$)$\leq a_{i_ui_u}^{(v-u)}$ (since $a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_u}$ is a term of $a_{i_ui_u}^{(v-u)}$) $\leq a_{i_ui_u}$ (because A is transitive)$= 0$ (since A is irreflexive), we have $a_{i_ui_{u+1}}\cdots a_{i_{v-1}i_v} = 0$, and so $a_{ii_1}a_{i_1i_2}\cdots a_{i_{n-1}j} = 0$ (by Lemma 4.2). Hence, $a_{ij}^{(n)} = 0$. Therefore, $A^n = 0$. This proves the conclusion. ∎

Proposition 4.6 generalizes Proposition 3.3 of Tan [34] and Corollary 3.1 of Han [16].

At the end of this section, some properties and characterizations for powers of generalized fuzzy matrices are showed from adjoint matrix.

Lemma 4.3. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii} \geq a_{jk}$ $(\forall i, j, k \in \underline{n})$, then $A^{n-1} \geq adj(A)$.*

Proof. In order to get $A^{n-1} \geq adj(A)$, we need to prove $a_{ij}^{(n-1)} \geq det(A_{ji})$.

By the hypothesis $a_{ii} \geq a_{jk}(\forall i, j, k \in \underline{n})$, then $a_{11} = a_{22} = \cdots = a_{nn}$.

First, we consider the case $i = j$. We see that $det(A_{ii})$ can be obtained from $det(A)$ by replacing a_{ii} by 1 and all other row-i entries $a_{ik}(k \neq i)$ by 0. Thus,

$$det(A_{ii}) = \sum_{\sigma \in s_n, \sigma(i)=i} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{i-1,\sigma(i-1)} a_{i+1,\sigma(i+1)} \cdots a_{n\sigma(n)},$$

where s_n denotes the symmetric group of all permutations of the indices $\{1, 2, ..., n\}$. We consider any term $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{i-1,\sigma(i-1)} a_{i+1,\sigma(i+1)} \cdots a_{n\sigma(n)}$ of $det(A_{ii})$, since

$$\begin{aligned} & a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{i-1,\sigma(i-1)} a_{i+1,\sigma(i+1)} \cdots a_{n\sigma(n)} \\ & \quad \leq a_{11} a_{22} \cdots a_{i-1,i-1} a_{i+1,i+1} \cdots a_{nn} (\text{by the hypothesis} a_{ii} \geq a_{jk} \ (\forall i, j, k \in \underline{n})) \\ & \quad = a_{ii}^{n-1} \ (\text{since } a_{11} = a_{22} = \cdots = a_{nn}) \\ & \quad \leq a_{ii}^{(n-1)} \ (\text{since } a_{ii}^{n-1} \text{ is a term of } a_{ii}^{(n-1)}), \end{aligned}$$

we have $det(A_{ii}) \leq a_{ii}^{(n-1)}$. On the other hand, since $a_{11} a_{22} \cdots a_{i-1,i-1} a_{i+1,i+1} \cdots a_{nn}$ is a term of the expansion of $det(A_{ii})$, we have $a_{11} a_{22} \cdots a_{i-1,i-1} a_{i+1,i+1} \cdots a_{nn} = a_{ii}^{n-1} \leq det(A_{ii})$. Thus, $a_{ii}^{n-1} \leq det(A_{ii}) \leq a_{ii}^{(n-1)}$. Since

$$\begin{aligned} a_{ii}^{(n-1)} & = \sum_{1 \leq i_1, \cdots, i_{n-2} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} i} \\ & \leq \sum_{1 \leq i_1, \cdots, i_{n-2} \leq n} a_{ii} a_{i_1 i_1} \cdots a_{i_{n-2} i_{n-2}} (\text{by the hypothesis } a_{ii} \geq a_{jk} (\forall i, j, k \in \underline{n})) \\ & = a_{ii}^{n-1} \ (\text{since } a_{11} = a_{22} = \cdots = a_{nn}), \end{aligned}$$

we can get $a_{ii}^{(n-1)} \leq a_{ii}^{n-1}$.

From above, we can get $a_{ii}^{n-1} \leq det(A_{ii}) \leq a_{ii}^{(n-1)} \leq a_{ii}^{n-1}$, and so $det(A_{ii}) = a_{ii}^{(n-1)} = a_{ii}^{n-1}$.

In the following, we consider the case $i \neq j$. Similarly, we note that $det(A_{ji})$ can be obtained from $det(A)$ by replacing a_{ji} by 1 and all other row-j entries $a_{jk}(k \neq i)$ by 0. Thus, we have

$$det(A_{ji}) = \sum_{\sigma \in s_n, \sigma(j)=i} a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n\sigma(n)},$$

where s_n denotes the symmetric group of all permutations of the indices $\{1, 2, ..., n\}$. Since $\sigma \in s_n$ and $\sigma(j) = i$, there must be an integer $t \geq 0$ such that $\sigma^{t+1}(i) = j$ and $i, \sigma(i), \cdots, \sigma^t(i), j$ are different each other.

We consider any term $a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n\sigma(n)}$ of $det(A_{ji})$, since

$$\begin{aligned} & a_{1\sigma(1)} a_{2\sigma(2)} \cdots a_{j-1,\sigma(j-1)} a_{j+1,\sigma(j+1)} \cdots a_{n\sigma(n)} \\ & \quad \leq a_{i\sigma(i)} a_{\sigma(i)\sigma^2(i)} \cdots a_{\sigma^t(i)j} a_{jj}^{n-t-2} \ (\text{by the hypothesis } a_{ii} \geq a_{jk} (\forall i, j, k \in \underline{n})) \\ & \quad \leq a_{ij}^{(n-1)} \ (\text{since } a_{i\sigma(i)} a_{\sigma(i)\sigma^2(i)} \cdots a_{\sigma^t(i)j} a_{jj}^{n-t-2} \text{ is a term of } a_{ij}^{(n-1)}), \end{aligned}$$

we have $det(A_{ji}) \leq a_{ij}^{(n-1)}$.

From above, we can get $det(A_{ji}) \leq a_{ij}^{(n-1)}(\forall i, j \in \underline{n})$, i.e., $adj(A) \leq A^{n-1}$. This proves the conclusion. ∎

Lemma 4.3 generalizes Theorem 4 of Duan [6].

Theorem 4.3. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii} \geq a_{jk}$ $(\forall i, j, k \in \underline{n})$, then $A^{n-1} = adj(A)$.*

Proof. By Lemma 4.3, we have $det(A_{ji}) \leq a_{ij}^{(n-1)}(\forall i, j \in \underline{n})$. Thus, we need only to show that $a_{ij}^{(n-1)} \leq det(A_{ji})$.

If $i = j$. By the proof of Lemma 4.3, we have $det(A_{ii}) = a_{ii}^{(n-1)}$. Then $a_{ii}^{(n-1)} \leq det(A_{ii})$.

If $i \neq j$. Let $a_{ij}^{(n-1)} = \sum\limits_{1 \leq i_1, \cdots, i_{n-2} \leq n} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j}$. We consider any term $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j}$ of $a_{ij}^{(n-1)}$. If the subscripts $i, i_1, ..., i_{n-2}, j$ are pairwise different, we can see that $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j}$ serves as a term of $det(A_{ji})$, and so

$$a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j} \leq det(A_{ji}).$$

If $\exists i_u = i_v$ ($u < v$ and $i_u, i_v \in \{i, i_1, ..., i_{n-2}, j\}$), deleting $a_{i_u i_{u+1}} \cdots a_{i_{v-1} i_v}$ from $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j}$, we obtain a new term $a_{ii_1} \cdots a_{i_{u-1} i_u} a_{i_v i_{v+1}} \cdots a_{i_{n-2} j}$. If there are still two identical numbers in the subscripts $i, i_1, ..., i_{u-1}, i_u, i_{v+1}, ..., i_{n-2}, j$, we apply the deleting method used in the above. This method can be applied repeatedly until the subscripts left are pairwise different. Finally, we can get a term $a_{ij_1} a_{j_1 j_2} \cdots a_{j_m j}$ such that $i, j_1, j_2, ..., j_m, j$ are pairwise different. Thus, $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j} \leq a_{ij_1} a_{j_1 j_2} \cdots a_{j_m j} a_{jj}^{n-m-2}$ (by $a_{ii} \geq a_{jk}$ $(\forall i, j, k \in \underline{n})$).

Let $\{k_1, k_2, ..., k_{n-m-2}\} = \underline{n} / \{i, j_1, j_2, ..., j_m, j\}$, then

$$\begin{aligned} a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j} &\leq a_{ij_1} a_{j_1 j_2} \cdots a_{j_m j} a_{jj}^{n-m-2} \\ &= a_{ij_1} a_{j_1 j_2} \cdots a_{j_m j} a_{k_1 k_1} a_{k_2 k_2} \cdots a_{k_{n-m-2} k_{n-m-2}} \\ &\text{(since } a_{11} = a_{22} = \cdots = a_{nn}) \\ &\leq det(A_{ji}) \text{ (since } a_{ij_1} a_{j_1 j_2} \cdots a_{j_m j} a_{k_1 k_1} a_{k_2 k_2} \cdots a_{k_{n-m-2} k_{n-m-2}} \\ &\text{is a term of } det(A_{ji})). \end{aligned}$$

Hence, $a_{ii_1} a_{i_1 i_2} \cdots a_{i_{n-2} j} \leq det(A_{ji})$. Therefore, $a_{ij}^{(n)} \leq det(A_{ji})$.

From above, we have $det(A_{ji}) = a_{ij}^{(n-1)}$, i.e., $adj(A) = A^{n-1}$. This completes the proof. ∎

Theorem 4.3 generalizes Theorem 6 of Duan [6].

Theorem 4.4. *Let $A = (a_{ij}) \in M_n(L)$. If the entries of A satisfy $a_{ii} \geq a_{jk}$ and $a_{ii} a_{jk} = a_{jk}$ $(\forall i, j, k \in \underline{n})$, then*

(1) *$adj(A)$ is idempotent;*

(2) *A is idempotent* $\Leftrightarrow adj(A) = A$;

(3) $adj(adj(A)) = adj(A)$;

(4) *if* $A \leq A^2$, *then* $Aadj(A) = adj(A)A = adj(A)$.

Proof. (1) By Theorems 4.2 and 4.3, we can get $(adj(A))^2 = (A^{n-1})^2 = A^{2n-2} = A^{n-1} = adj(A)$.

(2) By (1), the proof is trivial.

(3) By (2), the proof is obvious.

(4) Let $C = Aadj(A)$. Then $c_{ij} = \sum_{k=1}^{n} a_{ik}det(A_{jk}) \geq a_{ii}det(A_{ji}) = det(A_{ji})$ (by $a_{ii}a_{jk} = a_{jk}(\forall i, j, k \in \underline{n})$), i.e., $Aadj(A) \geq adj(A)$. On the other hand, by (1), Theorem 4.3 and the hypothesis $A \leq A^2$, we have $Aadj(A) \leq A^{n-1}adj(A) = (adj(A))^2 = adj(A)$. Therefore, $Aadj(A) = adj(A)$. The proof of the equality $adj(A)A = adj(A)$ is similar. The proof is completed. ∎

Theorem 4.4 generalizes Corollary 2 of Duan [6].

Theorem 4.5. *Let* $A = (a_{ij}) \in M_n(L)$. *If the entries of* A *satisfy* $a_{ik} = a_{jk}$ *for any* $i, j, k \in \underline{n}$, *then*

(1) *the entries of* $(adjA)^T = B = (b_{ij})$ *satisfy* $b_{ik} = b_{jk}$ *for any* $i, j, k \in \underline{n}$;

(2) *the entries of* $A(adjA) = C = (c_{ij})$ *satisfy* $c_{ik} = c_{jk}$ *for any* $i, j, k \in \underline{n}$ *and* $A(adjA) = (c_{ij}) = (det(A))$.

Proof. (1) By $B = (b_{ij}) = (adjA)^T$, we have

$$b_{ij} = det(A_{ij}) = \sum_{\sigma \in s_n, \sigma(i)=j} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{i-1,\sigma(i-1)}a_{i+1,\sigma(i+1)} \cdots a_{n\sigma(n)} \text{ and}$$

$$b_{kj} = det(A_{kj}) = \sum_{\sigma \in s_n, \sigma(k)=j} a_{1\sigma(1)}a_{2\sigma(2)} \cdots a_{k-1,\sigma(k-1)}a_{k+1,\sigma(k+1)} \cdots a_{n\sigma(n)}, \ (k \neq i).$$

Since $a_{ik} = a_{jk}$ for any $i, j, k \in \underline{n}$ and the numbers $\sigma(t)$ of columns cannot be changed in the two expansions of b_{ij} and b_{kj}, we can get $b_{ij} = b_{kj}$.

(2) By $a_{ik} = a_{jk}$, we have $A_{jk} = A_{ik}$, and so $det(A_{jk}) = det(A_{ik})$ for any $i, j, k \in \underline{n}$. Thus $c_{ij} = \sum_{k=1}^{n} a_{ik}det(A_{jk}) = \sum_{k=1}^{n} a_{ik}det(A_{ik}) = det(A)$. Therefore, $A(adjA) = (det(A))$. This completes the proof. ∎

Theorem 4.5 generalizes Proposition 4.6 [9].

Finally, we characterize the nilpotency with determinant zero from their cut matrices.

Let $A = (a_{ij}) \in M_n(L)$ and $k \in L$. The $k-cut$ of A means the $\{0, 1\}-$matrix and $A(k) = (a_{ij}(k)) \in M_n(L)$ defined as

$$a_{ij}(k) = \begin{cases} 1, & a_{ij} \geq k \\ 0, & \text{otherwise} \end{cases} \quad \text{for } i, j \in \underline{n}.$$

Proposition 4.7. *If $A \in M_n(L)$ and $det(A(k)) = 0$ for all $k \in L/\{0\}$, then $det(A) = 0$.*

Proof. Assume $det(A) > 0$. There exists a $\tau \in s_n$ such that $k = \prod_{i \in \underline{n}} a_{i\tau(i)} > 0$. Hence $det(A(k)) \geq \prod_{i \in \underline{n}} a_{i\tau(i)}(k) = 1$, i.e., $det(A(k)) > 0$. This is a contradiction. Therefore, $det(A) = 0$. This proves the conclusion. ∎

Proposition 4.7 generalizes Theorem 2.1 of Han [16].

Proposition 4.8. *If L has no nilpotent elements and $A \in M_n(L)$, then the following statements are equivalent:*

(1) $det(A) = 0$.

(2) $det(A(k)) = 0$ *for all* $k \in L/\{0\}$.

Proof. (1) ⇒ (2) Assume that $det(A(k)) > 0$ for some $k \in L/\{0\}$. Thus, $1 = det(A(k)) = \sum_{\sigma \in s_n} \prod_{i \in \underline{n}} a_{i\sigma(i)}(k)$ (since $A(k)$ is a $\{0,1\}$-matrix). Hence, there exists a $\tau \in s_n$ such that $\prod_{i \in \underline{n}} a_{i\tau(i)}(k) = 1$, and so $det(A) \geq \prod_{i \in \underline{n}} a_{i\tau(i)} \geq k^n$. Since L has no nilpotent elements, we have $k^n > 0$. Therefore, $det(A) \geq k^n > 0$, which is a contradiction.

(2) ⇒ (1) By Proposition 4.7, the proof is obvious. ∎

5. The invertibility of generalized fuzzy matrices

In this section, the invertibility of a generalized fuzzy matrix is investigated.

Definition 5.1. A matrix $A = (a_{ij}) \in M_n(L)$ is said to be

(1) right invertible if there exists a matrix $B \in M_n(L)$ such that $AB = I_n$;

(2) left invertible if there exists a matrix $B \in M_n(L)$ such that $BA = I_n$;

(3) invertible if A is both right and left invertible.

Definition 5.2. Let $A \in M_n(L)$. A mapping $f_A : L^n \longrightarrow L^n$ is defined by $f_A(x) = Ax$ for $x \in L^n$.

Theorem 5.1. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic and right invertible. Then $A^{d(A)} = I_n$.*

Proof. Since A is right invertible, we have $AX = I_n$ for some $X \in M_n(L)$. Since A is almost periodic, $A^{k(A)} = A^{k(A)+d(A)}$. Then $A^{d(A)} = A^{d(A)} \cdot I_n = A^{d(A)} \underbrace{(A \cdots (A(A}_{k\ times} X)X) \cdots X) = A^{d(A)} \cdot A^{k(A)} \cdot X^{k(A)} = A^{k(A)+d(A)} \cdot X^{k(A)} = A^{k(A)} \cdot X^{k(A)} = I_n$, and so $A^{d(A)} = I_n$. ∎

Theorem 5.2. *Let $A = (a_{ij}) \in M_n(L)$ be almost periodic and left invertible, then $A^{d(A)} = I_n$.*

Proof. Similar to that of Theorem 5.1. ∎

6. Conclusions

A path algebra is a special type of semiring but it generalizes Boolean algebra, fuzzy algebra, distributive lattice and incline. In this paper, we study matrices over path algebras. Firstly, we discuss compositions of matrices over path algebras and obtain a new matrix from given matrices. Secondly, we give some properties and characteristics of transitivity and convergence of powers of a matrix, and the adjoint matrix is also studied. Finally, the invertibility of a matrix is considered. The main results in the present paper are the generalizations of the corresponding results in the literatures which have studied the Boolean matrices, the fuzzy matrices, the lattice matrices and the incline matrices.

In this paper, L is always assumed to be a commutative path algebra. But, as for the non-commutative algebra, are this paper's conclusions still justified? It still deserves further study.

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References

[1] Baccelli, F., Mairesse, *Ergodic theorems for stochastic operators and discrete event networks*, in: J. Gunawardena (Ed.), Idempotency (Bristol, 1994) publications of the Newton Institute,Cambridge University Press, 11 (1998), 171-208.

[2] Carrè, B., *Groups and Networks*, Clarendon Press, Oxford, 1979.

[3] Cechlarova, K., *On the powers of matrices in bottleneck/fuzzy algebra*, Linear Algebra and its Applications, 246 (1996), 97-111.

[4] Cechlarova, K., *Powers of matrices over distributive lattices-a review*, Fuzzy Sets and Systems, 138 (2003), 627-641.

[5] Cho, H.H., Kim, H.K., *The competion index of a nearly reducible Boolean matrix*, Bulletin of the Korean Mathematical Society, 50 (2013), 2001-2011.

[6] Duan, J.S., *The transitive closure, convergence of powers and adjoint of generalized fuzzy matrices*, Fuzzy Sets and Systems, 145 (2004), 301-311.

[7] Dubois, D., Prade, H., *Fuzzy Sets and Systems*, Academiic Press, New York, 1980.

[8] Emam, E.G., Ragab, M.Z., *On the min-max composition of fuzzy matrices*, Fuzzy Sets and Systems, 75 (1995), 83-92.

[9] Emam, E.G., Ragab, 'M.Z., *The determinant and adjoint of a square fuzzy matrix*, Fuzzy Sets and Systems, 61 (1994), 297-307.

[10] Ghosh, S., *Matrices over semirings*, Information Sciences, 90 (24) (1996), 221-230.

[11] Goguen, J.A., *L-fuzzy set*, Journal of Mathematical Analysis and Applications, 18 (1967), 145-174.

[12] Golan, J.S., *Semirings and their Applications*, Kluwer Academic Publishers, Dordrecht, 1999.

[13] Gondran, M., *Path Algebra and Algorithms*, in: B. Roy(Ed.), Combinatorial Programming, Reidel, Dordrecht, 1975, 137-148.

[14] Han, S.C., Li, H.X., *Indices and periods of incline matrices*, Linear Algebra and its Applications, 387 (2004), 143-165.

[15] Han, S.C., Li, H.X., *Invertible incline matrices and Cramer's rule over inclines*, Linear Algebra and its Applications, 389 (2004), 121-138.

[16] Han, S.C., Li, H.X., Wang, Y.J., *On nilpotent incline matrices*, Linear Algebra and its Applications, 406 (2005), 201-217.

[17] Hashimoto, H., *Convergence of powers of a fuzzy transitive matrix*, Fuzzy Sets and Systems, 9 (1983), 153-160.

[18] Hashimato, H., *Decomposition of fuzzy matrices*, SIAM. J. Algebraic Discrete Methods, 6 (1985), 32-38.

[19] Hashimoto, H., *Transitivity of generalized fuzzy matrices*, Fuzzy Sets and Systems, 17 (1985), 83-90.

[20] Higuchi, S., Tanaka, K., Tamura, S., *Pattern classification based on fuzzy relations*, IEEE Transictions on Systems, Man, and Cybernetics, SMC-1 (1) (1971), 61-66.

[21] Imai, H., Miyakoshi, M., Okahara, Y., *The period of powers of a fuzzy matrix*, Fuzzy Sets and Systems, 109 (2000), 405-408.

[22] Jiang, J., Shu, L., Tian, X., *On generalized transitive matrices*, Journal of Applied Mathematics, vol. 2011, 15 pages, 2011.

[23] Kim, K.H., *Boolean Matrix Theory and Applications*, Marcel Dekker Inc., New York, 1982.

[24] Kim, K.H., Roush, F.W., *Generalized fuzzy matrices*, Fuzzy Sets and Systems, 4 (1980), 293-315.

[25] Klement, E.P., Mesiar, R., Pap, E., *Triangular Norms*, Kluwer Academic Publishers, Dordrecht, 2000.

[26] Li, J.X., *Periodicity of powers of fuzzy matrices (finite fuzzy relations)*, Fuzzy Sets and Systems, 48 (1992), 365-369.

[27] Minoux, M., *A generalization of the all minors matrix tree theorem to semirings*, Discrete Mathematics, 199 (1999), 139-150.

[28] Ovchinnikov, S.V., *Structure of fuzzy binary relations*, Fuzzy Sets and Systems, 6 (1981), 169-195.

[29] Reutenauer, C., Straubing, H., *Inversion of matrices over a commutative semiring*, Journal of Algebra, 88 (1984), 350-360, 1984.

[30] Rutherford, D.E., *The Cayley-Hamilton theorem for semiring*, Proceedings of the Royal society of Edinburgh. Section A. Mathematics, 66 (1964), 211-215.

[31] Tahani, V., *A fuzzy model of document retrieval systems*, Information Processing Management, 12 (1976), 177-187.

[32] Tan, Y.J., *On compositions of lattice matrices*, Fuzzy Sets and Systems, 129 (2002), 19-28.

[33] Tan, Y.J., *On generalized fuzzy matrices with periods*, Fuzzy Sets and Systems, 172 (2011), 87-103.

[34] Tan, Y.J., *On nilpotent matrices over distributive lattices*, Fuzzy Sets and Systems, 151 (2005), 421-433.

[35] Tan, Y.J., *On the powers of matrices over a distributive lattice*, Linear Algebra and its Applications, 336 (2001), 1-14.

[36] Tan, Y.J., *On transitivity of generalized fuzzy matrices*, Fuzzy Sets and Systems, 210 (2013), 69-88.

[37] Tan, Y.J., *On the transitive matrices over distributive lattices*, Linear Algebra and its Applications, 400 (2005), 169-191.

[38] Thomason, M.G., *Convergence of powers of a fuzzy matrix*, Journal of Mathematical Analysis and Applications, 57 (1977), 476-480.

[39] Zimmermann, U., *Linear and combinatorial optimization in ordered algebraic structures*, Annals of Discrete Mathematics, North Holland, 1981.

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