NEW SOLITARY WAVE AND MULTIPLE SOLITON SOLUTIONS FOR THE TIME-SPACE FRACTIONAL BOUSSINESQ EQUATION

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Abstract. The aim of this paper are in two fold: First, introduce a new simplified bilinear method based on a transformation method combined with the Hirota bilinear sense. Second, apply this new technique to study the time-space fractional Boussinesq equation. To the best of my knowledge, the proposed work present new N-soliton solutions of the fractional Boussinesq equation. These new exact solutions extend previous results and help us to explain the properties of multidimensional nonlinear solitary waves in shallow water. Parametric analysis is carried out in order to illustrate that the soliton amplitude, width and velocity are affected by the coefficient parameters in the equation.

Keywords: Hirota bilinear method, N-soliton solutions, modified Riemann-Liouville derivative, fractional Boussinesq equation.
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1. Introduction

It has been a great deal and efforts of interest in fractional differential equations. But, there were almost no practical applications of fractional calculus, and this area has been started as an abstract containing only mathematical manipulations of little or no use. The last three decades, the paradigm began to shift from pure mathematical formulations to applications in various fields. During the last decade fractional calculus has been applied to almost every field of science, engineering, and mathematics. Several fields of application of fractional differentiation and fractional integration are already well established, some others have just started. Fractional differential equations (FDEs) have found applications in many problems in physics and engineering [14], [16], [17], [18]. Phenomena in electromagnetic, acoustics, viscoelasticity, electrochemistry, and material science are also well described by differential equations of fractional order [19], [16], [20], [21]. The solution of differential equations of fractional order is much involved too. In general, there exists no method that yields an exact solution for nonlinear fractional differential equations. Only approximate solutions can be derived using linearization or perturbation method. Since most of the nonlinear FDEs cannot be solved exactly, approximate and numerical methods must be used. Not much work has been done on nonlinear problems and only a few numerical schemes have been proposed for solving nonlinear fractional differential equations. More recently, applications have included classes of nonlinear equation with multi-order fractional derivatives and this motivates researchers to develop a numerical scheme for their solution [22]. Numerical and analytical methods have included the Adomian decomposition method (ADM) [23], [24], the variational iteration method (VIM) [25], the homotopy perturbation method [26], [27], [28] and more recently the residual power series method [29]. In this work, a new simplified bilinear method based on a transformation method combined with the Hirota's bilinear sense will be introduced to study the fractional Boussinesq equation. Our goal from applying this method is to construct multiple regular soliton solutions and multiple singular soliton solutions. The introduced simplified algorithm derives the auxiliary functions, obtained in Hirota's method, without using the bilinear forms. Although there are studies for the classical Boussinesq equation [30] and some profound results have been established, it seems that detailed studies of the fractional Boussinesq equation are only beginning. To the best of my knowledge the proposed work will represent the first available $N-$ soliton solution of the time-space fractional Boussinesq equation.

2. Preliminaries

There are different definitions for fractional derivatives; for more details see [14]. In our paper we use the modified Riemann–Liouville derivative which was defined by Jumarie [15]

$$\begin{cases} D_x^\alpha f(x) &= \dfrac{1}{\Gamma(1-\alpha)}\displaystyle\int_0^x (x-y)^{-1-\alpha}[f(y)-f(0)]dy, \alpha < 0 \\ &= \dfrac{1}{\Gamma(1-\alpha)}\dfrac{d}{dx}\displaystyle\int_0^x (x-y)^{-1-\alpha}[f(y)-f(0)]dy, 0<\alpha<1 \\ &= [f^{(\alpha-n)}(x)]^{(n)}, n \le \alpha < n+1 \end{cases}$$

which has merits over the original one, for example, the α-order derivative of a constant is zero. The main properties of the modified Riemann–Liouville derivative were summarized in [15] and three useful formulas of them are given as follows:

$$\begin{cases} D_x^\alpha x^\beta &= \dfrac{\Gamma(1+\beta)}{\Gamma(1+\beta-\alpha)}x^{\beta-\alpha}, \beta > 0 \\ D_x^\alpha (u(x)v(x)) &= u(x)D_x^\alpha v(x) + v(x)D_x^\alpha u(x) \\ D_x^\alpha [f(v(x))] &= \dfrac{df}{du}D_x^\alpha u(x) = \left(\dfrac{du}{dx}\right)^\alpha D_u^\alpha f(u). \end{cases}$$

3. Multiple soliton solutions for the fractional Boussinesq equation

To derive N-soliton solutions of any completely integrable equation, we will mainly use the Hirota's direct method. The Hirota method relies on a transformation for considered equation to a bilinear form. The bilinear forms are usually used to enable us deriving the auxiliary function. It is remarkable to mention that it is not easy to find the bilinear form for many equations and sometimes it requires the introduction of new dependent and sometimes even independent variables. However, Hereman et al. [7], [8], [9], formally introduced the simplified algorithm to derive the auxiliary functions without using the bilinear forms. The Cole–Hopf transformation method combined with the simplified Hirota's sense is a powerful method to determine multiple soliton solutions and multiple singular soliton solutions for integrable systems[2], [3], [4], [5], [6], [10], [11], [12], [13].

In this section, we apply simplified bilinear method, to construct soliton solutions of fractional equation:

$$u_t^{(2\alpha)} + au_{xx} + 2b(uu_x^{(2\beta)} + (u_x^{(\beta)})^2) + cu_x^{(4\beta)} = 0, \tag{3.1}$$

where the coefficients a, b, c are real-valued parameters that are physically meaningful.

The following fractional transforms are introduced

$$T = \frac{p_1 t^\alpha}{\Gamma(1+\alpha)}, \quad X = \frac{p_2 x^\beta}{\Gamma(1+\beta)},$$

where p_1 and p_2 are constants. Using the above transforms, we can convert fractional derivatives into the following classical derivatives:

$$\frac{\partial^\alpha u}{\partial t^\alpha} = p_1 \frac{\partial u}{\partial T}, \quad \frac{\partial^\beta u}{\partial x^\beta} = p_2 \frac{\partial u}{\partial X},$$

see [1]. Therefore, equation (3.1) becomes

$$u_{TT} + au_{XX} + b(u^2)_{XX} + cu_{XXXX} = 0. \tag{3.2}$$

Now, we substitute

$$u(X,T) = e^{\theta_i(X,T)} \tag{3.3}$$

$$\theta_i(X,T) = k_i X - \omega_i T \tag{3.4}$$

into the linear terms of equation (3.2) to determine the dispersion relation as follows

$$\omega_i = ik_i\sqrt{a + ck_i^2}. \tag{3.5}$$

As a result, we obtain

$$\theta_i = k_i x - ik_i\sqrt{a + ck_i^2}T. \tag{3.6}$$

The single soliton solution of equation (3.2) is assumed to be

$$u(X,T) = R\,(\ln f)_{XX} \tag{3.7}$$

where the auxiliary function $f(X,T)$ is given by

$$f(X,T) = 1 + e^{\theta_1(X,T)} = 1 + e^{k_1 X - ik_1\sqrt{a+ck_1^2}T}. \tag{3.8}$$

Substituting (3.7) into equation (3.2) and solving for R we find $R = 6c/b$. By substituting $f(X,T)$ into (3.7), we obtain the single soliton solution

$$u(X,T) = \frac{6c}{b}k_1^2\frac{e^{\theta_1(X,T)}}{\left(1+e^{\theta_1(X,T)}\right)^2} = \frac{6c}{b}k_1^2\frac{e^{k_1X-ik_1\sqrt{a+ck_1^2}T}}{\left(1+e^{k_1X-ik_1\sqrt{a+ck_1^2}T}\right)^2}. \tag{3.9}$$

The solution in (3.9) can be written as

$$u(X,T) = \frac{3c}{2b}k_1^2\sec h^2\left(\frac{\theta_1(X,T)}{2}\right), \tag{3.10}$$

then we obtain the single-soliton solution $u(x,t)$

$$u(x,t) = \frac{3c}{2b}k_1^2\sec h^2\left(\frac{\theta_1(x,t)}{2}\right), \tag{3.11}$$

where

$$\theta_1(x,t) = k_1\frac{x^\beta}{\Gamma(1+\beta)} - \frac{ik_1\sqrt{a+ck_1^2}t^\alpha}{\Gamma(1+\alpha)}, \tag{3.12}$$

which is a bell-shaped solitary wave solution of the Boussinesq equation (3.1).

For two-soliton solutions, we set

$$f(X,T) = 1 + e^{\theta_1(X,T)} + e^{\theta_2(X,T)} + a_{12}e^{\theta_1(X,T)+\theta_2(X,T)}, \tag{3.13}$$

where θ_1 and θ_2 are defined in (3.6). Using (3.13) in

$$u(X,T) = \frac{6c}{b}(\ln f)_{XX}. \tag{3.14}$$

and substituting the result in equation (3.2), we obtain the following phase shift :

$$a_{12} = \frac{2ck_1^2 - 3ck_1k_2 - \sqrt{a+ck_1^2}\sqrt{a+ck_2^2} + 2ck_2^2 + a}{2ck_1^2 + 3ck_1k_2 - \sqrt{a+ck_1^2}\sqrt{a+ck_2^2} + 2ck_2^2 + a}. \tag{3.15}$$

The two-soliton solutions of the Boussinesq equation (3.2) are obtained in an explicit form as

$$\begin{aligned} u(X,T) &\\ = \frac{6c}{b}&\left[\begin{matrix} k_1^2e^{\theta_1(X,T)}+k_2^2e^{\theta_2(X,T)}+a_{12}\left(k_2^2e^{\theta_1(X,T)}+k_1^2e^{\theta_2(X,T)}\right)e^{\theta_1(X,T)+\theta_2(X,T)} \\ +\left[a_{12}(k_1+k_2)^2+(k_1-k_2)^2\right]e^{\theta_1(X,T)+\theta_2(X,T)} \end{matrix}\right] \Big/ \\ &\left[\left(1+e^{\theta_1(X,T)}+e^{\theta_2(X,T)}+a_{12}e^{\theta_1(X,T)+\theta_2(X,T)}\right)^2\right]. \end{aligned}$$

As a result, we obtain the two-soliton solutions of the fractional Boussinesq equation (3.1)

$$u(x,t) = \frac{6c}{b}\left[\begin{array}{c} k_1^2 e^{\theta_1(x,t)}+k_2^2 e^{\theta_2(x,t)}+a_{12}\left(k_2^2 e^{\theta_1(x,t)}+k_1^2 e^{\theta_2(x,t)}\right)e^{\theta_1(x,t)+\theta_2(x,t)} \\ +\left[a_{12}\left(k_1+k_2\right)^2+\left(k_1-k_2\right)^2\right]e^{\theta_1(x,t)+\theta_2(x,t)}\end{array}\right] \Big/ \left[\left(1+e^{\theta_1(x,t)}+e^{\theta_2(x,t)}+a_{12}e^{\theta_1(x,t)+\theta_2(x,t)}\right)^2\right] \tag{3.16}$$

where θ_1 and θ_2 are defined as

$$\theta_i(x,t) = k_i\frac{x^\beta}{\Gamma(1+\beta)} - \frac{ik_i\sqrt{a+ck_i^2}\,t^\alpha}{\Gamma(1+\alpha)},\ i=1,2. \tag{3.17}$$

The three-soliton solutions can be obtained by using

$$\begin{aligned} f(X,T) = 1 &+ e^{\theta_1(X,T)} + e^{\theta_2(X,T)} + e^{\theta_3(X,T)} + a_{12}e^{\theta_1(X,T)+\theta_2(X,T)} \\ &+ a_{13}e^{\theta_1(X,T)+\theta_3(X,T)} + a_{23}e^{\theta_2(X,T)+\theta_3(X,T)} \\ &+ a_{123}e^{\theta_1(X,T)+\theta_2(X,T)+\theta_3(X,T)}, \end{aligned} \tag{3.18}$$

where

$$a_{ij} = \frac{2ck_i^2 - 3ck_ik_j - \sqrt{a+ck_i^2}\sqrt{a+ck_j^2} + 2ck_j^2 + a}{2ck_i^2 + 3ck_ik_j - \sqrt{a+ck_i^2}\sqrt{a+ck_j^2} + 2ck_j^2 + a}. \tag{3.19}$$

Substituting (3.18) and (3.14) into equation (3.2), we find that

$$a_{123} = a_{12}a_{13}a_{23}. \tag{3.20}$$

Since (3.20) holds, we can use the results indicated in [31, 32] to conclude that the N-soliton solutions for the fractional Boussinesq equation (3.1) can be obtained for finite N, where $N \geq 1$.

4. Multiple singular soliton solutions for the fractional Boussinesq equation

To obtain a single singular soliton solution, we substitute

$$u(X,T) = e^{k_iX-\omega_iT}, \tag{4.1}$$

into the linear terms of equation (3.2). This gives the dispersion relation (3.5) and as a result we obtain

$$\theta_i(X,T) = k_iX - ik_i\sqrt{a+ck_i^2}\,T. \tag{4.2}$$

The singular single soliton solution of equation (3.2) is assumed to be

$$u(X,T) = R\left(\ln f\right)_{XX} \tag{4.3}$$

where the auxiliary function $f(X,T)$ is given by

$$f(X,T) = 1 - e^{\theta_1(X,T)} = 1 - e^{k_1 X - ik_1\sqrt{a+ck_1^2}T}. \tag{4.4}$$

Substituting (4.3) into equation (3.2) and solving for R, we find $R = \frac{6c}{b}$. Now, (4.3) gives the single singular soliton solution for the Boussinesq equation (3.2)

$$u(X,T) = -\frac{6c}{b}k_1^2 \frac{e^{k_1 X - ik_1\sqrt{a+ck_1^2}T}}{(e^{k_1 X - ik_1\sqrt{a+ck_1^2}T} - 1)^2}, \tag{4.5}$$

and so

$$u(x,t) = -\frac{6c}{b}k_1^2 \frac{e^{k_1\frac{x^\beta}{\Gamma(1+\beta)} - \frac{ik_1\sqrt{a+ck_1^2}t^\alpha}{\Gamma(1+\alpha)}}}{(e^{k_1\frac{x^\beta}{\Gamma(1+\beta)} - \frac{ik_1\sqrt{a+ck_1^2}t^\alpha}{\Gamma(1+\alpha)}} - 1)^2} = -\frac{3c}{2b}k_1^2 \csc h^2\left(\frac{\theta_1}{2}\right),$$

where θ_1 is defined by (3.12). Multiple singular soliton solutions for equation (3.2) can be expressed in the following form:

$$u(X,T) = \frac{6c}{b}(\ln f)_{XX}. \tag{4.6}$$

To determine the two singular soliton solutions explicitly, we substitute

$$f(X,T) = 1 - e^{\theta_1(X,T)} - e^{\theta_2(X,T)} + a_{12}e^{\theta_1(X,T)+\theta_2(X,T)}, \tag{4.7}$$

where $\theta_1(X,T)$, $\theta_2(X,T)$ are defined in (4.2) into (4.6) and substituting the result in equation (3.2), to obtain the phase shift a_{12} as in (3.15). As a result, we obtain the two singular soliton solutions of the fractional Boussinesq equation (3.1)

$$\begin{aligned} &u(x,t) \\ &= -\frac{6c}{b}\left[\begin{array}{c} k_1^2 e^{\theta_1(x,t)} + k_2^2 e^{\theta_2(x,t)} + a_{12}\left(k_2^2 e^{\theta_1(x,t)} + k_1^2 e^{\theta_2(x,t)}\right)e^{\theta_1(x,t)+\theta_2(x,t)} \\ -\left[a_{12}(k_1+k_2)^2 + (k_1-k_2)^2\right]e^{\theta_1(x,t)+\theta_2(x,t)} \end{array}\right] \Big/ \\ &\quad \left[\left(1 - e^{\theta_1(x,t)} - e^{\theta_2(x,t)} + a_{12}e^{\theta_1(x,t)+\theta_2(x,t)}\right)^2\right] \end{aligned}$$

where $\theta_1(x,t)$ and $\theta_2(x,t)$ are defined in (3.17).

For the singular three-soliton solutions we use

$$\begin{aligned} f(X,T) = 1 &- e^{\theta_1(X,T)} - e^{\theta_2(X,T)} - e^{\theta_3(X,T)} + a_{12}e^{\theta_1(X,T)+\theta_2(X,T)} \\ &+ a_{13}e^{\theta_1(X,T)+\theta_3(X,T)} + a_{23}e^{\theta_2(X,T)+\theta_3(X,T)} \\ &- a_{123}e^{\theta_1(X,T)+\theta_2(X,T)+\theta_3(X,T)}, \end{aligned} \tag{4.8}$$

where $\theta_i(X,T)$ $(i = 1,2,3)$ are defined in (4.2) and a_{ij} are defined in (3.19). Proceeding as before, we obtain$a_{123} = a_{12}a_{13}a_{23}$. The singular three-soliton solution can be obtained explicitly using (4.6) for the function f in (4.8).

5. Analysis of the parameters

The solution (3.11) gives a profile of bell-shaped solitary wave with soliton amplitude *amp* and width Δ can be expressed as

$$amp = \left|\frac{3c}{2b}k_1^2\right|, \qquad \Delta = \frac{2}{|k_1|}. \tag{5.1}$$

With the characteristic-line method [2], [3], [10], the characteristic line for each solitary wave can be defined by

$$x^{\beta} - \frac{i\Gamma(1+\beta)\sqrt{a+ck_i^2}t^{\alpha}}{\Gamma(1+\alpha)} = 0, \qquad i = 1, 2, 3, \ldots, \tag{5.2}$$

which can be derived from relations (3.5) and (3.6). Correspondingly, the velocity v of each solitary wave can be expressed by

$$v_i = \frac{1}{\beta}\frac{i\alpha\Gamma(1+\beta)\sqrt{a+ck_i^2}t^{\alpha-1}}{\Gamma(1+\alpha)}\left(\frac{i\Gamma(1+\beta)\sqrt{a+ck_i^2}t^{\alpha}}{\Gamma(1+\alpha)}\right)^{\frac{1}{\beta}-1}. \tag{5.3}$$

Further, the absolute value of velocity v determines the speed, namely, velocity in magnitude, and propagation direction of soliton is decided by the sign of v.

The analysis is illustrated by studying the time fractional Boussinesq equation

$$u_t^{(2\alpha)} + au_{xx} + b(u)^2_{xx} + cu_{xxxx} = 0, \tag{5.4}$$

From (3.11) we obtain the single-soliton solution $u(x,t)$

$$u(x,t) = \frac{3c}{2b}k_1^2 \sec h^2\left(\frac{\theta_1(x,t)}{2}\right), \tag{5.5}$$

where

$$\theta_i = k_i x - \frac{ik_i\sqrt{a+ck_i^2}t^{\alpha}}{\Gamma(1+\alpha)}. \tag{5.6}$$

For the single-soliton solution (5.5), it is obvious that the amplitude of the soliton is $\left|\frac{3c}{2b}k_1^2\right|$, which keeps invariant during the propagation. Following the characteristic-line method, the characteristic face for each solitary wave can be defined by

$$x = \frac{i\sqrt{a+ck_i^2}t^{\alpha}}{\Gamma(1+\alpha)}, \qquad i = 1, 2, \ldots, \tag{5.7}$$

which can be derived from relation (5.6). Correspondingly, the velocity of the wave at time t can be expressed as

$$v_i = \frac{i\alpha\sqrt{a+ck_i^2}t^{\alpha-1}}{\Gamma(1+\alpha)}. \tag{5.8}$$

Further, the absolute value of velocity v determines the speed, namely, velocity in magnitude, and propagation direction of soliton is decided by the sign of v.

We now analytically examine the effects of the parameters a, b, c on the behavior of the solitary waves. The soliton amplitude *amp* is dependent on the ratio of c to b. The solitonic amplitude increases with an increase in the ratio c/b. However, Expression (5.8) indicates that the propagation velocity of the solitary wave is influenced by the coefficient parameters a and c, while it is independent of the coefficient parameter b. It can be observed that the initial superposed solitons travel different distances over a period of time for the different choices of a and c, and it can be concluded that the larger the absolute value of v is the greater the speed. Moreover, if the sign of v reverses, the solitonic direction will be changed accordingly.

6. Conclusions

In this paper we succeeded in obtaining single-soliton, two-soliton, and three-soliton solutions of the time-space fractional Boussinesq model. Also, multiple-singular soliton solutions are obtained for the same model. The adopted method being used in this work is a simplified form of the bilinear method combining with a transformation method, where the needed auxiliary functions obtained in Hirota method can be derived without using bilinear forms. The obtained results reveal that the proposed method is effective and powerful tool in studying N-soliton and N-singular-soliton solutions for both classical and fractional nonlinear equations. Furthermore, based on the one-soliton solution in expression (3.11), we have carried out the parametric analysis in order to investigate the effects of the parameters a, b, c on the soliton amplitude, width and velocity. To our knowledge, the solutions we have constructed are new and different from those in the existing papers.

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