QUASI-FREDHOLM, SAPHAR SPECTRA FOR $C_0$ SEMIGROUPS GENERATORS

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Abstract. In this work, we show that the spectral inclusion of semigroups hold for Saphar, essentially Saphar and quasi-Fredholm spectra. Some stabilities results are also established.

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1. Introduction and preliminaries

Throughout, $X$ denotes a complex Banach space, let $A$ be a closed linear operator on $X$ with domain $D(A)$, we denote by $A^*$, $R(A)$, $N(A)$, $R^\infty(A) = \bigcap_{n \geq 0} R(A^n)$, $\sigma(A)$, respectively the adjoint, the range, the null space, the hyper-range, the spectrum of $A$.

Recall that a closed operator $A$ is said to be Kato operator or semi-regular if $R(A)$ is closed and $N(A) \subseteq R^\infty(A)$. Denote by $\rho_K(A)$:

$$\rho_K(A) = \{ \lambda \in \mathbb{C} : A-\lambda I \text{ is Kato } \}$$

the Kato resolvent and $\sigma_K(A) = \mathbb{C}\backslash \rho_K(A)$ the Kato spectrum of $A$. It is well known that $\rho_K(A)$ is an open subset of $\mathbb{C}$.

For subspaces $M$, $N$ of $X$ we write $M \subseteq^e N$ ($M$ is essentially contained in $N$) if there exists a finite-dimensional subspace $F \subset X$ such that $M \subseteq N + F$.

A closed operator $S$ is called a generalized inverse of $A$ if $R(A) \subseteq D(S)$, $R(S) \subseteq D(A)$, $ASA = A$ on $D(A)$ and $SAS = S$ on $D(S)$, which equivalent to the fact that $R(A) \subseteq D(S)$, $R(S) \subseteq D(A)$, $ASA = A$ on $D(A)$.

A closed operator $A$ is called a Saphar operator if $A$ has a generalized inverse and $N(A) \subseteq R^\infty(A)$, which equivalent to the fact that $A$ is Kato operator and has a generalized inverse, see [6].
If we assume in the definition above that $N(A) \subseteq R^\infty(A)$, $A$ is said to be a essentially Saphar operator. The Saphar and essentially Saphar spectra are defined by

$$
\sigma_{Sap}(A) = \{ \lambda \in \mathbb{C} : A - \lambda \text{ is not Saphar} \}.
$$

$$
\sigma_{eSap}(A) = \{ \lambda \in \mathbb{C} : A - \lambda \text{ is not essentially Saphar} \}.
$$

$\sigma_{Sap}(A)$ is a compact non empty set of $\mathbb{C}$ and we have

$$
\partial \sigma(A) \subseteq \sigma_K(A) \subseteq \sigma_{Sap}(A) \subseteq \sigma(A).
$$

A one-parameter family $\{T(t)\}_{t \geq 0}$ of operators on $X$ is called a $C_0$-semigroup of operators if:

1. $T(0) = I$.
2. $T(t+s) = T(t)T(s)$, $\forall t, s \geq 0$.
3. $\lim_{t \to 0} T(t)x = x$, $\forall x \in X$.

$\{T(t)\}_{t \geq 0}$ has a unique infinitesimal generator $A$ defined in the domain $D(A)$ by:

$$
D(A) = \left\{ x \in X : \lim_{t \to 0} \frac{T(t)x - x}{t} \text{ exists} \right\}
$$

$$
Ax = \lim_{t \to 0} \frac{T(t)x - x}{t}, \forall x \in D(A).
$$

Recall that for all $t \geq 0$, $T(t)$ is a bounded linear operator in $X$ and $A$ is a closed operator. Details for all this may be found in [7], [4].

Next, we introduce the following operator acting on $X$ and depending on the parameters $\lambda \in \mathbb{C}$ and $t \geq 0$:

$$
B_\lambda(t)x = \int_0^t e^{\lambda(t-s)}T(s)x ds, x \in X.
$$

It is well known that $B_\lambda(t)$ is a bounded linear operator on $X$ [7, 4] and we have:

$$
(e^\lambda - T(t))^n x = (\lambda - A)^n B^n_\lambda(t)x, \forall x \in X, n \in \mathbb{N}
$$

$$
(e^\lambda - T(t))^n x = B^n_\lambda(t)(\lambda - A)^n x, \forall x \in D(A^n), n \in \mathbb{N};
$$

$$
R^\infty(e^\lambda - T(t)) \subseteq R^\infty(\lambda - A);
$$

$$
N((\lambda - A)^n) \subseteq N(e^\lambda - T(t))^n.
$$

2. Spectral inclusions and stability for Saphar spectrum

Lemma 2.1 [8] Let $\{T(t)\}_{t \geq 0}$ a $C_0$-semigroup on $X$ with infinitesimal generator $A$. For $\lambda \in \mathbb{C}$ and $t \geq 0$, let $F_\lambda(t)x = \int_0^t e^{-\lambda s}B_\lambda(s)x ds$, then:
1. There exist a \( M \geq 1 \) and \( \omega > Re(\lambda) \) such that
\[
F_\lambda(t) \leq \frac{M}{(\omega - Re(\lambda))^2} e^{(\omega - Re(\lambda))t}.
\]

2. \( \forall x \in X, \ F_\lambda(t)x \in D(A) \) and \( (\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI \) with
\[
G_\lambda(t) = e^{-\lambda t} I.
\]

3. The operators \( F_\lambda(t), \ G_\lambda(t) \) and \( B_\lambda(t) \) are pairwise commute and for all \( x \in D(A) \):
\[
\begin{align*}
(\lambda - A)F_\lambda(t)x &= F_\lambda(t)(\lambda - A)x \\
(\lambda - A)G_\lambda(t)x &= G_\lambda(t)(\lambda - A)x \\
(\lambda - A)B_\lambda(t)x &= B_\lambda(t)(\lambda - A)x
\end{align*}
\]

**Lemma 2.2** Let \( \{T(t)\}_{t \geq 0} \) a \( C_0 \)-semigroup on \( X \) with infinitesimal generator \( A \). Then for all \( t > 0 \) we have:

\[
e^{\lambda t} - T(t) \text{ has a generalized inverse} \implies \lambda - A \text{ has a generalized inverse.}
\]

**Proof.** Suppose that \( e^{\lambda t} - T(t) \) has a generalized inverse then there exists a \( S \in \mathcal{B}(X) \) such that:
\[
(e^{\lambda t} - T(t))S(e^{\lambda t} - T(t)) = e^{\lambda t} - T(t)
\]
According to Lemma 2.1, we have \( (\lambda - A)F_\lambda(t) + G_\lambda(t)B_\lambda(t) = tI \). Then:
\[
\begin{align*}
t(\lambda - A) &= (\lambda - A)F_\lambda(t)(\lambda - A) + B_\lambda(t)G_\lambda(t)(\lambda - A) \\
&= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)B_\lambda(t)G_\lambda(t) \\
&= (\lambda - A)F_\lambda(t)(\lambda - A) + (e^{\lambda t} - T(t))G_\lambda(t) \\
&= (\lambda - A)F_\lambda(t)(\lambda - A) + (e^{\lambda t} - T(t))S(e^{\lambda t} - T(t))G_\lambda(t) \\
&= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)B_\lambda(t)S(\lambda - A)B_\lambda(t)G_\lambda(t) \\
&= (\lambda - A)F_\lambda(t)(\lambda - A) + (\lambda - A)B_\lambda(t)SB_\lambda(t)G_\lambda(t)(\lambda - A) \\
&= (\lambda - A)[F_\lambda(t) + B_\lambda(t)SB_\lambda(t)G_\lambda(t)](\lambda - A).
\]

Hence \( \lambda - A \) has a generalized inverse. \( \blacksquare \)

**Theorem 2.1** Let \( \{T(t)\}_{t \geq 0} \) be a \( C_0 \)-semigroup on \( X \) with infinitesimal generator \( A \). Then for all \( t > 0 \):
\[
e^{\lambda\sigma_{\text{Sap}}(A)} \subseteq \sigma_{\text{Sap}}(T(t)), \quad e^{\lambda\sigma^c_{\text{Sap}}(A)} \subseteq \sigma^c_{\text{Sap}}(T(t)).
\]
Proof. Assume that $e^{\lambda t} - T(t)$ is a Saphar operator, then $e^{\lambda t} - T(t)$ has a generalized inverse and $N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))$.

By Lemma 2.2, $\lambda - A$ has a generalized inverse, and we have:

$$N(\lambda - A) \subseteq N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))^\infty \subseteq R(\lambda - A)^\infty.$$ 

Therefore, $\lambda - A$ is a Saphar operator.

Let $M$ a finite dimensional subspace of $X$. We have

$$N(\lambda - A) \subseteq N(e^{\lambda t} - T(t)) \subseteq R(e^{\lambda t} - T(t))^\infty + M \subseteq R(\lambda - A)^\infty + M.$$ 

Hence $e^{\lambda t} - T(t)$ is an essentially Saphar operator implying that $\lambda - A$.

Recall that a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ on $X$ with infinitesimal generator $A$. The following assertions are equivalent:

1. $\sigma(A) \cap i\mathbb{R} = \emptyset$,
2. $\sigma_K(A) \cap i\mathbb{R} = \emptyset$,
3. $\sigma_{Sap}(A) \cap i\mathbb{R} = \emptyset$.

If the infinitesimal generator $A$ verified one of this properties, $\{T(t)\}_{t \geq 0}$ is strongly stable.

Proof. Since $\sigma_K(A) \subset \sigma_{Sap}(A) \subset \sigma(A)$, the result is immediately comes from [2, Corollary 2.1].

Recall that a $C_0$-semigroup $\{T(t)\}_{t \geq 0}$ is said uniformly exponentially stable if there exists $\epsilon > 0$ such that

$$\lim_{t \to \infty} e^{\epsilon t}||T(t)|| = 0.$$ 

Proposition 2.2 Let $\{T(t)\}_{t \geq 0}$ a $C_0$-semigroup on $X$ with infinitesimal generator $A$, $\Gamma$ is the unit circle in $\mathbb{C}$. The following assertions are equivalent:

1. $\{T(t)\}_{t \geq 0}$ is uniformly exponentially stable,
2. there exists $t_0 > 0$ such that $\sigma_K(T(t_0)) \cap \Gamma = \emptyset$,
3. there exists $t_0 > 0$ such that $\sigma_{Sap}(T(t_0)) \cap \Gamma = \emptyset$. 

Proof. Since $\sigma_K(T(t_0)) \subset \sigma_{Sap}(T(t_0)) \subset \sigma(T(t_0))$, the result is immediately comes from [2, Corollary 2.1].
Proof. 1) \(\iff\) 2): see [2, Corollary 2.2].

1) \(\implies\) 3) is obvious, see [4].

3) \(\implies\) 1) Suppose that there exists \(t_0 > 0\) such that \(\sigma_{Sap}(T(t_0)) \cap \Gamma = \emptyset\). Since \(\sigma_K(T(t_0)) \subseteq \sigma_{Sap}(T(t_0))\), hence \(\sigma_K(T(t_0)) \cap \Gamma = \emptyset\). By [2, Corollary 2.2] \(\{T(t)\}_{t \geq 0}\) is uniformly exponentially stable.

\[\]

3. Spectral inclusion for quasi-Fredholm spectrum

Recall from [5] some definitions:

**Definition 3.1** Let \(T\) a closed linear operator on \(X\) and let
\[
\Delta(T) = \{n \in \mathbb{N}, \forall m \geq n, R(T^n) \cap N(T) = R(T^m) \cap N(T)\}
\]
The degree of stable iteration \(\text{dis}(T)\) of \(T\) is defined as \(\text{dis}(T) = \inf \Delta(T)\), where \(\text{dis}(T) = \infty\) if \(\Delta(T) = \emptyset\).

**Definition 3.2** Let \(T\) a closed linear operator on \(X\). \(T\) is called a quasi-Fredholm operator of degree \(d\) if there exists an integer \(d \in \mathbb{N}\) such that:

1. \(\text{dis}(T) = d\);
2. \(R(T^n)\) is closed in \(X\) for all \(n \geq d\);
3. \(R(T) + N(T^n)\) is closed in \(X\) for all \(n \geq d\).

The quasi-Fredholm spectrum is defined by:
\[
\sigma_{qF}(T) = \{\lambda \in \mathbb{C} : T - \lambda I\text{ is not a quasi-Fredholm}\}.
\]

**Proposition 3.3** Let \(\{T(t)\}_{t \geq 0}\) a \(C_0\)-semigroup on \(X\) with infinitesimal generator \(A\). Then:
\[
\text{dis}(\lambda - A) \leq \text{dis}(e^{\lambda t} - T(t)).
\]

**Proof.** If \(\text{dis}(e^{\lambda t} - T(t)) = +\infty\) obvious.

If \(\text{dis}(e^{\lambda t} - T(t)) = d \in \mathbb{N}^*\) then for all \(n \geq d\), we have:
\[
R((e^{\lambda t} - T(t))^n) \cap N((e^{\lambda t} - T(t))) = R((e^{\lambda t} - T(t))^d) \cap N(e^{\lambda t} - T(t)).
\]
Then, for all \(n \geq d\), we have:
\[
R((\lambda - A)^n) \cap N(\lambda - A) = R((\lambda - A)^d) \cap N(\lambda - A).
\]

Indeed, let \(y \in R((\lambda - A)^d) \cap N(\lambda - A)\). Then there exists \(x \in X\) such that \(y = (\lambda - A)^d x\) and, by Lemma 2.1, \((\lambda - A)^d F_d(t) + B_d^+(t) G_d(t) = I\) implies that \((\lambda - A)^d x = (\lambda - A)^(2d) F_d(t)x + (e^{\lambda t} - T(t))^d G_d(t)x\), since \(y \in N(\lambda - A)\),
Let there therefore, according to Lemma 2.1 there exists 

\[ (\lambda - A)^n B^0(\lambda)z = (\lambda - A)^n B^0(\lambda)z \in R((\lambda - A)^n) \cap N(\lambda - A) \]

then, \( dis(\lambda - A) \leq d \).

If \( d = 0 \), for all \( n \geq d \), we have:

\[ R((\lambda - A)^n) \cap N((\lambda - A)^n) = N(\lambda - A) \]

Then, \( \forall n \in \mathbb{N} \),

\[ N((\lambda - A)^n) \subset R((\lambda - A)^n) \]

Then

\[ N(\lambda - A) \subset N((\lambda - A)^n) \subset R((\lambda - A)^n) \subset R((\lambda - A)^n) \]

Hence

\[ N(\lambda - A) \cap R((\lambda - A)^n) = N(\lambda - A) \cap R((\lambda - A)^0) \]

Therefore, \( dis(\lambda - A) = 0 \).

\[ \text{Proposition 3.4} \]

Let \( \{ T(t) \}_{t \geq 0} \) a \( C_0 \)-semigroup on \( X \) with infinitesimal generator \( A \). If \( R((\lambda - A)^n) \) is closed for all \( n \geq d \), then \( R((\lambda - A)^n) \) is closed for all \( n \geq d \).

\[ \text{Proof.} \] Let \( y_n = (\lambda - A)^n x_n \to y \), as \( n \to \infty \), we show that \( y \in R((\lambda - A)^n) \). According to Lemma 2.1 there exists \( F_d(t) \) and \( G_d(t) \) to bounded linear operators such that

\[ (1) \quad (\lambda - A)^d F_d(t) + B^d(\lambda) G_d(t) = I. \]

\[ B^0(\lambda) y_n = B^0(\lambda)(\lambda - A)^n x_n = (\lambda - A)^n x_n \in R((\lambda - A)^n). \]

Since \( B^0(\lambda) y_n \to B^0(\lambda) y \), then there exists \( z \in X \) such that

\[ B^0(\lambda) y = (\lambda - A)^n z. \]

By (1):

\[ (\lambda - A)^n x_n = (\lambda - A)^2 F_n(t) x_n + (\lambda - A)^n G_n(t) x_n \to \infty. \]

We have:

\[ y = (\lambda - A)^n F_n(t) y + (\lambda - A)^n G_n(t) z \]

\[ = (\lambda - A)^n [ F_n(t) y + B^0(\lambda) z] \in R((\lambda - A)^n). \]

\[ \text{Proposition 3.5} \]

Let \( \{ T(t) \}_{t \geq 0} \) a \( C_0 \)-semigroup on \( X \) with infinitesimal generator \( A \) and \( d \in \mathbb{N} \). If \( R((\lambda - A)^n) \) is closed in \( X \), then \( R(\lambda - A) + N((\lambda - A)^d) \) is closed.
Proof. Suppose that $R(e^{\lambda t} - T(t)) + N((e^{\lambda t} - T(t))^d)$ is closed in $X$.

Let $y_n = (\lambda - A)x_n + z_n \to y$, as $n \to \infty$, $z_n \in N((\lambda - A)^d)$.

Then

$$B^d(t)y_n = B^d_\lambda(t)(\lambda - A)x_n + B^d_\lambda(t)z_n \in R(e^{\lambda t} - T(t)) + N((e^{\lambda t} - T(t))^d),$$

then

$$B^d_\lambda(t)y \in R(e^{\lambda t} - T(t)) + N((e^{\lambda t} - T(t))^d)$$

and

$$B^d_\lambda(t)y = ((e^{\lambda t} - T(t))x + z \text{ with } z \in N((e^{\lambda t} - T(t)^d))$$

Then

$$y = (\lambda - A)^{2d}F_d(t)y + B^d_\lambda(t)G_d(t)y$$

$$= (\lambda - A)^{2d}F_d(t)y + G_d(t)[B^d_\lambda(t)(e^{\lambda t} - T(t))x + B^d_\lambda(t)z]\]$$

$$= (\lambda - A)[(\lambda - A)^{2d-1}F_d(t)y + G_d(t)B^{d-1}_\lambda(t)]\]

$$+ G_d(t)B^d_\lambda(t)z \in R(\lambda - A) + N((\lambda - A)^d).$$

Corollary 3.1 Let $\{T(t)\}_{t \geq 0}$ a $C_0$-semigroup on $X$ with infinitesimal generator $A$. If $R(e^{\lambda t} - T(t))$ is closed in $X$, then $R(\lambda - A)$ is closed.

Theorem 3.2 Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on $X$ with infinitesimal generator $A$. Then, for all $t > 0$,

$$e^{t\sigma_F(A)} \subseteq \sigma_F(T(t)).$$

Proof. Direct consequence of the three last proposition.

If $\text{dis}(T) = 0$, $T$ to be a Kato operator and by using the Corollary 3.1 and Proposition 3.3 we have the result of A. Elkoutri and M.A. Taoudi [2]:

Corollary 3.2 Let $\{T(t)\}_{t \geq 0}$ be a $C_0$-semigroup on $X$ with infinitesimal generator $A$. Then for all $t > 0$:

$$e^{t\sigma_K(A)} \subseteq \sigma_K(T(t)).$$

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References


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