SIMPLE ITERATIVE TECHNIQUE FOR SOLVING SOME MODELS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS USING HAAR WAVELET

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Abstract. We present here, simple iterative technique for solving some models of nonlinear partial differential equations using Haar wavelet. Numerical examples are given to establish the efficiency and accuracy of the present method.

Keywords: nonlinear partial differential equation; Haar wavelet method; function approximation.
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1. Introduction

Nonlinear partial differential equations has a wide variety in scientific and engineering applications. Mostly, mathematical models can be expressed in terms of nonlinear partial differential equations. For solving such mathematical models different approaches and techniques were established. Wavelet technique is one of the easiest and accurate technique for solving a variety of such models. Wavelet, being a powerful mathematical tool, has been widely used in image digital processing, quantum field theory and numerical analysis. In this paper, we study the physical behaviour of numerical solution of nonlinear partial differential equations. Consider the most general nonlinear partial differential equation of the form:

$$f\left(t, x, u, \frac{\partial u}{\partial x}, \frac{\partial u}{\partial t}\right) = 0, \tag{1}$$

subject to the initial and boundary conditions:

$$u(x,0) = f(x), \quad 0 \le x \le L, \tag{2}$$

and

$$u(0,t) = g_1(t), \quad u(L,t) = g_2(t), \quad t \ge 0. \tag{3}$$

In recent years, being a powerful mathematical tool, Haar wavelets have been used extensively in signal processing and in research of different branches of science. The main disadvantage of Haar wavelets is their discontinuity and at the points of discontinuity, derivatives do not exist. Due to this shortcoming, Haar wavelets are not applied directly for solving differential equations. There are two possibilities for overcoming these shortcomings. First, to regularize the piecewise constant Haar functions with interpolation splines: this technique has been applied by Cattani in [1], [2]. But, by this technique, it is difficult to find the solution easily and simplicity of Haar wavelets gets lost. Another possibility, which is proposed by Chen and Hsiao in [3], [4] is that they recommended to expand the highest order derivative appearing in the differential equation into the Haar series, instead of the function itself. The other derivatives (and the function) are obtained through integrations. Chen and Hsiao established the possibilities of their method by solving linear systems of ordinary and partial differential equations. Using the application of Haar analysis into the dynamical systems, Chen and Hsiao [3], had first derived an operational matrix of integration based on Haar wavelets. Numerical solutions of Telegraph equation, Fisher's equation and generalized Burger-Huxley equation using Haar wavelets have been presented in [5], [6] and [7] respectively. A survey of numerical solutions of differential equation is presented in [8]. Lepik [9], [10], [11], [12] had solved differential and integral equations using Haar wavelet. The numerical solutions of wave equation and convection-diffusion equation using Haar wavelets have been presented in [13], [14] respectively.

In Section 2, we briefly describe Haar wavelet. In Section 3, we have described Haar wavelet method for solving nonlinear partial differential equations. Function approximation has been presented in Section 4. In Section 5, numerical examples have been solved using the present method to illustrate the efficiency and accuracy of the method.

2. Haar wavelet

The Haar functions are an orthogonal family of switched rectangular waveforms where amplitudes can differ from one function to another. They are defined in the interval $[0,1]$ in general. Haar wavelet is a sequence of rescaled square shaped functions which together forms a wavelet family or basis. The Haar wavelet function $h_i(x)$ is defined in the interval $[\alpha,\gamma]$ as:

$$h_i(x) = \begin{cases} 1, & \alpha \le x < \beta, \\ -1, & \beta \le x < \gamma, \\ 0, & \text{elsewhere}, \end{cases} \tag{4}$$

where

$$\alpha = \frac{k}{m}, \ \beta = \frac{k+0.5}{m}, \ \gamma = \frac{k+1}{m}, \ \ m = 2^j \text{ and } j = 0, 1, 2, ..., J.$$

J denotes the level of resolution. The integer $k = 0, 1, 2, ..., m-1$ is the translation parameter. The index i is calculated as: $i = m + k + 1$. The minimal value of $i = 2$. The maximal value of i is 2^{j+1}.

The collocation points are calculated as:

$$x_l = \frac{l - 0.5}{2M}, \qquad l = 1, 2, 3, ..., 2M. \tag{5}$$

The Haar coefficient matrix is defined as:

$$H(i, l) = (h_i(x_l)). \tag{6}$$

This is a $2M \times 2M$ square matrix. The Haar coefficient matrix of order 8, is:

$$H_8(x) = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & -1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & -1 & -1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & -1 \end{pmatrix}. \tag{7}$$

Integrating (4), we obtain:

$$q_i(x) = \int_0^x h_i(t)dt, \tag{8}$$

using collocation points, it is defined as:

$$Q(i, l) = (q_i(x_l)), \tag{9}$$

which is again $2M \times 2M$ square matrix. Chen and Hsiao [3], presented this matrix in the form $Q_n = P_n H_n$, where $P_n H_n$ is interpreted as the product of the matrices, P_n and H_n, called Haar integration and Haar coefficient matrix, respectively. There are two possibilities for calculating the integrals of the Haar wavelets. In the first case, Chen and Hsiao [3], introduced the operational matrices of integration. But, in the second case, these integrals are calculated directly as discussed by Lepik in [12]. We follow the work as discussed in [12] for calculating integrals. The operational matrix P, which is $2M \times 2M$, is calculated as below:

$$P_{1,i}(x) = \int_0^x h_i(x)dx, \tag{10}$$

and

$$P_{n+1,i}(x) = \int_0^x P_{n,i}(x)dx, \quad n = 1, 2, 3, ... \tag{11}$$

The integrals of Haar wavelets considered here are:

$$P_{1,i}(x) = \begin{cases} x - \alpha, & \alpha \le x < \beta, \\ \gamma - x, & \beta \le x < \gamma, \\ 0, & \text{elsewhere}, \end{cases} \tag{12}$$

$$P_{2,i}(x) = \begin{cases} \frac{1}{2}(x-\alpha)^2, & \alpha \le x < \beta, \\ \frac{1}{2}[(x-\alpha)^2 - 2(x-\beta)^2], & \beta \le x < \gamma, \\ \frac{1}{2}[(x-\alpha)^2 - 2(x-\beta)^2 + (x-\gamma)^2], & \gamma \le x < 1, \\ 0, & \text{elsewhere}, \end{cases} \tag{13}$$

In general,

$$P_{n,i}(x) = \begin{cases} \frac{1}{n!}(x-\alpha)^n, & \alpha \le x < \beta, \\ \frac{1}{n!}[(x-\alpha)^n - 2(x-\beta)^n], & \beta \le x < \gamma, \\ \frac{1}{n!}[(x-\alpha)^n - 2(x-\beta)^n + (x-\gamma)^n], & \gamma \le x < 1, \\ 0, & \text{elsewhere}, \end{cases} \tag{14}$$

Further, Haar wavelets are orthogonal as shown below:

$$\int_0^1 h_m(x)h_n(x)dx = \begin{cases} 1, & m = n, \\ 0, & m \ne n. \end{cases} \tag{15}$$

3. Haar wavelet method [12]

Consider the nonlinear partial differential equation (1) with initial and boundary conditions. Assume that $\dot{u}'(x,t)$ can be expanded in terms of Haar wavelets as below:

$$\dot{u}'(x,t) = \sum_{i=1}^{2M} a_i h_i(x), \qquad t\epsilon(t_s, t_{s+1}]. \tag{16}$$

Here ($^{\bullet}$) means differentiation with respect to t and ($'$) means differentiation with respect to x. Integrating the above equation with respect to t from t_s to t and then, with respect to x; from 0 to x, we obtain successively:

$$u'(x,t) = u'(x,t_s) + (t-t_s)\sum_{i=1}^{2M} a_i h_i(x), \tag{17}$$

$$u(x,t) = u(0,t) + u(x,t_s) - u(0,t_s) + (t-t_s)\sum_{i=1}^{2M} a_i P_{1,i}(x). \tag{18}$$

Differentiating above equation with respect to t, we obtain:

$$\dot{u}(x,t) = \dot{u}(0,t) + \sum_{i=1}^{2M} a_i P_{1,i}(x). \tag{19}$$

Substituting $x \to x_l$ and $t \to t_{s+1}$ from Equation (16) to Equation (19), we successively obtain:

$$\dot{u}'(x_l,t_{s+1}) = \sum_{i=1}^{2M} a_i h_i(x_l), \tag{20}$$

$$u'(x_l,t_{s+1}) = u'(x_l,t_s) + (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i h_i(x_l), \tag{21}$$

$$u(x_l,t_{s+1}) = u(0,t_{s+1}) + u(x_l,t_s) - u(0,t_s) + (t_{s+1}-t_s)\sum_{i=1}^{2M} a_i P_{1,i}(x_l), \tag{22}$$

$$\dot{u}(x_l,t_{s+1}) = \dot{u}(0,t_{s+1}) + \sum_{i=1}^{2M} a_i P_{1,i}(x_l). \tag{23}$$

For iteration, substitute $\dot{u}(x,t) = \dot{u}(x,t_{s+1})$, $u(x,t) = u(x,t_s)$, $u'(x,t) = u'(x,t_s)$ and $u''(x,t) = u''(x,t_s)$ in Equation (1). After substitution the Equation (1) takes the form:

$$f(t,x,u(x,t_s),\frac{\partial u}{\partial x}(x,t_s),\frac{\partial u}{\partial t}(x,t_{s+1})) = 0. \tag{24}$$

Wavelet coefficients are obtained by substituting the values from Equation (20) to Equation (23) in (24). The numerical solution is obtained from (22) by substituting the wavelet coefficients.

4. Function approximation

The function $u(x,t) \in L^2(0,1)$ can be approximate as:

$$u(x,t) = \sum_{i=0}^{\infty} a_i h_i(x), \tag{25}$$

where the coefficient a_i are determined as:

$$a_i = 2^j \int_0^1 u(x,t) h_i(x) dx, \tag{26}$$

where $i = 2^j + k$, $j \geq 0$ and $0 \leq k < 2^j$. The series expansion of $u(x,t)$ contains infinite terms. If $u(x,t)$ is piecewise constant by itself, or may be approximated as piecewise constant during each subinterval, then $u(x,t)$ will be terminated at finite number of terms, that is:

$$u(x,t) = \sum_{i=0}^{m-1} a_i h_i(x) = a_m^T h_m(x), \tag{27}$$

where $a_m^T = [a_0, a_1, ..., a_{m-1}]$, $h_m = [h_0, h_1, ..., h_{m-1}]^T$ and T is transpose.

5. Numerical examples

Example 1: Consider the hyperbolic nonlinear partial differential equation of the form:

$$\frac{\partial u}{\partial t} = u\left(\frac{\partial u}{\partial x}\right), \qquad 0 < x < 1, \quad t \geq 0, \tag{28}$$

subject to the initial conditions $u(x,0) = \dfrac{x}{10}$. The exact solution of the problem is given by:

$$u(x,t) = -\frac{x}{t-10}. \tag{29}$$

The process is started with $u(x_l, 0) = \dfrac{x_l}{10}$. Absolute errors so obtained are presented in Table 1, for $t = 0.01$, $t = 0.1$, $t = 0.2$, $t = 0.3$ and $J = 4$. The value of $\Delta t = 0.001$.

Table 1: Absolute errors of Example 1 for different values of t.

xL/64	$t = 0.01$	$t = 0.1$	$t = 0.2$	$t = 0.3$
1	1.5662E-010	1.6020E-009	3.2863E-009	5.0574E-009
3	4.6985E-010	4.8060E-009	9.8590E-009	1.5172E-008
5	7.8309E-010	8.0100E-009	1.6432E-008	2.5287E-008
7	1.0963E-009	1.1214E-008	2.3004E-008	3.5402E-008
57	8.9272E-009	9.1314E-008	1.8732E-007	2.8827E-007
59	9.2404E-009	9.4518E-008	1.9389E-007	2.9839E-007
61	9.5537E-009	9.7722E-008	2.0047E-007	3.0850E-007
63	9.8669E-009	1.0093E-007	2.0704E-007	3.1862E-007

Example 2: Consider the hyperbolic nonlinear partial differential equation of the form:

$$\frac{\partial u}{\partial t} = x^2 - \frac{1}{4}\left(\frac{\partial u}{\partial x}\right)^2, \qquad 0 < x < 1, \qquad t \geq 0, \tag{30}$$

subject to the initial condition $u(x, 0) = 0$. The exact solution of the problem is given below:

$$u(x, t) = x^2 \tanh(t). \tag{31}$$

The process is started with $u(x_l, 0) = 0$. Absolute errors so obtained are presented in Table 2, for $t = 0.01$, $t = 0.1$, $t = 0.2$, $t = 0.3$ and $J = 4$. The value of $\Delta t = 0.001$.

Table 2: Absolute errors of Example 2 for different values of t.

xL/64	$t = 0.01$	$t = 0.1$	$t = 0.2$	$t = 0.3$
1	6.3982E-011	6.1035E-008	4.7988E-007	1.5792E-006
3	1.1992E-010	2.4614E-007	1.9445E-006	6.1841E-006
5	6.2541E-010	4.0055E-007	2.8610E-006	8.2946E-006
7	7.3893E-011	4.9182E-007	3.5299E-006	9.0496E-006
57	4.2191E-008	3.4936E-006	1.4759E-005	3.2748E-005
59	3.7054E-008	4.6152E-006	1.6860E-005	3.4737E-005
61	4.8018E-008	4.0442E-006	1.6479E-005	3.5813E-005
63	4.2555E-008	5.1879E-006	1.9363E-005	4.0680E-005

Conclusion

From above discussions, it is concluded here that the Haar wavelet method is more accurate, simple, fast and computationally efficient for solving nonlinear partial differential equations. The above examples demonstrates the simplicity of the Haar wavelet method. For getting the necessary accuracy the number of calculation points may be increased.

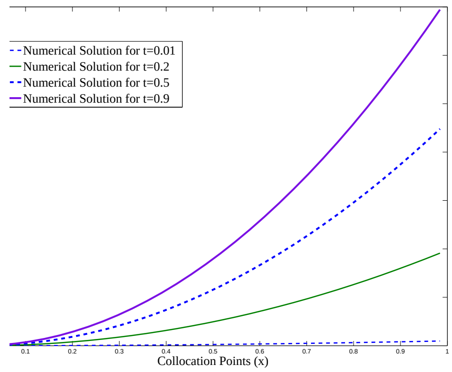

Figure 1: Numerical solutions of Example 2 at different t with $J = 4$ and $\Delta t = 0.001$.

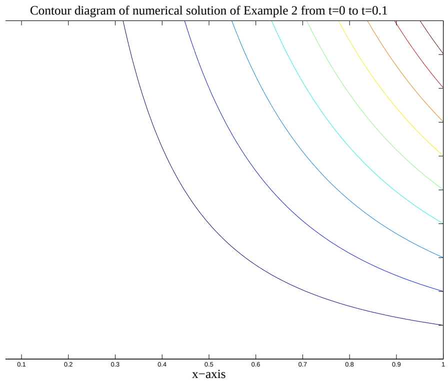

Figure 2: Contour diagram of numerical solutions of Example 2 from $t = 0 - 0.1$.

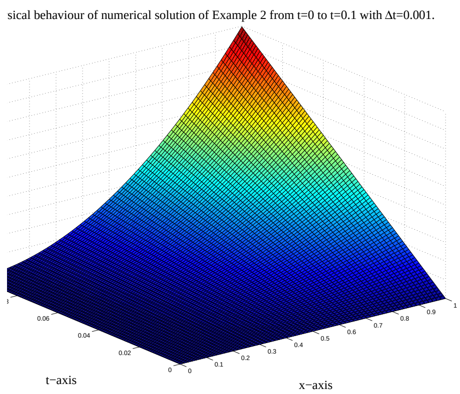

Figure 3: Physical behavior of numerical solutions of Example 2 from $t = 0 - 0.1$ with $\Delta t = 0.001$ and $J = 4$.

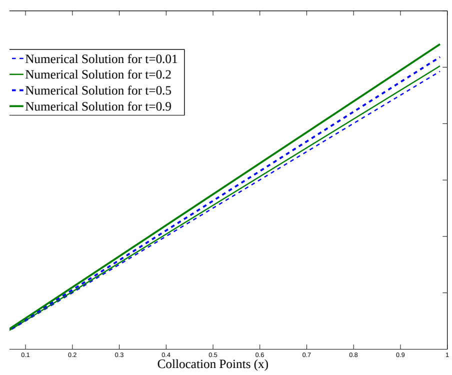

Figure 4: Numerical solutions of Example 1 at different t with $J = 4$ and $\Delta t = 0.001$.

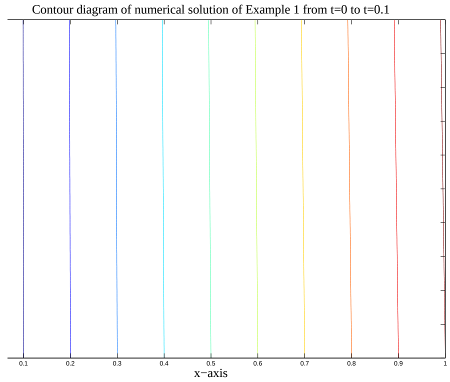

Figure 5: Contour diagram of numerical solutions of Example 1 from $t = 0 - 0.1$.

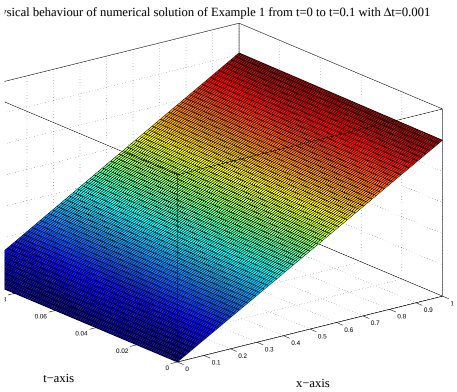

Figure 6: Physical behavior of numerical solutions of Example 1 from $t = 0 - 0.1$ with $\Delta t = 0.001$ and $J = 4$.

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