PASSAGE OF PROPERTY \((gw)\) FROM TWO OPERATORS TO THEIR TENSOR PRODUCT

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Abstract. A Banach space operator satisfies property \((gw)\) if the complement of its B-Weyl essential approximate point spectrum in its approximate point spectrum is the set of isolated eigenvalues of the operator. We give necessary and/or sufficient conditions ensuring the passage of property \((gw)\) from two Banach space operators \(A\) and \(B\) to their tensor product. In particular, we present a revised version of Theorem 2.3 in [20].

Keywords: property \((gw)\), generalized \(a\)-Weyl’s theorem, tensor product.

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1. Introduction and preliminary

Let \(T \in \mathcal{L}(X)\) be a bounded linear operator acting on a Banach space \(X\). We denote by \(T^*, N(T), R(T), \sigma(T), \sigma_p(T)\) and \(\sigma_{ap}(T)\) denote the adjoint, the null space, the range, the spectrum, the point spectrum and the approximate point spectrum of \(T\) respectively. If the range \(R(T)\) of \(T\) is closed and \(\alpha(T) := \dim N(T) < \infty\) (resp. \(\beta(T) := \text{codim} R(T) < \infty\)) then \(T\) is called an upper (resp. a lower) semi-Fredholm operator. If \(T\) is either an upper or a lower semi-Fredholm operator, then \(T\) is called a semi-Fredholm operator, and the index of \(T\) is defined by

\[
\text{ind}(T) = \alpha(T) - \beta(T).
\]
If both $\alpha(T)$ and $\beta(T)$ are finite, then $T$ is called a Fredholm operator. If $T$ is Fredholm of index zero, then $T$ is said to be a Weyl operator. The Weyl spectrum of $T$ is defined by

$$\sigma_w(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not Weyl} \}$$

and the Weyl essential approximate point spectrum is defined by

$$\sigma_{SF+}^w(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not upper semi-Fredholm with } \text{ind}(T - \lambda) \leq 0 \}.$$ 

For each nonnegative integer $n$ let $T_{[n]}$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$ (in particular $T_{[0]} = T$). If for some $n$, $R(T^n)$ is closed and $T_{[n]}$ is an upper (resp. lower) semi-Fredholm operator then $T$ is called an upper (resp. lower) semi-B-Fredholm operator. A semi-B-Fredholm operator is an upper or lower semi-Fredholm operator. If, moreover, $T_{[n]}$ is a Fredholm operator then $T$ is called a B-Fredholm operator. From [13, Proposition 2.1] if $T_{[n]}$ is a semi-Fredholm operator then $T_{[m]}$ is also a semi-Fredholm operator for each $m \geq n$, and $\text{ind}(T_{[m]}) = \text{ind}(T_{[n]})$. Then the index of a semi-B-Fredholm operator is defined as the index of the semi-Fredholm operator $T_{[n]}$ (see [12, 13]). $T \in \mathcal{L}(X)$ is said to be a B-Weyl operator if it is a B-Fredholm operator of index zero. The B-Weyl spectrum $\sigma_{BW}(T)$ of $T$ is defined by

$$\sigma_{BW}(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not a B-Weyl operator} \}.$$ 

We shall denote by $SBF{^-}_+(X)$ (or $SBF{^+}_-(X)$) the class of all $T$ upper semi-B-Fredholm operators such that $\text{ind}(T) \leq 0$ (respectively, $T$ lower semi-B-Fredholm operators such that $\text{ind}(T) \geq 0$). The spectrum associated with $SBF{^-}_+(X)$ is called the B-Weyl essential approximate point spectrum and is denoted by

$$\sigma_{SBF^+}^w(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin SBF^+_- (X) \},$$

while the spectrum associated with $SBF^+_-(X)$ is denoted by

$$\sigma_{SBF^-}^w(T) = \{ \lambda \in \mathbb{C} : T - \lambda I \notin SBF^-_+ (X) \}.$$ 

The ascent $a(T)$ and the descent $d(T)$ of $T$ are given by

$$a(T) = \inf \{ n : N(T^n) = N(T^{n+1}) \} \text{ and } d(T) = \inf \{ n : R(T^n) = R(T^{n+1}) \},$$

with $\inf \emptyset = \infty$. It is well-known that if $a(T)$ and $d(T)$ are both finite then they are equal, see [17, Proposition 38.3]. An operator $T \in \mathcal{L}(X)$ is said to be Browder if it is Weyl of finite ascent and descent. Let $\sigma_b(T)$ be the Browder spectrum defined by

$$\sigma_b(T) = \{ \lambda \in \mathbb{C} : T - \lambda \text{ is not a Browder operator} \}.$$ 

Following Coburn [15], Weyl’s theorem holds for $T$ if $\sigma(T) \setminus \sigma_w(T) = E^0(T)$ where $E^0(T) := \{ \lambda \in \text{iso}\sigma(T) : 0 < a(T - \lambda) < \infty \}$. Here and elsewhere in this paper, for $K \subset \mathbb{C}$, $\text{iso}K$ is the set of isolated points of $K$. 

We say that $a$-Weyl’s theorem holds for $T$ if $\sigma_{ap}(T) \setminus \sigma_{SF^{-}}(T) = E_a^0(T)$ where $E_a^0(T) := \{\lambda \in \text{iso}\sigma_{ap}(T) : 0 < \alpha(T - \lambda) < \infty\}$.

$T$ is said to have the property $(w)$ if $\sigma_a(T) \setminus \sigma_{SF^{-}}(T) = E_a^0(T)$. Originally introduced by Rakočević in [19], property $(w)$ has been intensively studied in the recent past, see [2], [3], [4], [5], [6]. Following Berkani and Koliha [10], [12], generalized Weyl’s theorem holds of $T$ if $\sigma(T) \setminus \sigma_{BW}(T) = E(T)$ where $E(T) := \{\lambda \in \text{iso}\sigma(T) : \alpha(T - \lambda) > 0\}$; and generalized $a$-Weyl’s theorem holds for $T$ if $\sigma_{ap}(T) \setminus \sigma_{SBF^{-}}(T) = E_a(T)$ where $E_a(T) := \{\lambda \in \text{iso}\sigma_{ap}(T) : \alpha(T - \lambda) > 0\}$.

Operators satisfying property $(gw)$, an extension of property $(w)$ have been introduced and studied in [9], [11], [7]. $T$ is said to have the property $(gw)$, if $\sigma_{ap}(T) \setminus \sigma_{SBF^{-}}(T) = E(T)$. Note that the property $(gw)$ implies generalized Weyl’s theorem but $(gw)$ is not intermediate between Weyl’s theorems and generalized $a$-Weyl’s theorem, see [9].

The problem of transferring Weyl’s theorem, a-Weyl’s theorem and Property $(w)$ from operators $A$ and $B$ to their tensor product $A \otimes B$ was considered in [16], [20], [18]. The main objective of this paper is to study the transfer of property $(gw)$ from a bounded linear operator $A$ acting on a Banach space $X$ and a bounded linear operator $B$ acting on a Banach space $Y$ to their tensor product $A \otimes B$.

In Section 2, after having recalled some preliminary definitions and facts, we give necessary and sufficient condition for transferring property $(gw)$ from isoloid operators $A$ and $B$ to their tensor product $A \otimes B$ in term of the $B$-Weyl essential approximate point spectrum equality. Section 3 is devoted to characterize the transference of generalized $a$-Weyl’s theorem from a-isloid operators $A$ and $B$ to their tensor product $A \otimes B$.

2. Property $(gw)$ and tensor product

Let $T \in \mathcal{L}(X)$, then $T$ is said to be isoloid, if $\text{iso}\sigma(T) = E(T)$. In [20, Theorem 2.3], it was stated that if $A$ and $B$ are isoloid operators that possess property $(gw)$, then $A \otimes B$ satisfies property $(gw)$ if and only if

$$\sigma_{SBF^{-}}(A \otimes B) = \sigma_{SBF^{-}}(A) \sigma_{ap}(B) \cup \sigma_{ap}(A) \sigma_{SBF^{-}}(B).$$

We point out that Theorem 2.3 in [20] is accurate as shown by the following example.

**Example 2.1** Let $A$ be a nonzero nilpotent operator and let $B$ be a quasi-nilpotent which is not nilpotent. Then it is easy to see that

$$\sigma_{ap}(A) = \{0\}, \sigma_{SBF^{-}}(A) = \emptyset \text{ and } \sigma_{ap}(B) = \sigma_{SBF^{-}}(B) = \{0\}.$$

Hence $A$ and $B$ satisfy property $(gw)$. Since $A \otimes B$ is nilpotent then $0$ is a pole and then $\sigma_{SBF^{-}}(A \otimes B) = \emptyset$. Hence $A \otimes B$ satisfies property $(gw)$.

$$\sigma_{SBF^{-}}(A) \sigma(B) \cup \sigma(A) \sigma_{SBF^{-}}(B) = \{0\} \neq \sigma_{SBF^{-}}(A \otimes B).$$

Here $0 \in \text{iso}\sigma(A \otimes B)$. 


However, under the assumption $0 \notin \text{iso}(A \otimes B)$, we have the following result.

**Theorem 2.2** Let $X$ and $Y$ two Banach spaces and consider $A \in \mathcal{L}(X)$ and $B \in \mathcal{L}(Y)$ such that $A$ and $B$ are isoloid and $0 \notin \text{iso}(A \otimes B)$. If property (gw) holds for $A$ and $B$, then the following statements are equivalent.

(i) $A \otimes B$ satisfies property (gw).

(ii) $\sigma_{SBF^+}(A \otimes B) = \sigma_{SBF^+}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF^+}(B)$.

**Proof.** (ii) $\Rightarrow$ (i): Since $A$ and $B$ obey property (gw), then

$$\sigma_{ap}(A) \backslash_{SBF^+} (A) = E(A) \text{ and } \sigma_{ap}(B) \backslash_{SBF^+} (B) = E(B).$$

Assume that

$$\sigma_{SBF^+}(A \otimes B) = \sigma_{SBF^+}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF^+}(B).$$

Let $\lambda \in E(A \otimes B)$. Then there exists $\mu \in \text{iso}(A)$ and $\nu \in \text{iso}(B)$ such that $\lambda = \mu \nu$. Since $A$ and $B$ are isoloid, then $\mu \in E(A)$ and $\nu \in E(B)$. Hence $\mu \notin \sigma_{SBF^+}(A)$ and $\nu \notin \sigma_{SBF^+}(B)$. Then $\lambda \notin \sigma_{SBF^+}(A \otimes B)$. Thus

$$E(A \otimes B) \subseteq \sigma_{ap}(A \otimes B) \backslash \sigma_{SBF^+}(A \otimes B).$$

Conversely, suppose that $\lambda \notin \sigma_{ap}(A \otimes B) \backslash \sigma_{SBF^+}(A \otimes B)$, then there exists $\mu \in \sigma_{ap}(A) \backslash \sigma_{SBF^+}(A)$ and $\nu \in \sigma_{ap}(B) \backslash \sigma_{SBF^+}(B)$ such that $\lambda = \mu \nu$. Since

$$A \otimes B = (A - \mu) \otimes B + \mu I \otimes (B - \nu),$$

then we can see that $\lambda \in E(A \otimes B)$. Therefore $A \otimes B$ satisfies property (gw).

(i) $\Rightarrow$ ii): Assume that $A \otimes B$ satisfies property (gw). Let

$$\lambda \in E(A \otimes B) = \sigma_{ap}(A \otimes B) \backslash_{SBF^+} (A \otimes B).$$

Since $0 \notin \text{iso}(A \otimes B)$, then $\lambda \notin 0$. Hence $\lambda \in \text{iso}(A \otimes B) = \text{iso}(A) \text{iso}(B)$. That is $\lambda = \mu \nu$ with $\mu \in \text{iso}(A)$ and $\nu \in \text{iso}(B)$. Since $A$ and $B$ are isoloid, then $\mu \in E(A) = \sigma_{ap}(A) \backslash_{SBF^+} (A)$ and $\nu \in E(B) = \sigma_{ap}(B) \backslash_{SBF^+} (B)$, and hence $\lambda = \mu \nu \notin \sigma_{SBF^+}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF^+}(B)$. Thus

$$\sigma_{SBF^+}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF^+}(B) \subseteq \sigma_{SBF^+}(A \otimes B).$$

Conversely, let $\lambda \in \sigma_{ap}(A \otimes B) \backslash (\sigma_{SBF^+}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF^+}(B))$, then for $\lambda = \mu \nu$ we have that $\mu \in \sigma_{ap}(A)$ and $\nu \in \sigma_{ap}(B)$, hence $\mu \in E(A)$ and $\nu \in E(B)$. Thus $\lambda = \mu \nu \in E(A \otimes B) = \sigma_{ap}(A \otimes B) \backslash_{SBF^+} (A \otimes B)$. Finally,

$$\sigma_{SBF^+}(A \otimes B) = \sigma_{SBF^+}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF^+}(B).$$
Remark 2.3 Let $T \in \mathcal{L}(X)$ without eigenvalue, then $\sigma_{SBF^-}(T) = \sigma_{ap}(T)$. Indeed, let $\lambda \in \sigma_{ap}(T) \setminus \sigma_{SBF^-}(T)$. Without loss of generality we may assume that $\lambda = 0$. Then there exists some positive integer $n$ such that $T_{[n]}$ is upper semi-Fredhom. Hence $R(T_{[n]}) = R(T^{n+1})$ is closed. Since $T$ has no eigenvalue, then $N(T^{n+1}) = \{0\}$. Thus $0 \notin \sigma_{ap}(T^{n+1})$ which is a contradiction. Therefore, $\sigma_{SBF^-}(T) = \sigma_{ap}(T)$.

In particular $T$ satisfies generalized $a$-Weyl's theorem.

Also, the assumption "$A$ and $B$ are isoid" is crucial in Theorem 2.2, as shown by the following example.

Example 2.4 Let $I_1$ and $I_2$ be the identities acting on $\mathbb{C}$ and $I_2$, respectively. Let $S_1$ and $S_2$ defined on $I_2$ by

$$S_1(x_1, x_2, \ldots) = \left(0, \frac{1}{3}x_1, \frac{1}{3}x_2, \ldots \right) \text{ and } S_2(x_1, x_2, \ldots) = \left(0, \frac{1}{2}x_1, \frac{1}{3}x_2, \ldots \right).$$

Let $A = I_1 \oplus S_1$ and $B = S_2 - I_2$. Clearly,

$$\sigma(A) = \left\{ \lambda \in \mathbb{C} : |\lambda| \leq \frac{1}{3} \right\} \cup \{1\} \text{ and } \sigma(B) = \{-1\}.$$

We have $\sigma(B) = \sigma_{ap}(B) = \sigma_{BW}(B) = \{-1\}$. Since $B$ and $B^*$ have the SVEP, then by [1, Corollary 3.53] and [1, Theorem 3.66]

$$\sigma_{SBF^-}(B) = \sigma_{BW}(B) = \sigma(B) = \{-1\}.$$

Now, $\sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{2}\} \cup \{1\}$. We claim that $\sigma_{SBF^-}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\}$.

Indeed, we have that $\sigma_{SBF^-}(A) \subseteq \sigma_{ap}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} \cup \{1\}$ and $1 \notin \sigma_{SBF^-}(A)$. If there exists $\lambda \in \mathbb{C}$, $|\lambda| = \frac{1}{3}$ such that $A - \lambda(I_1 \oplus I_2)$ is upper semi-B-Fredholm of negative index, then $S_1 - \lambda I_2$ is upper semi-B-Fredholm of negative index. Since $S_1$ has no eigenvalue, then it follows from Remark 2.3 that $\lambda \notin \sigma_{ap}(S_1) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\}$. This is a contradiction. Since $A$ and $B$ have the SVEP, then from [1, Corollary 3.72], we conclude that

$$\sigma_{SBF^-}(A^2) = \sigma_{SBF^-}(A)^2 = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{9}\} \text{ and } \sigma_{SBF^-}(B^2) = \sigma_{SBF^-}(B)^2 = \{1\}.$$

Now,

$$\sigma_{ap}(A) \setminus \sigma_{SBF^-}(A) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} \cup \{1\} \setminus \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{3}\} = \{1\} = E(A)$$

and

$$\sigma_{ap}(B) \setminus \sigma_{SBF^-}(B) = \emptyset = E(B).$$

Also,

$$\sigma_{ap}(A^2) \setminus \sigma_{SBF^-}(A^2) = \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{9}\} \cup \{1\} \setminus \{\lambda \in \mathbb{C} : |\lambda| = \frac{1}{9}\} = \{1\} = E(A^2)$$

and

$$\sigma_{ap}(B^2) \setminus \sigma_{SBF^-}(B^2) = \emptyset = E(B^2).$$

Thus the property $(gw)$ holds for $A$, $B$, $A^2$ and $B^2$. 

In the other hand, since $A \otimes B$ has no eigenvalue then it follows from Remark 2.3 that
\[ \sigma_{SBF^-}(A \otimes B) = \{ \lambda \in \mathbb{C} : |\lambda| = \sigma_{ap}(A \otimes B) = \{ \lambda \in \mathbb{C} : |\lambda| = \frac{1}{3} \} \cup \{-1\} \]
and
\[ \sigma_{SBF^-}(A^2 \otimes B^2) = \{ \lambda \in \mathbb{C} : |\lambda| = \sigma_{ap}(A^2 \otimes B^2) = \{ \lambda \in \mathbb{C} : |\lambda| = \frac{1}{9} \} \cup \{1\} \].
Hence
\[ \sigma_{SBF^-}(A \otimes B) = \sigma_{SBF^-}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF^-}(B), \]
and
\[ \sigma_{SBF^-}(A^2 \otimes B^2) = \sigma_{SBF^-}(A^2)\sigma_{ap}(B^2) \cup \sigma_{ap}(A^2)\sigma_{SBF^-}(B^2). \]
Also
\[ \sigma_{ap}(A \otimes B) \setminus \sigma_{SBF^-}(A \otimes B) = \emptyset = E(A \otimes B) \]
and
\[ \sigma_{ap}(A^2 \otimes B^2) \setminus \sigma_{SBF^-}(A^2 \otimes B^2) = \emptyset \neq 1 = E(A^2 \otimes B^2). \]
Thus the property $(gw)$ holds for $A \otimes B$ but not for $A^2 \otimes B^2$. Note here that $B$ and $B^2$ are not isoloid.

The following example show that there exists two operators $A, B \in \mathcal{L}(X)$ such that $A \otimes B$ satisfies the property $(gw)$ but $A$ and $B$ do not satisfy the property $(gw)$.

**Example 2.5** Let $B = U + U^*$ where $U$ is the unilateral shift on $l_2$. Since $B$ is self-adjoint, then
\[ \sigma(B) = \sigma_{ap}(B) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \} \]
and from [1],
\[ \sigma_{BW}(B) = \sigma_{SBF^-}(B) = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}. \]
Hence
\[ \sigma_{ap}(B) \setminus \sigma_{SBF^-}(B) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \} \setminus \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}. \]
Since $E(B) = \emptyset$, then property $(gw)$ fails for $B$. In the other hand, if $I$ is the identity acting on $l_2$, then $I \otimes B$ is self-adjoint, hence
\[ \sigma(I \otimes B) = \sigma_{ap}(I \otimes B) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}. \]
\[ \sigma_{BW}(I \otimes B) = \{ \lambda \in \mathbb{C} : |\lambda| = 1 \}, \text{ and } \sigma_{SBF^-}(I \otimes B) = \{ \lambda \in \mathbb{C} : |\lambda| \leq 1 \}. \]
Hence,
\[ \sigma_{ap}(I \otimes B) \setminus \sigma_{SBF^-}(I \otimes B) = \emptyset = E(I \otimes B). \]
Thus $I \otimes B$ satisfies property $(gw)$. 

3. Generalized \(a\)-Weyl’s theorem and tensor product

Recall that given \(T \in \mathcal{L}(X)\), \(T\) is said to be \(a\)-isoloid, if \(isos_{ap}(T) = E_{a}(T)\).

**Theorem 3.1** Let \(X\) and \(Y\) two Banach spaces. Suppose that \(A \in \mathcal{L}(X)\) and \(B \in \mathcal{L}(Y)\) are \(a\)-isoloid and satisfy generalized \(a\)-Weyl’s theorem. If \(0 \notin isos_{ap}(A \otimes B)\) and \(\sigma_{SBF_{+}^{-}}(A \otimes B) = \sigma_{SBF_{+}^{-}}(A) \cup \sigma_{ap}(B) \cup \sigma_{ap}(A) \sigma_{SBF_{+}^{-}}(B)\), then \(A \otimes B\) satisfies generalized \(a\)-Weyl’s theorem.

**Proof.** Let \(\lambda \in \sigma_{ap}(A \otimes B) \setminus \sigma_{SBF_{+}^{-}}(A \otimes B)\). Assume for the sake of contradiction that \(\lambda \in acc_{ap}(A \otimes B)\). Since \(\lambda \in acc(\sigma_{ap}(A) \sigma_{ap}(B))\), it follows that

\[
\lambda \in acc_{ap}(A) \sigma_{ap}(B) \cup \sigma_{ap}(A) acc_{ap}(B).
\]

Since generalized \(a\)-Weyl’s theorem holds for both \(A\) and \(B\) then

\[
\lambda \in \sigma_{SBF_{+}^{-}}(A) \sigma_{ap}(B) \cup \sigma_{ap}(A) \sigma_{SBF_{+}^{-}}(B).
\]

By hypothesis, we get that \(\lambda \in \sigma_{SBF_{+}^{-}}(A \otimes B)\). This is a contradiction. Hence \(\lambda \in isos_{ap}(A \otimes B)\). Since \(A \otimes B - \lambda\) is upper semi B-Fredholm, then \(\lambda \in E_{a}(A \otimes B)\). For the reverse inclusion, let \(\lambda \in E_{a}(A \otimes B)\). By hypothesis, \(\lambda \neq 0\). With the same argument as in the proof of Theorem 2.2 we get \(\lambda = \mu \nu\) with \(\mu \in isos_{ap}(A)\) and \(\nu \in isos_{ap}(B)\). Since \(A\) and \(B\) are \(a\)-isoloid, then \(\mu \in E_{a}(A)\) and \(\nu \in E_{a}(B)\). Hence from the fact that generalized \(a\)-Weyl’s theorem holds for \(A\) and \(B\), we deduce that

\[
\mu \in \sigma_{ap}(A) \setminus \sigma_{SBF_{+}^{-}}(A)\text{ and } \nu \in \sigma_{ap}(B) \setminus \sigma_{SBF_{+}^{-}}(B).
\]

Thus

\[
\lambda \notin \sigma_{ap}(A) \sigma_{SBF_{+}^{-}}(B) \cup \sigma_{SBF_{+}^{-}}(A) \sigma_{ap}(B).
\]

By hypothesis, we conclude that \(\lambda \notin \sigma_{SBF_{+}^{-}}(A \otimes B)\), and hence

\[
\lambda \in \sigma_{ap}(A \otimes B) \setminus \sigma_{SBF_{+}^{-}}(A \otimes B).
\]

\[\blacksquare\]

4. Concluding remarks

A bounded linear operator \(T\) is said to satisfy \(a\)-Browder’s theorem if \(\sigma_{ap}(T) \setminus \sigma_{SBF_{+}^{-}}(T) = \pi_{0}^{a}(T)\); where \(\pi_{0}^{a}(T)\) is the set of all left pole of finite rank defined by

\[
\pi_{0}^{a}(T) = \{\lambda \in \mathbb{C} : a(T - \lambda) < \infty; R(T^{a(T - \lambda) + 1}) \text{ is closed and } a(T - \lambda) < \infty\}.
\]

It is well known that \(a\)-weyl’s theorem implies \(a\)-Browder’s theorem and the reverse is not true.

\(T\) is said to satisfy the generalized \(a\)-Browder’s theorem if \(\sigma_{ap}(T) \setminus \sigma_{SBF_{+}^{-}}(T) = \pi_{0}^{a}(T)\); where \(\pi_{0}^{a}(T)\) is the set of all left pole defined by \(\pi_{0}^{a}(T) = \{\lambda \in \mathbb{C} : a(T - \lambda) < \infty \text{ and } R(T^{a(T - \lambda) + 1}) \text{ is closed}\}\). Generalized \(a\)-Weyl’s theorem implies
generalized \(a\)-Browder’s theorem and the reverse is not true. Recently we proved that \(a\)-Browder’s theorem is equivalent to generalized \(a\)-Browder’s theorem [8].

In [16, Theorem 1] it was shown that if \(A \in \mathcal{L}(X)\) and \(B \in \mathcal{L}(Y)\) are isoloid operators which satisfy property \((w)\), then equality

\[
\sigma_{SF_+}(A \otimes B) = \sigma_{SF_+}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SF_+}(B)
\]

is equivalent to \(A \otimes B\) satisfies \(a\)-Browder’s theorem.

In [14, Remark 4.6] Boasso, Duggal and Jeon asked:

If \(A\) and \(B\) satisfy generalized \(a\)-Browder’s theorem, does the equality

\[
\sigma_{SBF_+}(A \otimes B) = \sigma_{SBF_+}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+}(B)
\]

equivalent to \(A \otimes B\) satisfies generalized \(a\)-Browder’s theorem?

In [20, Theorem 2.1] it is apparent that the author answered this question positively. However, it is not true as shown by the following example.

**Example 4.1** Let \(A\) be a nonzero nilpotent operator and let \(B\) be a quasi-nilpotent which is not nilpotent. Then \(A\) and \(B\) satisfies generalized \(a\)-Browder’s theorem. Hence

\[
\sigma(A) = \{0\}, \quad \sigma_{SBF_+}(A) = \emptyset \quad \text{and} \quad \sigma(B) = \sigma_{SBF_+}(B) = \{0\}.
\]

Then

\[
\sigma_{SBF_+}(A)\sigma(B) \cup \sigma(A)\sigma_{SBF_+}(B) = \{0\}.
\]

However, since \(A \otimes B\) is nilpotent then 0 is a pole and then \(\sigma_{SBF_+}(A \otimes B) = \emptyset\). Here \(A \otimes B\) satisfies generalized \(a\)-Browder’s theorems.

**Remarks 4.2**

(1) In [20, Theorem 2.2], it was stated that if \(A\) and \(B\) are isoloid and satisfy generalized \(a\)-Weyl’s theorem, then \(\sigma_{SBF_+}(A \otimes B) = \sigma_{SBF_+}(A)\sigma_{ap}(B) \cup \sigma_{ap}(A)\sigma_{SBF_+}(B)\) implies that \(A \otimes B\) satisfies generalized \(a\)-Weyl’s theorem. In the proof, the author used Theorem 2.1 which is false as mentioned above.

(2) In the proof of [20, Lemma 2.1], the author used the following equivalence "\(A \otimes B - \frac{1}{n}I\) is injective if and only if \(A\) and \(B\) are injective". We would like to point out that this equivalence is not true. For instance, if \(A\) and \(B\) are nonzero nilpotent operators then \(A\) and \(B\) are not injective however \(A \otimes B - \frac{1}{n}I\) is injective.
References


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