FUZZY PARAMETERIZED FUZZY SOFT NORMAL SUBGROUPS OF GROUPS

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Abstract. In the present paper, we redefine the concept of $FPFS$-sets in algebra systems, which is a novel way that is different from the definition of $FPFS$-sets in hemirings by Liu [10]. Based on the definition, we put forth $FPFS$-groups, $FPFS$-normal subgroups and FP-equivalent fuzzy soft normal subgroups of groups. Further, some properties and characterizations are investigated. Finally, aggregate fuzzy normal subgroups of groups are given.

Keyword: $FPFS$-sets; $FPFS$-groups; $FPFS$-normal subgroups; FP-equivalent fuzzy soft normal subgroups; Aggregate fuzzy normal subgroups.
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1. Introduction

In the real world, there are many problems that we met are full of indeterminacy and vagueness. Facing so many uncertain data, classical methods are not always successful, the reason is that these methods are designed for certain situations and the actual situations that under consideration are often more complex. In 1965, Zadeh proposed the fuzzy set theory, which could as a mathematical approach to deal with inexact and uncertain knowledge. After that, fuzzy set theory is progressing rapidly. In 1971, Rosenfeld [17] applied this concept to the theory of groupoids and groups. Especially Liu [11] investigated the fuzzy isomorphism theorems of groups. However, fuzzy set theory have its inherent difficulties which was pointed out in [15]. To overcome the difficulties, in 1999, Molodtsov [15] first put forward the concept of soft set theory as a new mathematical tool for dealing with uncertainty and vagueness. It has been proven useful in many other fields. Moreover, it has opened a new direction, new exploration, new path of thinking to

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mathematicians, engineers, computer scientists and many other researchers and so on. Up to the present, the research on soft sets is progressing rapidly. Several researchers have point out several directions for the applications of soft sets, more details (see e.g. [1], [5], [8], [9], [13], [14], [16]), and some of them also investigated the operations on the theory of soft sets. In particular, an adjustable approach to fuzzy soft set based decision making was applied by Feng [7]. Also, Maji [12] applied the soft set theory to the fuzzy set theory and investigated some related properties of them.

Following the discovery of the soft set theory, some researchers applied the theory to the algebraic structure. In 2007, Aktas et al. [1] applied the notion of soft sets to the theory of groups and investigated some properties of them. Further, Yang [18] given the notions of fuzzy soft semigroups and fuzzy soft ideals and discussed fuzzy soft images and fuzzy soft inverse images of fuzzy soft semigroups (ideals). In particular, Zhan investigated the ideal theory on hemirings, most of relevant conclusions have been already demonstrated in Zhan's book, which is referred to [19].

Recently, Çağman et al. [3], [4] put forward the concepts of fuzzy parameterized fuzzy soft set ($FPFS$-sets) and fuzzy parameterized soft set (FP-soft sets), respectively. Moreover, Liu [10] applied this theory to hemirings and some basic properties are investigated. The main purpose of this paper is to give a novel definition of $FPFS$-sets by a new way and study some related properties. This paper is organized as follows: we recall some concepts and results on groups and $FPFS$-sets in Section 2. In Section 3, we introduce the concepts of $FPFS$-normal subgroups and FP-equivalent fuzzy soft normal subgroups by making the set of parameters to be an algebraic structure. In addition, some properties and characterizations are investigated. Finally, we give the aggregate fuzzy normal subgroups of groups in Section 4.

2. Preliminaries

Definition 2.1 [17] A fuzzy subset μ of a group G is said to be a fuzzy subgroup of G if

(1) $\mu(xy) \geq \min\{\mu(x), \mu(y)\}$,

(2) $\mu(x^{-1}) \geq \mu(x)$,

holds for all $x, y \in G$.

Equivalently, $\mu(xy^{-1}) \geq \mu(x) \wedge \mu(y)$ for all $x, y \in G$.

Definition 2.2 [6] A fuzzy subset μ of a group G is said to be a fuzzy normal subgroup of G if the following axioms hold:

$$\mu(xyx^{-1}) \geq \mu(y) \text{ for all } x, y \in G.$$

Equivalently, $\mu(xyx^{-1}) = \mu(y)$ for all $x, y \in G$ or $\mu(xy) = \mu(yx)$ for all $x, y \in G$.

The following concepts are referred to [3].

Definition 2.3 Let U be an initial universe, E be the set of all parameters and X be a fuzzy set over E with the membership function $\mu_X : E \to [0,1]$ and $\gamma_X(x)$ be a fuzzy set over E for all $x \in E$, $F(U)$ be the set of all fuzzy set of U. Then a fuzzy parameterized fuzzy soft set Γ_X on U is defined by a function $\gamma_X(x)$ representing a mapping

$$\gamma_X : E \to F(U) \text{ such that } \gamma_X(x) = \emptyset \text{ if } \mu_X(x) = 0.$$

Here γ_X is called fuzzy approximate function of the fuzzy parameterized fuzzy soft set Γ_X, and the value $\gamma_X(x)$ is a fuzzy set called x-element of the fuzzy parameterized fuzzy soft set for all $x \in E$. Thus a fuzzy parameterized fuzzy soft set Γ_X over U can be represented by the set of ordered pairs

$$\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x))) : x \in E, \gamma_X(x) \in F(U), \mu_X(x) \in [0,1]\}.$$

A fuzzy parameterized fuzzy soft set is briefly said to be an $FPFS$-set. The set of all $FPFS$-sets over U is denoted by $FPFS(U)$.

Definition 2.4 Let $\Gamma_X \in FPFS(U)$.

(1) If $\gamma_X(x) = \emptyset$ for all $x \in E$, then Γ_X is called an X-empty $FPFS$-set, denoted by $\Gamma_{\emptyset_X}$.

(2) If $X = \emptyset$, then the Γ_X is called an empty $FPFS$-set, denoted by $\Gamma_\emptyset$.

(3) If $\mu_X(x) = 1$ and $\gamma_X(x) = U$ for all $x \in E$, then Γ_X is called an X-universal $FPFS$-set, denoted by Γ_X.

(4) If $X = E$, then the X-universal $FPFS$-set is called an universal $FPFS$-set, denoted by Γ_E.

Definition 2.5 Let $\Gamma_X, \Gamma_Y \in FPFS(U)$.

(1) Γ_X is an $FPFS$-subset of Γ_Y, denoted by $\Gamma_X \widetilde{\subseteq} \Gamma_Y$, if $\mu_X(x) \leq \mu_Y(x)$ and $\gamma_X(x) \subseteq \gamma_Y(x)$ for all $x \in E$.

(2) Γ_X and Γ_Y are FP-equal, denoted by $\Gamma_X = \Gamma_Y$, if $\mu_X(x) = \mu_Y(x)$ and $\gamma_X(x) = \gamma_Y(x)$ for all $x \in E$.

Definition 2.6 Let $\Gamma_X \in FPFS(U)$. Then the complement of Γ_X, denoted by $\Gamma_X^{\tilde{c}}$, is an $FPFS$-set defined by

$$\mu_X^{\tilde{c}}(x) = 1 - \mu(x) \text{ and } \gamma_Y^{\tilde{c}}(x) = U \setminus \gamma_X(x).$$

Definition 2.7 Let $\Gamma_X, \Gamma_Y \in FPFS(U)$.

(1) The intersection of Γ_X and Γ_Y, denoted by $\Gamma_X \widetilde{\cap} \Gamma_Y$, is defined by $\mu_{X\widetilde{\cap}Y}(x) = \min\{\mu_X(x), \mu_Y(x)\}$ and $\gamma_{X\widetilde{\cap}Y}(x) = \gamma_X(x) \cap \gamma_Y(x)$ for all $x \in E$.

(2) The union of Γ_X and Γ_Y, denoted by $\Gamma_X \widetilde{\cup} \Gamma_Y$, is defined by $\mu_{X\widetilde{\cup}Y}(x) = \max\{\mu_X(x), \mu_Y(x)\}$ and $\gamma_{X\widetilde{\cap}Y}(x) = \gamma_X(x) \cup \gamma_Y(x)$ for all $x \in E$.

3. Fuzzy parameterized fuzzy soft normal subgroups

In this section, we give the definitions of $FPFS$-groups ($FPFS$-normal subgroups) and FP-equivalent fuzzy normal subgroups of groups, then some related properties and characterizations of them are investigated.

Definition 3.1 Let G_1 and G_2 be two groups as an initial universe and a set of all parameters, respectively. Let X be a fuzzy set over G_2 with $\mu_X : G_2 \to [0,1]$ and γ_X be a fuzzy set over G_1 for all $x \in G_2$. Assume that $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\} \in FPFS(G_1)$. Then Γ_X is called a fuzzy parameterized fuzzy soft group (fuzzy parameterized fuzzy soft normal subgroup) (briefly, $FPFS$-group or $FPFS$-normal subgroup) over G_1 if for all $x \in G_2$, X and $\gamma_X(x)$ are two fuzzy subgroups (fuzzy normal subgroups) of G_2 and G_1, respectively.

Example 3.2

(1) Let $G_1 = S_3 = \{e, (12), (23), (13), (123), (132)\}$ be a group as an initial universe and $G_2 = \{e, (12)\}$ be a group as a set of all parameters, respectively. If $X = \{0.5/e, 0.4/(12)\}$,

$$\begin{aligned} \gamma_X(e) &= \{0.9/e, 0.8/(12), 0.4/(13), 0.3/(23), 0.5/(123), 0.5/(132)\}, \\ \gamma_X((12)) &= \{0.8/e, 0.7/(12), 0.4/(13), 0.4/(23), 0.5/(123), 0.5/(132)\}, \end{aligned}$$

for all $x \in G_2$, we can verify that X and $\gamma_X(x)$ are two fuzzy subgroups of G_2 and G_1, respectively. So Γ_X is an $FPFS$-group over G_1.

(2) Let $G_1 = G_2 = \{e, (12)\}$ be two groups as an initial universe and a set of all parameters, respectively. If $X=\{0.5/e, 0.4/(12)\}$, $\gamma_X(e)=\{0.9/e, 0.8/(12)\}$, $\gamma_X((12)) = \{0.8/e, 0.7/(12)\}$, for all $x \in G_2$, we can verify that X and $\gamma_X(x)$ are two fuzzy normal subgroups of G_2 and G_1, respectively. So Γ_X is an $FPFS$-normal subgroup over G_1.

Proposition 3.3 *Let Γ_X and Γ_Y be two $FPFS$-normal subgroups over G_1. Then their intersection $\Gamma_X \widetilde{\cap} \Gamma_Y$ is also an $FPFS$-normal subgroup over G_1.*

Proof. We can write $\Gamma_X \widetilde{\cap} \Gamma_Y = \Gamma_{X \widetilde{\cap} Y}$. Since Γ_X and Γ_Y are two $FPFS$-normal subgroups over G_1, it follows that X and Y are two fuzzy normal subgroups of G_2, then for all $x, y \in G_2$,

$$\begin{aligned} \mu_{X \widetilde{\cap} Y}(xyx^{-1}) &= \min\{\mu_X(xyx^{-1}), \mu_Y(xyx^{-1})\} \\ &\geq \min\{\mu_X(y), \mu_Y(y)\} \\ &= \mu_{X \widetilde{\cap} Y}(y). \end{aligned}$$

So $X \cap Y$ is a fuzzy normal subgroup of G_2. Now we shall prove $\gamma_X(x) \cap \gamma_Y(x)$ is a fuzzy normal subgroup of G_1.

For all $x \in G_2$, $s, t \in G_1$,

$$\begin{aligned}(\mu_{\gamma_X(x)} \cap \mu_{\gamma_Y(x)})(sts^{-1}) &= \min\{\mu_{\gamma_X(x)}(sts^{-1}), \mu_{\gamma_Y(x)}(sts^{-1})\} \\ &\geq \min\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(t)\} \\ &= (\mu_{\gamma_X(x)} \cap \mu_{\gamma_Y(x)})(t).\end{aligned}$$

Hence $\mu_{\gamma_X(x)} \cap \mu_{\gamma_Y(x)}$ is a fuzzy normal subgroup of G_1, that is to say $\gamma_{X\widetilde{\cap}Y} = \gamma_X(x) \cap \gamma_Y(x)$ is a fuzzy normal subgroup of G_1. Therefore, $\Gamma_X \widetilde{\cap} \Gamma_Y$ is an $FPFS$-normal subgroup over G_1. ∎

We know that the intersection of all $FPFS$-normal subgroups over a group G_1 is also an $FPFS$-normal subgroup over G_1. Then we would consider whether the union of $FPFS$-normal subgroups over G_1 is also an $FPFS$-normal subgroup over G_1.

Definition 3.4 Let Γ_X and Γ_Y be two $FPFS$-normal subgroups over G_1. Then we said the sequence of values are ordered, if for any $x, y \in G_2$, $s, t \in G_1$, $\mu_X(x) \geq \mu_X(y)$, $\gamma_X(x)(s) \geq \gamma_X(x)(t)$ implies $\mu_Y(x) \geq \mu_Y(y)$, $\gamma_Y(x)(s) \geq \gamma_Y(x)(t)$.

Proposition 3.5 *Let Γ_X and Γ_Y be two $FPFS$-normal subgroups over G_1 with ordered sequence of values. Then their union $\Gamma_X \widetilde{\cup} \Gamma_Y$ is still an $FPFS$-normal subgroup over G_1.*

Proof. We can write $\Gamma_X \widetilde{\cup} \Gamma_Y = \Gamma_{X\widetilde{\cup}Y}$. For all $x, y \in G_2$, $s, t \in G_1$. Let

$$\begin{aligned}P_1 &= \max\{\min\{\mu_X(x), \mu_X(y)\}, \min\{\mu_Y(x), \mu_Y(y)\}\}, \\ P_2 &= \min\{\max\{\mu_X(x), \mu_Y(x)\}, \max\{\mu_X(y), \mu_Y(y)\}\}, \\ S_1 &= \max\{\min\{\mu_{\gamma_X(x)}(s), \mu_{\gamma_X(x)}(t)\}, \min\{\mu_{\gamma_Y(x)}(s), \mu_{\gamma_Y(x)}(t)\}\}, \\ S_2 &= \min\{\max\{\mu_{\gamma_X(x)}(s), \mu_{\gamma_Y(x)}(s)\}, \max\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(t)\}\}.\end{aligned}$$

We know that $P_1 \geq P_2$. In fact, since the sequence of values are ordered. Let $\mu_X(x) \geq \mu_X(y)$, $\gamma_X(x)(s) \geq \gamma_X(x)(t)$. We have $\mu_Y(x) \geq \mu_Y(y)$, $\gamma_Y(x)(s) \geq \gamma_Y(x)(t)$. It follows that $P_1 = \max\{\mu_X(y), \mu_Y(y)\}$.
(1) if $\max\{\mu_X(x), \mu_Y(x)\} > \max\{\mu_X(y), \mu_Y(y)\}$, then $P_2 = \max\{\mu_X(y), \mu_Y(y)\}$, in this case, we get $P_1 = P_2$;
(2) if $\max\{\mu_X(x), \mu_Y(x)\} \leq \max\{\mu_X(y), \mu_Y(y)\}$, then $P_2 = \max\{\mu_X(x), \mu_Y(x)\}$, we get $P_1 \geq P_2$. Thus, in any case, we have $P_1 \geq P_2$. Similarly, we have $S_1 \geq S_2$.

Now, we have

$$\begin{aligned}\max\{\mu_X(x), \mu_Y(x)\} &\geq \max\{\mu_X(y), \mu_Y(y)\}, \\ \max\{\mu_{\gamma_X(x)}(s), \mu_{\gamma_Y(x)}(s)\} &\geq \max\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(t)\}.\end{aligned}$$

$$\begin{aligned}\mu_{X\widetilde{\cup}Y}(xyx^{-1}) &= \max\{\mu_X(xyx^{-1}), \mu_Y(xyx^{-1})\} \\ &\geq \max\{\min\{\mu_X(x), \mu_X(y)\}, \min\{\mu_Y(x), \mu_Y(y)\}\} \\ &\geq \min\{\max\{\mu_X(x), \mu_Y(x)\}, \max\{\mu_X(y), \mu_Y(y)\}\} \\ &= \max\{\mu_X(y), \mu_Y(y)\} \\ &= \mu_{X\widetilde{\cup}Y}(y).\end{aligned}$$

So $X \cup Y$ is a fuzzy normal subgroup of G_2. In a similar way, we have $(\mu_{\gamma_X(x)} \cup \mu_{\gamma_Y(x)})(st^{-1}s) \geq (\mu_{\gamma_X(x)} \cup \mu_{\gamma_Y(x)})(t)$. That is $(\gamma_{X\widetilde{\cup}Y})(x) = \gamma_X(x) \cup \gamma_Y(x)$ is a fuzzy normal subgroup of G_1.
Therefore, $\Gamma_X \widetilde{\cup} \Gamma_Y$ is an $FPFS$-normal subgroup over G_1. ∎

Definition 3.6 [2] Let μ and ν be fuzzy sets in a set G. The Cartesian product of μ and ν is defined by

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}, \forall x, y \in G.$$

Definition 3.7 The multiplication of Γ_X and Γ_Y, denoted by $\Gamma_X \widetilde{\times} \Gamma_Y$, is defined by $\mu_{X\widetilde{\times}Y}(x, y) = \min\{\mu_X(x), \mu_Y(y)\}$ and $\gamma_{X\widetilde{\times}Y}(x) = \gamma_X(x) \times \gamma_Y(x)$ for all $x \in G_2$.

Proposition 3.8 *Let Γ_X and Γ_Y be two $FPFS$-normal subgroups over G_1. Then $\Gamma_X \widetilde{\times} \Gamma_Y$ is an $FPFS$-normal subgroup over $G_1 \times G_1$.*

Proof. We can write $\Gamma_X \widetilde{\times} \Gamma_Y = \Gamma_{X\widetilde{\times}Y}$. Since Γ_X and Γ_Y are two $FPFS$-normal subgroups over G_1, it follows that X and Y are two fuzzy normal subgroups of G_2, then for all $x, y, z, w \in G_2$,

$$\begin{aligned} \mu_{X\widetilde{\times}Y}(xyx^{-1}, zwz^{-1}) &= \min\{\mu_X(xyx^{-1}), \mu_Y(zwz^{-1})\} \\ &\geq \min\{\mu_X(y), \mu_Y(w)\} \\ &= \mu_{X\widetilde{\times}Y}(y, w). \end{aligned}$$

So $X \times Y$ is a fuzzy normal subgroup of $G_2 \times G_2$. Now we shall prove $\gamma_X(x) \times \gamma_Y(x)$ is a fuzzy normal subgroup of $G_1 \times G_1$.

For all $x \in G_2$, $s, t, a, b \in G_1$,

$$\begin{aligned} (\mu_{\gamma_X(x)} \times \mu_{\gamma_Y(x)})(sts^{-1}, aba^{-1}) &= \min\{\mu_{\gamma_X(x)}(sts^{-1}), \mu_{\gamma_Y(x)}(aba^{-1})\} \\ &\geq \min\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(b)\} \\ &= (\mu_{\gamma_X(x)} \times \mu_{\gamma_Y(x)})(t, b). \end{aligned}$$

Hence $\mu_{\gamma_X(x)} \times \mu_{\gamma_Y(x)}$ is a fuzzy normal subgroup of $G_1 \times G_1$, that is to say $\gamma_{X\widetilde{\times}Y} = \gamma_X(x) \times \gamma_Y(x)$ is a fuzzy normal subgroup of $G_1 \times G_1$. Therefore, $\Gamma_X \widetilde{\times} \Gamma_Y$ is an $FPFS$-normal subgroup over $G_1 \times G_1$. ∎

Definition 3.9 Let Γ_X be an $FPFS$-normal subgroup over G_1, Γ_Y be an $FPFS$-group over G_1. Then Γ_X is said to be an $FPFS$-normal subgroup of Γ_Y, if for all $x \in G_2$, $\mu_X(x) \leq \mu_Y(x)$ and γ_X is an $FPFS$-subset of γ_Y.

Example 3.10 Assume that $G_1 = \{1, -1, i, -i\}$ is a group, where $i^2 = -1$ and $G_2 = \{1, -1\}$ is a set of parameters. If $X = \{0.3/1, 0.1/-1\}$, $\gamma_X(1) = \{0.5/1, 0.5/-1, 0.3/i, 0.3/-i\}$, $\gamma_X(-1) = \{0.5/1\}$, and $Y = \{0.4/1, 0.2/-1\}$, $\gamma_Y(1) = \{0.6/1, 0.6/-1, 0.4/i, 0.4/-i\}$, $\gamma_Y(-1) = \{0.5/1, 0.5/-1, 0.3/i, 0.3/-i\}$, then Γ_X is an $FPFS$-normal subgroup over G_1, Γ_Y is an $FPFS$-group over G_1 and Γ_X is an $FPFS$-normal subgroup of Γ_Y.

Theorem 3.11 *Let $\Gamma_X, \Gamma_Y, \Gamma_Z$ be $FPFS$-groups over G_1. If Γ_X is an $FPFS$-normal subgroup of Γ_Y and Γ_Y is an $FPFS$-normal subgroup of Γ_Z, then Γ_X is an $FPFS$-normal subgroup of Γ_Z.*

Proof. The proof is obvious and is omitted. ∎

Definition 3.12 Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) | x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ and $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y)) | y \in K_2, \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0,1]\}$ be $FPFS$-sets over groups G_1 and K_1, respectively. If $f : G_1 \to K_1$ and $g : G_2 \to K_2$ are two functions, then (f, g) is called an FP-fuzzy soft homomorphism from Γ_X to Γ_Y, denoted by $\Gamma_X \sim \Gamma_Y$, if the following conditions are satisfied:

(1) f is an epimorphism from G_1 to K_1,
(2) g is a surjective mapping,
(3) $f(\gamma_X(x)) = \gamma_Y(g(x))$ and $\mu_X(x) = \mu_Y(g(x))$ for all $x \in G_2$.

In the above definition, if f is an isomorphism from G_1 to K_1 and g is a bijective mapping, then (f, g) is called an FP-fuzzy soft isomorphism from Γ_X to Γ_Y, denoted by $\Gamma_X \simeq \Gamma_Y$.

Example 3.13 Consider $G_1 = G_2 = \{1, -1\}$ and $K_1 = K_2 = \{e, (12)\}$. Define a homomorphism $f : G_1 \to K_1$ by $f(1) = e$, $f(-1) = (12)$ for all $s \in G_1$, and a mapping $g : G_2 \to K_2$ by $g(1) = e$, $g(-1) = (12)$ for all $x \in G_2$.

Let X be a fuzzy set over G_2 defined by $X = \{0.5/1, 0.5/-1\}$, Y be a fuzzy set over K_2 defined by $Y = \{0.5/e, 0.5/(12)\}$.

Let $\gamma_X : G_2 \to F(G_1)$ defined by

$$(\gamma_X(1))(s) = \begin{cases} 0.4, & s = 1, s \in G_1, \\ 0.1, & s = -1, s \in G_1. \end{cases}$$

$$(\gamma_X(-1))(s) = \begin{cases} 0.6, & s = 1, s \in G_1, \\ 0.2, & s = -1, s \in G_1. \end{cases}$$

$\gamma_Y : K_2 \to F(K_1)$ defined by

$$(\gamma_Y(e))(k) = \begin{cases} 0.4, & k = e, k \in K_1, \\ 0.1, & k = (12), k \in K_1. \end{cases}$$

$$(\gamma_Y(12))(k) = \begin{cases} 0.6, & k = e, k \in K_1 \\ 0.2, & k = (12), k \in K_1. \end{cases}$$

It is clear that Γ_X and Γ_Y are $FPFS$-sets over G_1 and K_1, respectively. We can immediately see that $\mu_X(x) = \mu_Y(g(x))$ and we can deduce that $f(\gamma_X(x)) = \gamma_Y(g(x))$ for all $x \in G_2$. Hence (f, g) is an FP- fuzzy soft homomorphism from Γ_X to Γ_Y.

Lemma 3.14 [10] *Let $f : G \to K$ be an epimorphism of groups and μ be a fuzzy normal subgroup of G, then $f(\mu)$ is a fuzzy normal subgroup of K.*

Theorem 3.15 *Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2,\ \gamma_X(x) \in F(G_1),\ \mu_X(x) \in [0,1]\}$ be an $FPFS$-normal subgroup over group G_1 and $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y)) \mid y \in K_2,\ \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0,1]\}$ be an $FPFS$-set over group K_1. If Γ_X is FP-fuzzy soft homomorphic to Γ_Y, then Γ_Y is an $FPFS$-normal subgroup over K_1.*

Proof. Let (f,g) be an FP-fuzzy soft homomorphism from Γ_X to Γ_Y. Since Γ_X is an $FPFS$-normal subgroup over group G_1, $f(G_1) = K_1$, $g(G_2) = K_2$ and for all $x \in G_2$, X, $\gamma_X(x)$ are two fuzzy normal subgroups of G_2 and G_1, respectively. Now, for all $y \in K_2$, there exists $x \in G_2$ such that $g(x) = y$. Since (f,g) is an FP-fuzzy soft homomorphism from Γ_X to Γ_Y, so $\gamma_Y(y) = \gamma_Y(g(x)) = f(\gamma_X(x))$ and $\mu_Y(y) = \mu_Y(g(x)) = \mu_X(x)$. By Lemma 3.14, we have that Y and $\gamma_Y(y)$ are two fuzzy normal subgroups of K_2 and K_1, respectively. Hence Γ_Y is an $FPFS$-normal subgroup over K_1. ∎

Definition 3.16 Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2,\ \gamma_X(x) \in F(G_1),\ \mu_X(x) \in [0,1]\}$ be an $FPFS$-normal subgroup over group G_1. Then Γ_X is said to be FP-equivalent fuzzy soft normal subgroup over G_1 if, for any $x, y \in G_2$, $\mu_X(x) = \mu_X(y)$, we have $\gamma_X(x) = \gamma_X(y)$.

Example 3.17 Assume that $G_1 = \{1, -1, i, -i\}$ is a group, where $i^2 = -1$, and $G_2 = \{e, (123), (132)\}$ is a group as a set of parameters and X is a fuzzy set over G_2 defined by $X = \{0.5/e, 04/(123), 0.4/(132)\}$. Let γ_X be defined by $\gamma_X(e) = \{0.8/1, 0.8/-1, 0.7/i, 0.7/-i\}$, $\gamma_X(123) = \{0.6/1, 0.6/-1, 0.4/i, 0.4/-i\}$, $\gamma_X(132) = \{0.6/1, 0.6/-1, 0.4/i, 0.4/-i\}$. It is clearly that Γ_X is an FP-equivalent fuzzy soft normal subgroup over G_1.

Theorem 3.18 *Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) | x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ be an FP-equivalent fuzzy soft normal subgroup over group G_1 and $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y)) | y \in K_2, \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0,1]\}$ be an $FPFS$-set over group K_1. If Γ_X is FP-fuzzy soft homomorphic to Γ_Y, then Γ_Y is an FP-equivalent fuzzy soft normal subgroup over K_1.*

Proof. Let (f,g) be an FP-fuzzy soft homomorphism from Γ_X to Γ_Y. Since Γ_X is an FP-equivalent fuzzy soft normal subgroup over group G_1, we have $\gamma_X(x_1) = \gamma_X(x_2)$, if $\mu_X(x_1) = \mu_X(x_2)$ for any $x_1, x_2 \in G_2$. By Theorem 3.15, we have Γ_Y is an $FPFS$-normal subgroup over K_1. Now, for all $y_1, y_2 \in K_2$, then there exist $x_1, x_2 \in G_2$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since $\mu_Y(y_1) = \mu_Y(g(x_1)) = \mu_X(x_1)$, $\mu_Y(y_2) = \mu_Y(g(x_2)) = \mu_X(x_2)$, then $\mu_Y(y_1) = \mu_Y(y_2)$. And we have $\gamma_Y(y_1) = \gamma_Y(g(x_1)) = f(\gamma_X(x_1)) = f(\gamma_X(x_2)) = \gamma_Y(g(x_2)) = \gamma_Y(y_2)$, hence Γ_Y is an FP-equivalent fuzzy soft normal subgroup over K_1. ∎

Definition 3.19 Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) | x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ be an $FPFS$-normal subgroup over group G_1. Then Γ_X is said to be FP-increasing fuzzy soft normal subgroup over G_1 if, for any $x, y \in G_2$, $\mu_X(x) \le \mu_X(y)$, we have $\gamma_X(x) \subseteq \gamma_X(y)$, and Γ_X is said to be FP-decreasing fuzzy soft normal subgroup over G_1 if, for any $x, y \in G_2$, $\mu_X(x) \le \mu_X(y)$, we have $\gamma_X(x) \supseteq \gamma_X(y)$.

Example 3.20 Let $G_1 = \{e, (13)\}$, $G_2 = \{e, (123), (132)\}$ and X be a fuzzy set over G_2 defined by $X = \{0.6/e, 03/(123), 0.3/(132)\}$, γ_X be defined by $\gamma_X(e) = \{0.7/e, 0.5/(13)\}$, $\gamma_X(123) = \{0.2/e, 0.1/(13)\}$, $\gamma_X(132) = \{0.2/e, 0.1/(13)\}$. It is clear that Γ_X is an FP-increasing fuzzy soft normal subgroup over G_1.

Theorem 3.21 *Let* $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x))|x \in G_2,\ \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ *be an FP-increasing (decreasing) fuzzy soft normal subgroup over group* G_1 *and* $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y))|y \in K_2, \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0,1]\}$ *be an $FPFS$-set over group K_1. If Γ_X is FP-fuzzy soft homomorphic to Γ_Y, then Γ_Y is an FP-increasing (decreasing) fuzzy soft normal subgroup over K_1.*

Proof. Let (f, g) be an FP-fuzzy soft homomorphism from Γ_X to Γ_Y. Since Γ_X is an FP-increasing fuzzy soft normal subgroup over group G_1, we have $\gamma_X(x_1) \subseteq \gamma_X(x_2)$, if $\mu_X(x_1) \leq \mu_X(x_2)$ for any $x_1, x_2 \in G_2$. By Theorem 3.15, we have Γ_Y is an $FPFS$-normal subgroup over K_1. Now, for all $y_1, y_2 \in K_2$, then there exist $x_1, x_2 \in G_2$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since $\mu_Y(y_1) = \mu_Y(g(x_1)) = \mu_X(x_1)$, $\mu_Y(y_2) = \mu_Y(g(x_2)) = \mu_X(x_2)$, then $\mu_Y(y_1) \leq \mu_Y(y_2)$. And we have $\gamma_Y(y_1) = \gamma_Y(g(x_1)) = f(\gamma_X(x_1)) \subseteq f(\gamma_X(x_2)) = \gamma_Y(g(x_2)) = \gamma_Y(y_2)$, hence Γ_Y is an FP-increasing fuzzy soft normal subgroup over K_1. ∎

4. Aggregate fuzzy normal subgroups

An aggregate fuzzy set of an $FPFS$-set has been defined by Çağman et al. (see [3]). They also defined $FPFS$-aggregation operator that produced an aggregate fuzzy set from an $FPFS$-set and its fuzzy parameter set.

Definition 4.1 Let $\Gamma_X \in FPFS(G_1)$. Then $FPFS$-aggregation operator, denoted by $FPFS_{agg}$, is defined by

$$FPFS_{agg} : F(G_2) \times FPFS(G_1) \to F(G_1),$$

$$FPFS_{agg}(X, \Gamma_X) = \Gamma_X^*.$$

where

$$\Gamma_X^* = \{\mu_{\Gamma_X^*}(u)/u|u \in G_1\},$$

which is a fuzzy set over G_1. The value Γ_X^* is called an aggregate fuzzy set of the Γ_X. Here the membership degree $\mu_{\Gamma_X^*}(u)$ of u is defined as follows

$$\mu_{\Gamma_X^*}(u) = \frac{1}{|G_2|} \sum_{x \in G_2} \mu_X(x)\mu_{\gamma_X(x)}(u),$$

where $|G_2|$ is the cardinality of G_2.

Theorem 4.2 *Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x))|x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ be an $FPFS$-normal subgroup over group G_1. Then the aggregate fuzzy set Γ_X^* of Γ_X is a fuzzy normal subgroup of G_1.*

Proof. Since Γ_X is an $FPFS$-normal subgroup over group G_1, so $\gamma_X(x)$ is a fuzzy normal subgroup of G_1 for all $x \in G_2$. Equivalently, $\mu_{\gamma_X(x)}(sts^{-1}) \geq \mu_{\gamma_X(x)}(t)$ for all $s, t \in G_1$. Then

$$\begin{aligned}\mu_{\Gamma_X^*}(sts^{-1}) &= \frac{1}{|G_2|}\sum_{x\in G_2}\mu_X(x)\mu_{\gamma_X(x)}(sts^{-1}) \\ &\geq \frac{1}{|G_2|}\sum_{x\in G_2}\mu_X(x)\mu_{\gamma_X(x)}(t) \\ &= \mu_{\Gamma_X^*}(t).\end{aligned}$$

Then Γ_X^* is a fuzzy normal subgroup of G_1. ■

Remark 4.3 (1) The above Γ_X^* is called an aggregate fuzzy normal subgroup of $FPFS$-normal subgroup Γ_X; (2) Γ_X^* is a fuzzy normal subgroup of G_1, but Γ_X is not always $FPFS$-normal subgroup of G_1.

Example 4.4

(1) Let $G_1 = M_n$ be a matrix group, A, B be a lower triangular matrix and a diagonal matrix. And let $G_2 = \{e, (12)\}$, the parameters e, (12) stand for "lower triangular" and "diagonal", respectively. And X is a fuzzy set over G_2 defined by

$$\mu_X(x) = \begin{cases} 0.5, & x = e, \\ 0.3, & x = (12). \end{cases}$$

Let γ_X be defined by

$$\mu_{\gamma_X(e)}(r) = \begin{cases} 0, & r \text{ is not a lower triangular matrix,} \\ 1, & r \text{ is a lower triangular matrix.} \end{cases}$$

$$\mu_{\gamma_X(12)}(r) = \begin{cases} 0, & r \text{ is not a diagonal matrix,} \\ 1, & r \text{ is a diagonal matrix.} \end{cases}$$

It is clear that Γ_X is an $FPFS$-normal subgroup over G_1. The aggregate fuzzy set can be found as

$$\mu_{\Gamma_X^*}(u) = \begin{cases} 0.2, & u \in B, \\ 0.1, & u \in A \setminus B, \\ 0, & \text{otherwise.} \end{cases}$$

We can verify that Γ_X^* is a fuzzy normal subgroup of G_1. Then Γ_X^* is called an aggregate fuzzy normal subgroup of $FPFS$-normal subgroup Γ_X.

(2) Let $G_1 = \{1, -1, i, -i\}$ be a group, where $i^2 = -1$, $G_2 = \{e, (12)\}$, X be a fuzzy set over G_2 defined by $X = \{0.3/e, 0.4/(12)\}$, γ_X be defined by $\gamma_X(e) = \{0.8/1, 0.8/-1, 0.6/i, 0.6/-i\}$, $\gamma_X(12) = \{0.7/1, 0.7/-1, 0.5/i, 0.5/-i\}$. So $\Gamma_X^* = \{0.26/1, 0.26/-1, 0.19/i, 0.19/-i\}$. It is clear that Γ_X^* is a fuzzy normal subgroup of G_1, but Γ_X is not $FPFS$-normal subgroup of G_1 because of X is not a fuzzy normal subgroup of G_2.

Proposition 4.5 *Γ_X is an $FPFS$-normal subgroup of G_1 if and only if Γ_X^* is a fuzzy normal subgroup of G_1 and X is a fuzzy normal subgroup of G_2.*

Proof. Let Γ_X be an $FPFS$-normal subgroup of G_1, it is obvious that Γ_X^* is a fuzzy normal subgroup of G_1 and X is a fuzzy normal subgroup of G_2.

Conversely, let Γ_X^* be a fuzzy normal subgroup of G_1 and X be a fuzzy normal subgroup of G_2. Assume that Γ_X is not an $FPFS$-normal subgroup of G_1, so $\gamma_X(x)$ is not a fuzzy normal subgroup of G_1 for all $x \in G_2$ or X is not a fuzzy normal subgroup of G_2, then Γ_X^* is not a fuzzy normal subgroup of G_1 or X is not a fuzzy normal subgroup of G_2. Which is a contradiction. So Γ_X is an $FPFS$-normal subgroup of G_1. The proof is complete. ■

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