FUZZY PARAMETERIZED FUZZY SOFT NORMAL SUBGROUPS OF GROUPS

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Abstract. In the present paper, we redefine the concept of FPFS-sets in algebra systems, which is a novel way that is different from the definition of FPFS-sets in hemirings by Liu [10]. Based on the definition, we put forth FPFS-groups, FPFS-normal subgroups and FP-equivalent fuzzy soft normal subgroups of groups. Further, some properties and characterizations are investigated. Finally, aggregate fuzzy normal subgroups of groups are given.

Keyword: FPFS-sets; FPFS-groups; FPFS-normal subgroups; FP-equivalent fuzzy soft normal subgroups; Aggregate fuzzy normal subgroups.

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1. Introduction

In the real world, there are many problems that we met are full of indeterminacy and vagueness. Facing so many uncertain data, classical methods are not always successful, the reason is that these methods are designed for certain situations and the actual situations that under consideration are often more complex. In 1965, Zadeh proposed the fuzzy set theory, which could as a mathematical approach to deal with inexact and uncertain knowledge. After that, fuzzy set theory is progressing rapidly. In 1971, Rosenfeld [17] applied this concept to the theory of groupoids and groups. Especially Liu [11] investigated the fuzzy isomorphism theorems of groups. However, fuzzy set theory have its inherent difficulties which was pointed out in [15]. To overcome the difficulties, in 1999, Molodtsov [15] first put forward the concept of soft set theory as a new mathematical tool for dealing with uncertainty and vagueness. It has been proven useful in many other fields. Moreover, it has opened a new direction, new exploration, new path of thinking to

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mathematicians, engineers, computer scientists and many other researchers and so on. Up to the present, the research on soft sets is progressing rapidly. Several researchers have pointed out several directions for the applications of soft sets, more details (see e.g. [1], [5], [8], [9], [13], [14], [16]), and some of them also investigated the operations on the theory of soft sets. In particular, an adjustable approach to fuzzy soft set based decision making was applied by Feng [7]. Also, Maji [12] applied the soft set theory to the fuzzy set theory and investigated some related properties of them.

Following the discovery of the soft set theory, some researchers applied the theory to the algebraic structure. In 2007, Aktas et al. [1] applied the notion of soft sets to the theory of groups and investigated some properties of them. Further, Yang [18] given the notions of fuzzy soft semigroups and fuzzy soft ideals and discussed fuzzy soft images and fuzzy soft inverse images of fuzzy soft semigroups (ideals). In particular, Zhan investigated the ideal theory on hemirings, most of relevant conclusions have been already demonstrated in Zhan’s book, which is referred to [19].

Recently, Çağman et al. [3], [4] put forward the concepts of fuzzy parameterized fuzzy soft set (FPFS-sets) and fuzzy parameterized soft set (FP-soft sets), respectively. Moreover, Liu [10] applied this theory to hemirings and some basic properties are investigated. The main purpose of this paper is to give a novel definition of FPFS-sets by a new way and study some related properties. This paper is organized as follows: we recall some concepts and results on groups and FPFS-sets in Section 2. In Section 3, we introduce the concepts of FPFS-normal subgroups and FP-equivalent fuzzy soft normal subgroups by making the set of parameters to be an algebraic structure. In addition, some properties and characterizations are investigated. Finally, we give the aggregate fuzzy normal subgroups of groups in Section 4.

2. Preliminaries

**Definition 2.1** [17] A fuzzy subset \( \mu \) of a group \( G \) is said to be a fuzzy subgroup of \( G \) if

\[
\begin{align*}
(1) \quad \mu(xy) & \geq \min\{\mu(x), \mu(y)\}, \\
(2) \quad \mu(x^{-1}) & \geq \mu(x),
\end{align*}
\]

holds for all \( x, y \in G \).

Equivalently, \( \mu(xy^{-1}) \geq \mu(x) \wedge \mu(y) \) for all \( x, y \in G \).

**Definition 2.2** [6] A fuzzy subset \( \mu \) of a group \( G \) is said to be a fuzzy normal subgroup of \( G \) if the following axioms hold:

\[
\mu(xyx^{-1}) \geq \mu(y) \text{ for all } x, y \in G.
\]

Equivalently, \( \mu(xyx^{-1}) = \mu(y) \) for all \( x, y \in G \) or \( \mu(xy) = \mu(yx) \) for all \( x, y \in G \).
The following concepts are referred to [3].

**Definition 2.3** Let $U$ be an initial universe, $E$ be the set of all parameters and $X$ be a fuzzy set over $E$ with the membership function $\mu_X : E \to [0,1]$ and $\gamma_X(x)$ be a fuzzy set over $E$ for all $x \in E$, $F(U)$ be the set of all fuzzy set of $U$. Then a fuzzy parameterized fuzzy soft set $\Gamma_X$ on $U$ is defined by a function $\gamma_X(x)$ representing a mapping

$$\gamma_X : E \to F(U)$$

such that $\gamma_X(x) = \emptyset$ if $\mu_X(x) = 0$.

Here $\gamma_X$ is called fuzzy approximate function of the fuzzy parameterized fuzzy soft set $\Gamma_X$, and the value $\gamma_X(x)$ is a fuzzy set called $x$-element of the fuzzy parameterized fuzzy soft set for all $x \in E$. Thus a fuzzy parameterized fuzzy soft set $\Gamma_X$ over $U$ can be represented by the set of ordered pairs

$$\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) : x \in E, \gamma_X(x) \in F(U), \mu_X(x) \in [0,1]\}.$$

A fuzzy parameterized fuzzy soft set is briefly said to be an FPFS-set. The set of all FPFS-sets over $U$ is denoted by $\text{FPFS}(U)$.

**Definition 2.4** Let $\Gamma_X \in \text{FPFS}(U)$.

1. If $\gamma_X(x) = \emptyset$ for all $x \in E$, then $\Gamma_X$ is called an $X$-empty FPFS-set, denoted by $\Gamma_{\emptyset_X}$.

2. If $X = \emptyset$, then the $\Gamma_X$ is called an empty FPFS-set, denoted by $\Gamma_{\emptyset}$.

3. If $\mu_X(x) = 1$ and $\gamma_X(x) = U$ for all $x \in E$, then $\Gamma_X$ is called an $X$-universal FPFS-set, denoted by $\Gamma_X$.

4. If $X = E$, then the $X$-universal FPFS-set is called an universal FPFS-set, denoted by $\Gamma_E$.

**Definition 2.5** Let $\Gamma_X, \Gamma_Y \in \text{FPFS}(U)$.

1. $\Gamma_X$ is an FPFS-subset of $\Gamma_Y$, denoted by $\Gamma_X \subseteq \Gamma_Y$, if $\mu_X(x) \leq \mu_Y(x)$ and $\gamma_X(x) \subseteq \gamma_Y(x)$ for all $x \in E$.

2. $\Gamma_X$ and $\Gamma_Y$ are FP-equal, denoted by $\Gamma_X = \Gamma_Y$, if $\mu_X(x) = \mu_Y(x)$ and $\gamma_X(x) = \gamma_Y(x)$ for all $x \in E$.

**Definition 2.6** Let $\Gamma_X \in \text{FPFS}(U)$. Then the complement of $\Gamma_X$, denoted by $\Gamma_X^c$, is an FPFS-set defined by

$$\mu_{\Gamma_X^c}(x) = 1 - \mu_X(x) \text{ and } \gamma_{\Gamma_X^c}(x) = U \setminus \gamma_X(x).$$

**Definition 2.7** Let $\Gamma_X, \Gamma_Y \in \text{FPFS}(U)$.

1. The intersection of $\Gamma_X$ and $\Gamma_Y$, denoted by $\Gamma_X \cap \Gamma_Y$, is defined by $\mu_{\Gamma_X \cap \Gamma_Y}(x) = \min\{\mu_X(x), \mu_Y(x)\}$ and $\gamma_{\Gamma_X \cap \Gamma_Y}(x) = \gamma_X(x) \cap \gamma_Y(x)$ for all $x \in E$.

2. The union of $\Gamma_X$ and $\Gamma_Y$, denoted by $\Gamma_X \cup \Gamma_Y$, is defined by $\mu_{\Gamma_X \cup \Gamma_Y}(x) = \max\{\mu_X(x), \mu_Y(x)\}$ and $\gamma_{\Gamma_X \cup \Gamma_Y}(x) = \gamma_X(x) \cup \gamma_Y(x)$ for all $x \in E$. 
3. Fuzzy parameterized fuzzy soft normal subgroups

In this section, we give the definitions of FPFS-groups (FPFS-normal subgroups) and FP-equivalent fuzzy normal subgroups of groups, then some related properties and characterizations of them are investigated.

Definition 3.1 Let $G_1$ and $G_2$ be two groups as an initial universe and a set of all parameters, respectively. Let $X$ be a fuzzy set over $G_2$ with $\mu_X : G_2 \rightarrow [0, 1]$ and $\gamma_X$ be a fuzzy set over $G_1$ for all $x \in G_2$. Assume that $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0, 1]\} \in FPFS(G_1)$. Then $\Gamma_X$ is called a fuzzy parameterized fuzzy soft group (fuzzy parameterized fuzzy soft normal subgroup) (briefly, FPFS-group or FPFS-normal subgroup) over $G_1$ if for all $x \in G_2$, $X$ and $\gamma_X(x)$ are two fuzzy subgroups (fuzzy normal subgroups) of $G_2$ and $G_1$, respectively.

Example 3.2

(1) Let $G_1 = S_3 = \{e, (12), (23), (13), (123), (132)\}$ be a group as an initial universe and $G_2 = \{e, (12)\}$ be a group as a set of all parameters, respectively. If $X = \{0.5/e, 0.4/(12)\}$,

$$\gamma_X(e) = \{0.9/e, 0.8/(12), 0.4/(13), 0.3/(23), 0.5/(123), 0.5/(132)\},$$

$$\gamma_X((12)) = \{0.8/e, 0.7/(12), 0.4/(13), 0.4/(23), 0.5/(123), 0.5/(132)\},$$

for all $x \in G_2$, we can verify that $X$ and $\gamma_X(x)$ are two fuzzy subgroups of $G_2$ and $G_1$, respectively. So $\Gamma_X$ is an FPFS-group over $G_1$.

(2) Let $G_1 = G_2 = \{e, (12)\}$ be two groups as an initial universe and a set of all parameters, respectively. If $X = \{0.5/e, 0.4/(12)\}$, $\gamma_X(e) = \{0.9/e, 0.8/(12)\}$, $\gamma_X((12)) = \{0.8/e, 0.7/(12)\}$, for all $x \in G_2$, we can verify that $X$ and $\gamma_X(x)$ are two fuzzy normal subgroups of $G_2$ and $G_1$, respectively. So $\Gamma_X$ is an FPFS-normal subgroup over $G_1$.

Proposition 3.3 Let $\Gamma_X$ and $\Gamma_Y$ be two FPFS-normal subgroups over $G_1$. Then their intersection $\Gamma_X \cap \Gamma_Y$ is also an FPFS-normal subgroup over $G_1$.

Proof. We can write $\Gamma_X \cap \Gamma_Y = \Gamma_X \cap \Gamma_Y$. Since $\Gamma_X$ and $\Gamma_Y$ are two FPFS-normal subgroups over $G_1$, it follows that $X$ and $Y$ are two fuzzy normal subgroups of $G_2$, then for all $x, y \in G_2$,

$$\mu_{X \cap Y}(xyx^{-1}) = \min\{\mu_X(xy^{-1}), \mu_Y(xy^{-1})\} \geq \min\{\mu_X(y), \mu_Y(y)\} = \mu_{X \cap Y}(y).$$

So $X \cap Y$ is a fuzzy normal subgroup of $G_2$. Now we shall prove $\gamma_X(x) \cap \gamma_Y(x)$ is a fuzzy normal subgroup of $G_1$. 
For all \(x \in G_2, \, s, t \in G_1\),

\[
(\mu_{\gamma_X(x)} \cap \mu_{\gamma_Y(x)})(sts^{-1}) = \min\{\mu_{\gamma_X(x)}(sts^{-1}), \mu_{\gamma_Y(x)}(sts^{-1})\}
\geq \min\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(t)\}
= (\mu_{\gamma_X(x)} \cap \mu_{\gamma_Y(x)})(t).
\]

Hence \(\mu_{\gamma_X(x)} \cap \mu_{\gamma_Y(x)}\) is a fuzzy normal subgroup of \(G_1\), that is to say \(\gamma_X \cap \gamma_Y = \gamma_X(x) \cap \gamma_Y(x)\) is a fuzzy normal subgroup of \(G_1\). Therefore, \(\Gamma_X \cap \Gamma_Y\) is an FPFS-normal subgroup over \(G_1\).

We know that the intersection of all FPFS-normal subgroups over a group \(G_1\) is also an FPFS-normal subgroup over \(G_1\). Then we would consider whether the union of FPFS-normal subgroups over \(G_1\) is also an FPFS-normal subgroup over \(G_1\).

**Definition 3.4** Let \(\Gamma_X\) and \(\Gamma_Y\) be two FPFS-normal subgroups over \(G_1\). Then we said the sequence of values are ordered, if for any \(x, y \in G_2, \, s, t \in G_1\), \(\mu_X(x) \geq \mu_X(y)\), \(\gamma_X(x)(s) \geq \gamma_X(x)(t)\) implies \(\nu_Y(x) \geq \nu_Y(y)\), \(\gamma_Y(x)(s) \geq \gamma_Y(x)(t)\).

**Proposition 3.5** Let \(\Gamma_X\) and \(\Gamma_Y\) be two FPFS-normal subgroups over \(G_1\) with ordered sequence of values. Then their union \(\Gamma_X \cup \Gamma_Y\) is still an FPFS-normal subgroup over \(G_1\).

**Proof.** We can write \(\Gamma_X \cup \Gamma_Y = \Gamma_X \cap \Gamma_Y\). For all \(x, y \in G_2, \, s, t \in G_1\). Let

\[
P_1 = \max\{\min\{\mu_X(x), \mu_X(y)\}, \min\{\nu_X(x), \nu_Y(y)\}\}
\]

\[
P_2 = \min\{\max\{\mu_X(x), \nu_Y(x)\}, \max\{\mu_Y(x), \nu_Y(y)\}\},
\]

\[
S_1 = \max\{\min\{\mu_{\gamma_X(x)}(s), \mu_{\gamma_X(x)}(t)\}, \min\{\mu_{\gamma_Y(x)}(s), \mu_{\gamma_Y(x)}(t)\}\},
\]

\[
S_2 = \min\{\max\{\mu_{\gamma_X(x)}(s), \mu_{\gamma_Y(x)}(s)\}, \max\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(t)\}\}.
\]

We know that \(P_1 \geq P_2\). In fact, since the sequence of values are ordered. Let \(\mu_X(x) \geq \mu_X(y)\), \(\gamma_X(x)(s) \geq \gamma_X(x)(t)\). We have \(\nu_X(x) \geq \nu_X(y)\), \(\gamma_X(x)(s) \geq \gamma_X(x)(t)\). It follows that \(P_1 = \max\{\mu_X(x), \nu_Y(y)\}\).

(1) if \(\max\{\mu_X(x), \nu_X(y)\} > \max\{\mu_X(x), \nu_Y(y)\}\), then \(P_2 = \max\{\mu_X(x), \nu_Y(y)\}\), in this case, we get \(P_1 = P_2\);

(2) if \(\max\{\mu_X(x), \nu_X(y)\} \leq \max\{\mu_X(x), \nu_Y(y)\}\), then \(P_2 = \max\{\mu_X(x), \nu_Y(x)\}\), we get \(P_1 \geq P_2\). Thus, in any case, we have \(P_1 \geq P_2\). Similarly, we have \(S_1 \geq S_2\).

Now, we have

\[
\max\{\mu_X(x), \mu_Y(x)\} \geq \max\{\mu_X(y), \mu_Y(y)\},
\]

\[
\max\{\mu_{\gamma_X(x)}(s), \mu_{\gamma_Y(x)}(s)\} \geq \max\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(t)\}.
\]

\[
\mu_{\gamma_X}(xyx^{-1}) = \max\{\mu_X(xy^{-1}), \nu_Y(xy^{-1})\}
\geq \max\{\min\{\mu_X(x), \mu_X(y)\}, \min\{\mu_Y(x), \mu_Y(y)\}\}
\geq \min\{\max\{\mu_X(x), \nu_Y(x)\}, \max\{\mu_X(y), \nu_Y(y)\}\}
= \max\{\mu_X(y), \nu_Y(y)\}
= \mu_{\gamma_X}(y).
\]
So $X \cup Y$ is a fuzzy normal subgroup of $G_2$. In a similar way, we have $(\mu_{\gamma_X(x)} \cup \mu_{\gamma_Y(x)})(st^{-1}s) \geq (\mu_{\gamma_X(x)} \cup \mu_{\gamma_Y(x)})(t)$. That is $(\gamma_{X \cup Y})(x) = \gamma_X(x) \cup \gamma_Y(x)$ is a fuzzy normal subgroup of $G_1$.

Therefore, $\Gamma_X \cup \Gamma_Y$ is an FPFS-normal subgroup over $G_1$. \hfill \blacksquare

**Definition 3.6** [2] Let $\mu$ and $\nu$ be fuzzy sets in a set $G$. The Cartesian product of $\mu$ and $\nu$ is defined by

$$(\mu \times \nu)(x, y) = \min\{\mu(x), \nu(y)\}, \forall x, y \in G.$$

**Definition 3.7** The multiplication of $\Gamma_X$ and $\Gamma_Y$, denoted by $\Gamma_X \times \Gamma_Y$, is defined by $\mu_{\gamma_X \gamma_Y}(x, y) = \min\{\mu_X(x), \mu_Y(y)\}$ and $\gamma_{X \gamma_Y}(x) = \gamma_X(x) \times \gamma_Y(x)$ for all $x \in G_2$.

**Proposition 3.8** Let $\Gamma_X$ and $\Gamma_Y$ be two FPFS-normal subgroups over $G_1$. Then $\Gamma_X \times \Gamma_Y$ is an FPFS-normal subgroup over $G_1 \times G_1$.

**Proof.** We can write $\Gamma_X \times \Gamma_Y = \Gamma_{X \times Y}$. Since $\Gamma_X$ and $\Gamma_Y$ are two FPFS-normal subgroups over $G_1$, it follows that $X$ and $Y$ are two fuzzy normal subgroups of $G_2$, then for all $x, y, z, w \in G_2$,

$$\mu_{X \times Y}(xy^{-1}, zwz^{-1}) = \min\{\mu_X(xy^{-1}), \mu_Y(zwz^{-1})\} \geq \min\{\mu_X(y), \mu_Y(w)\} = \mu_{X \times Y}(y, w).$$

So $X \times Y$ is a fuzzy normal subgroup of $G_2 \times G_2$. Now we shall prove $\gamma_X(x) \times \gamma_Y(x)$ is a fuzzy normal subgroup of $G_1 \times G_1$.

For all $x \in G_2$, $s, t, a, b \in G_1$,

$$(\mu_{\gamma_X(x)} \times \mu_{\gamma_Y(x)})(sts^{-1}, aba^{-1}) = \min\{\mu_{\gamma_X(x)}(sts^{-1}), \mu_{\gamma_Y(x)}(aba^{-1})\} \geq \min\{\mu_{\gamma_X(x)}(t), \mu_{\gamma_Y(x)}(b)\} = (\mu_{\gamma_X(x)} \times \mu_{\gamma_Y(x)})(t, b).$$

Hence $\mu_{\gamma_X(x)} \times \mu_{\gamma_Y(x)}$ is a fuzzy normal subgroup of $G_1 \times G_1$, that is to say $\gamma_{X \times Y} = \gamma_X(x) \times \gamma_Y(x)$ is a fuzzy normal subgroup of $G_1 \times G_1$. Therefore, $\Gamma_X \times \Gamma_Y$ is an FPFS-normal subgroup over $G_1 \times G_1$. \hfill \blacksquare

**Definition 3.9** Let $\Gamma_X$ be an FPFS-normal subgroup over $G_1$, $\Gamma_Y$ be an FPFS-group over $G_1$. Then $\Gamma_X$ is said to be an FPFS-normal subgroup of $\Gamma_Y$, if for all $x \in G_2$, $\mu_X(x) \leq \mu_Y(x)$ and $\gamma_X$ is an FPFS-subset of $\gamma_Y$.

**Example 3.10** Assume that $G_1 = \{1, -1, i, -i\}$ is a group, where $i^2 = -1$ and $G_2 = \{1, -1\}$ is a set of parameters. If $X = \{0.3/1, 0.1/ -1\}$, $\gamma_X(1) = \{0.5/1, 0.5/-1, 0.3/i, 0.3/-i\}$, $\gamma_X(-1) = \{0.5/1\}$, and $Y = \{0.4/1, 0.2/ -1\}$, $\gamma_Y(1) = \{0.6/1, 0.6/-1, 0.4/i, 0.4/-i\}$, $\gamma_Y(-1) = \{0.5/1, 0.5/-1, 0.3/i, 0.3/-i\}$, then $\Gamma_X$ is an FPFS-normal subgroup over $G_1$, $\Gamma_Y$ is an FPFS-group over $G_1$ and $\Gamma_X$ is an FPFS-normal subgroup of $\Gamma_Y$. 

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**Theorem 3.11** Let $\Gamma_X, \Gamma_Y, \Gamma_Z$ be FPFS-groups over $G_1$. If $\Gamma_X$ is an FPFS-normal subgroup of $\Gamma_Y$ and $\Gamma_Y$ is an FPFS-normal subgroup of $\Gamma_Z$, then $\Gamma_X$ is an FPFS-normal subgroup of $\Gamma_Z$.

**Proof.** The proof is obvious and is omitted. □

**Definition 3.12** Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x))| x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0, 1]\}$ and $\Gamma_Y = \{(\mu_Y(y)/y, \gamma_Y(y))| y \in K_2, \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0, 1]\}$ be FPFS-sets over groups $G_1$ and $K_1$, respectively. If $f : G_1 \to K_1$ and $g : G_2 \to K_2$ are two functions, then $(f, g)$ is called an FP-fuzzy soft homomorphism from $\Gamma_X$ to $\Gamma_Y$, denoted by $\Gamma_X \simeq \Gamma_Y$, if the following conditions are satisfied:

1. $f$ is an epimorphism from $G_1$ to $K_1$,
2. $g$ is a surjective mapping,
3. $f(\gamma_X(x)) = \gamma_Y(g(x))$ and $\mu_X(x) = \mu_Y(g(x))$ for all $x \in G_2$.

In the above definition, if $f$ is an isomorphism from $G_1$ to $K_1$ and $g$ is a bijective mapping, then $(f, g)$ is called an FP-fuzzy soft isomorphism from $\Gamma_X$ to $\Gamma_Y$, denoted by $\Gamma_X \simeq \Gamma_Y$.

**Example 3.13** Consider $G_1 = G_2 = \{1, -1\}$ and $K_1 = K_2 = \{e, (12)\}$. Define a homomorphism $f : G_1 \to K_1$ by $f(1) = e, f(-1) = (12)$ for all $s \in G_1$, and a mapping $g : G_2 \to K_2$ by $g(1) = e, g(-1) = (12)$ for all $x \in G_2$.

Let $X$ be a fuzzy set over $G_2$ defined by $X = \{0.5/1, 0.5/-1\}$, $Y$ be a fuzzy set over $K_2$ defined by $Y = \{0.5/e, 0.5/(12)\}$.

Let $\gamma_X : G_2 \to F(G_1)$ defined by

\[
(\gamma_X(1))(s) = \begin{cases} 
0.4, & s = 1, s \in G_1, \\
0.1, & s = -1, s \in G_1. 
\end{cases}
\]

\[
(\gamma_X(-1))(s) = \begin{cases} 
0.6, & s = 1, s \in G_1, \\
0.2, & s = -1, s \in G_1. 
\end{cases}
\]

$\gamma_Y : K_2 \to F(K_1)$ defined by

\[
(\gamma_Y(e))(k) = \begin{cases} 
0.4, & k = e, k \in K_1, \\
0.1, & k = (12), k \in K_1. 
\end{cases}
\]

\[
(\gamma_Y(12))(k) = \begin{cases} 
0.6, & k = e, k \in K_1 \\
0.2, & k = (12), k \in K_1. 
\end{cases}
\]

It is clear that $\Gamma_X$ and $\Gamma_Y$ are FPFS-sets over $G_1$ and $K_1$, respectively. We can immediately see that $\mu_X(x) = \mu_Y(g(x))$ and we can deduce that $f(\gamma_X(x)) = \gamma_Y(g(x))$ for all $x \in G_2$. Hence $(f, g)$ is an FP-fuzzy soft homomorphism from $\Gamma_X$ to $\Gamma_Y$.

**Lemma 3.14** [10] Let $f : G \to K$ be an epimorphism of groups and $\mu$ be a fuzzy normal subgroup of $G$, then $f(\mu)$ is a fuzzy normal subgroup of $K$. 


**Theorem 3.15** Let $\Gamma_X = \{ (\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1] \}$ be an FPFS-normal subgroup over group $G_1$ and $\Gamma_Y = \{ (\mu_Y(y)/y, \gamma_Y(y)) \mid y \in K_2, \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0,1] \}$ be an FPFS-set over group $K_1$. If $\Gamma_X$ is FP-fuzzy soft homomorphic to $\Gamma_Y$, then $\Gamma_Y$ is an FPFS-normal subgroup over $K_1$.

**Proof.** Let $(f, g)$ be an FP-fuzzy soft homomorphism from $\Gamma_X$ to $\Gamma_Y$. Since $\Gamma_X$ is an FPFS-normal subgroup over group $G_1$, $f(G_1) = K_1$, $g(G_2) = K_2$ and for all $x \in G_2$, $X$, $\gamma_X(x)$ are two fuzzy normal subgroups of $G_2$ and $G_1$, respectively. Now, for all $y \in K_2$, there exists $x \in G_2$ such that $g(x) = y$. Since $(f, g)$ is an FP-fuzzy soft homomorphism from $\Gamma_X$ to $\Gamma_Y$, so $\gamma_Y(y) = \gamma_Y(g(x)) = f(\gamma_X(x))$ and $\mu_Y(y) = \mu_Y(g(x)) = \mu_X(x)$. By Lemma 3.14, we have that $Y$ and $\gamma_Y(y)$ are two fuzzy normal subgroups of $K_2$ and $K_1$, respectively. Hence $\Gamma_Y$ is an FPFS-normal subgroup over $K_1$.

**Definition 3.16** Let $\Gamma_X = \{ (\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1] \}$ be an FPFS-normal subgroup over group $G_1$. Then $\Gamma_X$ is said to be FP-equivalent fuzzy soft normal subgroup over $G_1$ if, for any $x, y \in G_2$, $\mu_X(x) = \mu_Y(x)$, we have $\gamma_X(x) = \gamma_Y(y)$.

**Example 3.17** Assume that $G_1 = \{1, -1, i, -i\}$ is a group, where $i^2 = -1$, and $G_2 = \{e, (123), (132)\}$ is a group as a set of parameters and $X$ is a fuzzy set over $G_2$ defined by $X = \{0.5/e, 0.4/(123), 0.4/(132)\}$. Let $\gamma_X$ be defined by $\gamma_X(e) = \{0.8/1, 0.8/-1, 0.7/i, 0.7/-i\}$, $\gamma_X(123) = \{0.6/1, 0.6/-1, 0.4/i, 0.4/-i\}$, $\gamma_X(132) = \{0.6/1, 0.6/-1, 0.4/i, 0.4/-i\}$. It is clearly that $\Gamma_X$ is an FP-equivalent fuzzy soft normal subgroup over $G_1$.

**Theorem 3.18** Let $\Gamma_X = \{ (\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1] \}$ be an FP-equivalent fuzzy soft normal subgroup over group $G_1$ and $\Gamma_Y = \{ (\mu_Y(y)/y, \gamma_Y(y)) \mid y \in K_2, \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0,1] \}$ be an FPFS-set over group $K_1$. If $\Gamma_X$ is FP-fuzzy soft homomorphic to $\Gamma_Y$, then $\Gamma_Y$ is an FP-equivalent fuzzy soft normal subgroup over $K_1$.

**Proof.** Let $(f, g)$ be an FP-fuzzy soft homomorphism from $\Gamma_X$ to $\Gamma_Y$. Since $\Gamma_X$ is an FP-equivalent fuzzy soft normal subgroup over group $G_1$, we have $\gamma_X(x_1) = \gamma_X(x_2)$, if $\mu_X(x_1) = \mu_X(x_2)$ for any $x_1, x_2 \in G_2$. By Theorem 3.15, we have $\Gamma_Y$ is an FPFS-normal subgroup over $K_1$. Now, for all $y_1, y_2 \in K_2$, then there exist $x_1, x_2 \in G_2$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since $\mu_Y(y_1) = \mu_Y(g(x_1)) = \mu_X(x_1)$, $\mu_Y(y_2) = \mu_Y(g(x_2)) = \mu_X(x_2)$, then $\mu_Y(y_1) = \mu_Y(y_2)$ and we have $\gamma_Y(y_1) = \gamma_Y(g(x_1))) = f(\gamma_X(x_1)) = f(\gamma_X(x_2)) = \gamma_Y(g(x_2)) = \gamma_Y(y_2)$, hence $\Gamma_Y$ is an FP-equivalent fuzzy soft normal subgroup over $K_1$.

**Definition 3.19** Let $\Gamma_X = \{ (\mu_X(x)/x, \gamma_X(x)) \mid x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1] \}$ be an FPFS-normal subgroup over group $G_1$. Then $\Gamma_X$ is said to be FP-increasing fuzzy soft normal subgroup over $G_1$ if, for any $x, y \in G_2$, $\mu_X(x) \leq \mu_X(y)$, we have $\gamma_X(x) \leq \gamma_X(y)$, and $\Gamma_X$ is said to be FP-decreasing fuzzy soft normal subgroup over $G_1$ if, for any $x, y \in G_2$, $\mu_X(x) \leq \mu_X(y)$, we have $\gamma_X(x) \geq \gamma_X(y)$.
Example 3.20 Let $G_1 = \{e, (13)\}$, $G_2 = \{e, (123), (132)\}$ and $X$ be a fuzzy set over $G_2$ defined by $X = \{0.6/e, 0.3/(123), 0.3/(132)\}$, $\gamma_X$ be defined by $\gamma_X(e) = \{0.7/e, 0.5/(13)\}$, $\gamma_X(123) = \{0.2/e, 0.1/(13)\}$, $\gamma_X(132) = \{0.2/e, 0.1/(13)\}$. It is clear that $\Gamma_X$ is an $FP$-increasing fuzzy soft normal subgroup over $G_1$.

Theorem 3.21 Let $\Gamma_X = \{ (\mu_X(x)/x, \gamma_X(x)) | x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0, 1] \}$ be an $FP$-increasing (decreasing) fuzzy soft normal subgroup over group $G_1$ and $\Gamma_Y = \{ (\mu_Y(y)/y, \gamma_Y(y)) | y \in K_2, \gamma_Y(y) \in F(K_1), \mu_Y(y) \in [0, 1] \}$ be an $FPFS$-set over group $K_1$. If $\Gamma_X$ is $FP$-fuzzy soft homomorphic to $\Gamma_Y$, then $\Gamma_Y$ is an $FP$-increasing (decreasing) fuzzy soft normal subgroup over $K_1$.

Proof. Let $(f, g)$ be an $FP$-fuzzy soft homomorphism from $\Gamma_X$ to $\Gamma_Y$. Since $\Gamma_X$ is an $FP$-increasing fuzzy soft normal subgroup over group $G_1$, we have $\gamma_X(x_1) \subseteq \gamma_X(x_2)$, if $\mu_X(x_1) \leq \mu_X(x_2)$ for any $x_1, x_2 \in G_2$. By Theorem 3.15, we have $\Gamma_Y$ is an $FPFS$-normal subgroup over $K_1$. Now, for all $y_1, y_2 \in K_2$, then there exist $x_1, x_2 \in G_2$ such that $g(x_1) = y_1$, $g(x_2) = y_2$. Since $\mu_Y(y_1) = \mu_Y(g(x_1)) = \mu_X(x_1)$, $\mu_Y(y_2) = \mu_Y(g(x_2)) = \mu_X(x_2)$, then $\mu_Y(y_1) \leq \mu_Y(y_2)$. And we have $\gamma_Y(y_1) = \gamma_Y(g(x_1)) = f(\gamma_X(x_1)) \subseteq f(\gamma_X(x_2)) = \gamma_Y(g(x_2)) = \gamma_Y(y_2)$, hence $\Gamma_Y$ is an $FP$-increasing fuzzy soft normal subgroup over $K_1$.

4. Aggregate fuzzy normal subgroups

An aggregate fuzzy set of an $FPFS$-set has been defined by Çağman et al. (see [3]). They also defined $FPFS$-aggregation operator that produced an aggregate fuzzy set from an $FPFS$-set and its fuzzy parameter set.

Definition 4.1 Let $\Gamma_X \in FPFS(G_1)$. Then $FPFS$-aggregation operator, denoted by $FPFS_{agg}$, is defined by

$$FPFS_{agg} : F(G_2) \times FPFS(G_1) \rightarrow F(G_1),$$

$$FPFS_{agg}(X, \Gamma_X) = \Gamma_X^*,$$

where

$$\Gamma_X^* = \{ \mu_{\Gamma_X}^*(u)/u | u \in G_1 \},$$

which is a fuzzy set over $G_1$. The value $\Gamma_X^*$ is called an aggregate fuzzy set of the $\Gamma_X$. Here the membership degree $\mu_{\Gamma_X}^*(u)$ of $u$ is defined as follows

$$\mu_{\Gamma_X}^*(u) = \frac{1}{|G_2|} \sum_{x \in G_2} \mu_X(x) \mu_{\gamma_X(x)}(u),$$

where $|G_2|$ is the cardinality of $G_2$. 
Theorem 4.2 Let $\Gamma_X = \{(\mu_X(x)/x, \gamma_X(x))|x \in G_2, \gamma_X(x) \in F(G_1), \mu_X(x) \in [0,1]\}$ be an FPFS-normal subgroup over group $G_1$. Then the aggregate fuzzy set $\Gamma_X^*$ of $\Gamma_X$ is a fuzzy normal subgroup of $G_1$.

Proof. Since $\Gamma_X$ is an FPFS-normal subgroup over group $G_1$, so $\gamma_X(x)$ is a fuzzy normal subgroup of $G_1$ for all $x \in G_2$. Equivalently, $\mu_{\gamma_X(x)}(sts^{-1}) \geq \mu_{\gamma_X(x)}(t)$ for all $s, t \in G_1$. Then

$$\mu_{\Gamma_X^*}(u) = \frac{1}{|G_2|} \sum_{x \in G_2} \mu_X(x) \mu_{\gamma_X(x)}(sts^{-1})$$

$$\geq \frac{1}{|G_2|} \sum_{x \in G_2} \mu_X(x) \mu_{\gamma_X(x)}(t) = \mu_{\Gamma_X^*}(t).$$

Then $\Gamma_X^*$ is a fuzzy normal subgroup of $G_1$.

Remark 4.3 (1) The above $\Gamma_X^*$ is called an aggregate fuzzy normal subgroup of FPFS-normal subgroup $\Gamma_X$; (2) $\Gamma_X^*$ is a fuzzy normal subgroup of $G_1$, but $\Gamma_X$ is not always FPFS-normal subgroup of $G_1$.

Example 4.4

(1) Let $G_1 = M_n$ be a matrix group, $A, B$ be a lower triangular matrix and a diagonal matrix. And let $G_2 = \{e, (12)\}$, the parameters $e, (12)$ stand for “lower triangular” and “diagonal”, respectively. And $X$ is a fuzzy set over $G_2$ defined by

$$\mu_X(x) = \begin{cases} 0.5, & x = e, \\ 0.3, & x = (12). \end{cases}$$

Let $\gamma_X$ be defined by

$$\mu_{\gamma_X(e)}(r) = \begin{cases} 0, & r \text{ is not a lower triangular matrix,} \\ 1, & r \text{ is a lower triangular matrix.} \end{cases}$$

$$\mu_{\gamma_X(12)}(r) = \begin{cases} 0, & r \text{ is not a diagonal matrix,} \\ 1, & r \text{ is a diagonal matrix.} \end{cases}$$

It is clear that $\Gamma_X$ is an FPFS-normal subgroup over $G_1$. The aggregate fuzzy set can be found as

$$\mu_{\Gamma_X^*}(u) = \begin{cases} 0.2, & u \in B, \\ 0.1, & u \in A \setminus B, \\ 0, & \text{otherwise.} \end{cases}$$

We can verify that $\Gamma_X^*$ is a fuzzy normal subgroup of $G_1$. Then $\Gamma_X^*$ is called an aggregate fuzzy normal subgroup of FPFS-normal subgroup $\Gamma_X$. 

(2) Let \( G_1 = \{1, -1, i, -i\} \) be a group, where \( i^2 = -1 \), \( G_2 = \{e, (12)\} \), \( X \) be a fuzzy set over \( G_2 \) defined by \( X = \{0.3/e, 0.4/(12)\} \), \( \gamma_X \) be defined by \( \gamma_X(e) = \{0.8/1, 0.8/ -1, 0.6/i, 0.6/ -i\} \), \( \gamma_X(12) = \{0.7/1, 0.7/ -1, 0.5/i, 0.5/ -i\} \). So \( \Gamma_X^* = \{0.26/1, 0.26/ -1, 0.19/i, 0.19/ -i\} \). It is clear that \( \Gamma_X^* \) is a fuzzy normal subgroup of \( G_1 \), but \( \Gamma_X \) is not an FPFS-normal subgroup of \( G_1 \) because of \( X \) is not a fuzzy normal subgroup of \( G_2 \).

**Proposition 4.5** \( \Gamma_X \) is an FPFS-normal subgroup of \( G_1 \) if and only if \( \Gamma_X^* \) is a fuzzy normal subgroup of \( G_1 \) and \( X \) is a fuzzy normal subgroup of \( G_2 \).

**Proof.** Let \( \Gamma_X \) be an FPFS-normal subgroup of \( G_1 \), it is obvious that \( \Gamma_X \) is a fuzzy normal subgroup of \( G_1 \) and \( X \) is a fuzzy normal subgroup of \( G_2 \).

Conversely, let \( \Gamma_X^* \) be a fuzzy normal subgroup of \( G_1 \) and \( X \) be a fuzzy normal subgroup of \( G_2 \). Assume that \( \Gamma_X \) is not an FPFS-normal subgroup of \( G_1 \), so \( \gamma_X(x) \) is not a fuzzy normal subgroup of \( G_1 \) for all \( x \in G_2 \) or \( X \) is not a fuzzy normal subgroup of \( G_2 \), then \( \Gamma_X^* \) is not a fuzzy normal subgroup of \( G_1 \) or \( X \) is not a fuzzy normal subgroup of \( G_2 \). Which is a contradiction. So \( \Gamma_X \) is an FPFS-normal subgroup of \( G_1 \). The proof is complete.

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**References**


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