SEPARATION AXIOMS BETWEEN $T_0$ AND $T_1$ ON LATTICES AND LATTICE MODULES

Gulsen Ulucak
Department of Mathematics
Gebze Technical University
P.K 141 41400, Gebze-Kocaeli
Turkey
e-mail: gulsenulucak@gtu.edu.tr

Unsal Tekir
Department of Mathematics
Marmara University
Ziverbey, 34722, Goztepe, Istanbul
Turkey
e-mail: utekir@marmara.edu.tr

Kursat Hakan Oral
Department of Mathematics
Yildiz Technical University
34720, Istanbul
Turkey
e-mail: khoral@yildiz.edu.tr

Abstract. In this study, we characterize the lattices and lattice modules whose prime spectrum satisfy some of the separation axioms between $T_0$ and $T_1$. This characterizations are the notion of $pm$–lattice, $m$–lattice, $\varepsilon$–lattice, etc. Similarly, this characterizations are the notion of $PM$–lattice module, $\mu$–lattice module, $\varepsilon$–lattice module, etc.

Keywords: separation axioms, prime element, prime spectrum.

1. Introduction

A multiplicative lattice $L$ is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins and has compact greatest element $1_L$ (least element $0_L$) as a multiplicative identity (zero). Multiplicative lattices have been studied extensively by E.W. Johnson, C.Jayaram, the current authors, and others, see, for example, [8]-[17]. An element $a$ of a multiplicative lattice $L$ is called compact if $a \leq \bigvee b_\alpha$ implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee ... \vee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. Throughout this paper, $L$ denotes multiplicative
lattice and \( L_* \) denotes the set of all compact elements of \( L \). An element \( a \in L \) is said to be proper if \( a < 1_L \). An element \( p < 1_L \) in \( L \) is said to be prime if \( ab \leq p \) implies \( a \leq p \) or \( b \leq p \). An element \( m < 1_L \) in \( L \) is said to be maximal if \( m < x \leq 1_L \) implies \( x = 1_L \). By a minimal prime element over an element \( n \) of \( L \) (or a prime element minimal over \( n \)), we mean a prime element which is minimal in the collection of all prime elements \( p \) with \( n \leq p \). Minimal prime elements over \( 0_L \) are simply called the minimal prime elements. We define the dimension of \( L \), denoted with \( \dim(L) \), to be the maximal positive integer \( k \), if such exists, such that there exists a chain of prime different elements of \( L \) as \( p_0 < p_1 < \ldots < p_k \).

By a \( C \)-lattice we mean a (not necessarily modular) complete multiplicative lattice, with least element \( 0_L \) and compact greatest element \( 1_L \) (a multiplicative identity), which is generated under joins by multiplicatively closed subset \( C \) of compact elements. We note that in \( C \)-lattice, a finite product of compact elements is again compact. Since \( 1_L \) is compact element, then maximal elements exist in \( L \). Further, if \( L \) is a \( C \)-lattice, then there is a minimal prime element of \( L \) (See, [15]). If \( \dim(L) = 0 \), then \( L \) is said to be zero-dimensional \( C \)-lattice. Note that \( L \) is a zero-dimensional lattice if and only if every prime element of \( L \) is maximal.

We denote by \( \sigma(L) \) the set of prime elements of \( L \). We put \( V(a) = \{ p \in \sigma(L) \mid a \leq p \} \) for any \( a \in L \). Then we introduce a topology on \( \sigma(L) \) such that \( \{ V(a) \mid a \in L \} \) is the family of all closed sets. With the above definition the following axioms are hold:

1. \( V(0_L) = \sigma(L) \) and \( V(1_L) = \emptyset \).
2. \( \bigcap_{\alpha} V(a_\alpha) = V(\bigvee a_\alpha) \) for any index set \( \Delta \).
3. \( V(a) \cup V(b) = V(a \land b) = V(ab) \) (See, [16]).

We define \( \zeta(L) = \{ V(a) \mid a \in L \} \) as it is family of all closed sets that satisfies (i), (ii) and (iii) conditions for \( L \). Then there is a topology on \( \sigma(L) \), say \( \tau \), and \( \tau \) is called the Zariski topology. Any open set in \( \sigma(L) \) is denoted \( \sigma(L) - V(a) \) such that \( a \in L \) since \( V(a) \) is closed set in \( \sigma(L) \). Let \( D_a = \sigma(L) - V(a) \) for \( a \in L \) (See, [7]).

Let \( M \) be a complete lattice. Recall that \( M \) is a lattice module over the multiplicative lattice \( L \), or simply an \( L \)-module in case there is a multiplication between elements of \( L \) and \( M \), denoted by \( lB \) for \( l \in L \) and \( B \in M \), which satisfies the following properties:

1. \( (lb)B = l(bB) \);
2. \( \left( \bigvee_{\alpha} l_\alpha \right) \left( \bigvee_{\beta} B_\beta \right) = \bigvee_{\alpha,\beta} l_\alpha B_\beta \);
3. \( 1_L B = B \);
4. \( 0_L B = 0_M \).
for all \( l, l_\alpha, b \) in \( L \) and for all \( B, B_\beta \) in \( M \).

Let \( M \) be an L-module. The greatest element of \( M \) will be denoted by \( 1_M \). An element \( N \in M \) is said to be proper if \( N < 1_M \). An element \( N < 1_M \) in \( M \) is said to be prime if \( aX \leq N \) implies \( X \leq N \) or \( a1_M \leq N \) for \( a \in L \), \( X \in M \). If \( N \) is a prime element of \( M \), then \( (N : 1_M) \) is a prime element of \( L \) (See [10]). An element \( N < 1_M \) in \( M \) is said to be maximal if \( N < X \leq 1_M \) implies \( X = 1_M \). If \( 1_M \) is a compact element of \( M \), then \( M \) has a maximal element by (See, [6]). If \( N \) and \( K \) belong to \( M \), \( (N : K) \) is the join of all \( a \in L \) such that \( aK \leq N \). By a minimal prime element over an element \( N \) of \( M \) (or a prime element minimal over \( N \)), we mean a prime element which is minimal in the collection of all prime elements \( P \) with \( N \leq P \). Minimal prime elements over \( 0_M \) are simply called the minimal prime elements. We define the dimension of \( M \), denoted with \( \text{dim}(M) \), to be the maximal positive integer \( k \), if such exists, such that there exists a chain of prime distinguished elements of \( L \) as \( P_0 < P_1 < \cdots < P_k \).

Let \( \sigma(M) \) be the set of all prime elements of \( M \). Next we define \( V(N) = \{P \in \sigma(M) \mid (N : 1_M) \leq (P : 1_M)\} \). Then the following axioms are hold:

(i) \( V(0_M) = \sigma(M) \) and \( V(1_M) = \emptyset \).

(ii) \( \bigcap_{\alpha \in \Delta} V(N_\alpha) = V\left(\bigvee_{\alpha \in \Delta} (N_\alpha : 1_M)1_M\right) \) for any index set \( \Delta \).

(iii) \( V(N) \cup V(K) = V(N \wedge K) \) (See, [7]).

We define \( \zeta(M) = \{V(N) \mid N \text{ is an element of } M\} \) as it is family of all closed sets that satisfies (i), (ii) and (iii) conditions for \( M \). Then there is a topology on \( \sigma(M) \), say \( \tau \), and \( \tau \) is called the Zariski topology. Any open set in \( \sigma(M) \) is denoted by \( \sigma(L) - V(N) \) such that \( N \in M \) since \( V(N) \) is closed set in \( \sigma(M) \). We define \( X_\alpha = \sigma(M) - V(a1_M) \) for each \( a \in L \) and it is clear that every \( X_\alpha \) is an open set in \( \sigma(M) \) (See, [7]).

For a topological space \((X, \tau)\) and \( x \in X \), define the kernel of \( x \), \( \overline{\{x\}} \), as the set \( \{y \in X \mid x \in \{y\}\} \). The set \( \overline{\{x\}} - \{x\} \) is called shell of \( x \) and it will be denoted in this work by \( \{x\} \). The closure of a subset \( A \) of a topological space \((X, \tau)\) is the intersection of the members of the family of all closed sets containing \( A \). The set \( A \) is called a point closure if \( A = \{p\} \) for some \( p \in X \). A point \( x \) is an accumulation point of a subset \( A \) of a topological space \((X, \tau)\) if every neighborhood of \( x \) contains points of \( A \) other than \( x \). Then it is true that each neighborhood of a point \( x \) intersects \( A \) if and only if \( x \) is either a point of \( A \) or an accumulation point of \( A \). The set of all accumulation points of a set \( A \) is called the derived set of \( A \) and it will be denoted by \( A' \). The closure of any set is the union of the set and the derived set of it (For more detail, see [5]).

With the Zariski topology on \( \sigma(L) \), since \( \{p\} = V(p) \) for \( p \in \sigma(L) \), it is clear that \( \{p\} - \{p\} = \{q \in \sigma(L) \mid p \in V(q) \text{ and } p \neq q\} \) and \( \{p\}' = V(p) - \{p\} = \{q \in \sigma(L) \mid p < q\} \). For this topology, \( \{p\} \) is a closed set if and only if \( p \) is a maximal element of \( L \) (See, [7]).
With the Zariski topology on \( \sigma(M) \), the above definitions and \( \overline{\{P\}} = V(P) = \{Q \in \sigma(M) : (P : 1_M) \leq (Q : 1_M)\} \) for \( P \in \sigma(M) \), it is clear that \( \overline{\{P\}} = \{Q \in \sigma(M) \mid P \in V(Q) \text{ and } P \neq Q\} \) and \( \overline{\{P\}'} = V(P) \setminus \{P\} = \{Q \in \sigma(M) \mid (P : 1_M) \leq (Q : 1_M) \text{ and } P \neq Q\} \) (see [7], Proposition 25).

An \( L \)-module \( M \) is called a multiplication lattice module if for every element \( N \in M \) there exists an element \( a \in L \) such that \( N = a1_M \). In this case, \( N = (N : 1_M)1_M \) (see [10] for more detail). If the set \( \{P\} \) is a closed in \( \sigma(M) \) then \( P \) is a maximal element of \( M \). The converse is also true, when \( M \) is a multiplication lattice \( L \)-module (See [7]).

Throughout this paper, we assume that \( 1_L \) and \( 1_M \) are compact element in \( L \) and \( M \), respectively.

2. Separation axioms between \( T_0 \) and \( T_1 \) on \( \sigma(L) \)

**Definition 1** A topological space \((X, \tau)\) is:

1. \( T(\beta) \) if for any \( x \in X \), \( \{x\}' \) is empty or singleton.
2. \( T(\beta') \) if for any \( x \in X \), \( \overline{\{x\}} \) is empty or singleton.
3. \( T(\varepsilon) \) if for any \( x, y \in X \), \( x \neq y \), \( \{x\}' \cap \{y\}' \) is empty or singleton.
4. \( T_{UD} \) if for any \( x \in X \), \( \{x\}' \) is the union of disjoint closed sets.
5. \( T_D \) if for any \( x \in X \), \( \{x\}' \) is a closed set.
6. \( T_{DD} \) if it is \( T_D \) and for any \( x, y \in X \), \( x \neq y \), \( \{x\}' \cap \{y\}' = \emptyset \).
7. \( T_Y \) if for any \( x, y \in X \), \( x \neq y \), \( \overline{\{x\}} \cap \overline{\{y\}} \) is empty or singleton.
8. \( T_{YS} \) for any \( x, y \in X \), with \( x \neq y \), \( \overline{\{x\}} \cap \overline{\{y\}} \) is either empty or \( \{x\} \) or \( \{y\} \).

**Definition 2** \( L \) is said to be \( pm \)-lattice if every prime element is contained a unique maximal element.

**Proposition 1** \( \sigma(L) \) is a \( T(\beta) \)-space if and only if \( L \) is a \( pm \)-lattice and \( \dim(L) \leq 1 \).

**Proof.** (\( \Rightarrow \) : Let \( \sigma(L) \) be a \( T(\beta) \)-space. Assume that \( L \) is not a \( pm \)-lattice. Then there is a prime element \( p \) such that \( p < k \) and \( p < t \) with \( k \) and \( t \) two different maximal elements. So \( k, t \in V(p) \) and since \( p \neq k \) and \( p \neq t \), then \( k, t \in \{p\}' \), a contradiction.

Assume that \( \dim(L) \geq 2 \). Then there exist prime elements \( p, q, k \) such that \( p < q < k \). So \( q, k \in \{p\}' \), a contradiction.

(\( \Leftarrow \) : Let \( L \) be a \( pm \)-lattice and \( \dim(L) \leq 1 \). Suppose that \( \sigma(L) \) is not a \( T(\beta) \)-space. Then \( \{p\}' \neq \emptyset \) and \( \{p\}' \) is not singleton for some \( p \in \sigma(L) \). Thus there are two different prime elements \( k, t \) with \( k, t \in \{p\}' \). So \( p < k \) and \( p < t \). Since \( \dim(L) \leq 1 \), then \( k \) and \( t \) are maximal elements, a contradiction. \( \blacksquare \)
Definition 3 A lattice $L$ is said to be $m$–lattice if every prime element contains a unique minimal prime element of $L$.

Proposition 2 Let $L$ be a $C$–lattice. Then $\sigma(L)$ is a $T(\beta')$–space if and only if $L$ is an $m$–lattice and $\dim(L) \leq 1$.

Proof. ($\Rightarrow$) : Let $\sigma(L)$ be a $T(\beta')$–space. Assume that $L$ is not an $m$–lattice. Then there is a prime element $p$ such that $u < p$ and $v < p$ with $u$ and $v$ two different minimal prime elements. Hence $p \in V(u)$ and $p \in V(v)$, that is, $p \in \{u\}$ and $p \in \{v\}$. Since $p \neq u$ and $p \neq v$, then $u, v \in \{p\} = \{p\} \setminus \{v\}$, a contradiction.

Assume that $\dim(L) \geq 2$. Then there exist prime elements $p, q, k$ such that $p < q < k$. Thus $k \in V(p)$ and $k \in V(q)$, that is, $k \in \{p\}$ and $k \in \{q\}$. Therefore, $p, q \in \{k\}$, a contradiction.

($\Leftarrow$) : Let $L$ be a $m$–lattice and $\dim(L) \leq 1$. Suppose that $\sigma(L)$ is not a $T(\beta')$–space. Then $\{p\} \neq \emptyset$ and $\{p\}$ is not singleton for some $p \in \sigma(L)$. Thus there are two different prime elements $k, t$ with $k, t \in \{p\}$. So $k < p$ and $t < p$. Since $\dim(L) \leq 1$, then $k$ and $t$ are minimal prime elements, a contradiction. 

Definition 4 Let $m, n$ be maximal elements of $L$ and $k, t$ be minimal prime elements of $L$. Then $\rho_<(m, n) = \{p \in \sigma(L) \mid p < m \land n\}$ and $\rho_>(k, t) = \{p \in \sigma(L) \mid k < p \land t < p\}$

Definition 5 A lattice $L$ is called $\varepsilon$–lattice if $\rho_<(m, n)$ and $\rho_>(k, t)$ are empty or singleton for any different maximal elements $m, n$ of $L$ and different minimal prime elements $k, t$ of $L$.

Proposition 3 Let $L$ be a $C$–lattice. Then $\sigma(L)$ is a $T(\varepsilon)$–space if and only if $L$ is an $\varepsilon$–lattice and $\dim(L) \leq 2$.

Proof. ($\Rightarrow$) : Let $\sigma(L)$ be a $T(\varepsilon)$–space. If $L$ is not an $\varepsilon$–lattice, then $\rho_<(m, n)$ or $\rho_>(k, t)$ is neither empty nor singleton for any different maximal elements $m, n$ of $L$ and different minimal prime elements $k, t$ of $L$. Assume that $\rho_<(m, n)$ is neither empty nor singleton. Then there are different prime elements $p, q$ such that $p, q \in \rho_<(m, n)$. Thus $p < m \land n$ and $q < m \land n$, that is, $p < m$ and $p < n$, $q < m$ and $q < n$. Then $m, n \in \{p\}'$ and $m, n \in \{q\}'$. Hence $m, n \in \{p\}' \land \{q\}'$, a contradiction. Suppose that $\rho_>(k, t)$ is neither empty nor singleton. Then there exist different prime elements $u, v$ such that $u, v \in \rho_>(k, t)$. Hence $k < u, t < u$ and $k < v, t < v$. Therefore, $u, v \in \{k\}'$ and $u, v \in \{t\}'$, a contradiction. If $\dim(L) \geq 3$, then there are $p, q, k, t \in \sigma(L)$ with $p < q < k < t$. So $k, t \in \{p\}' \land \{q\}'$, which is a contradiction.

($\Leftarrow$) : Let $L$ be a $\varepsilon$–lattice and $\dim(L) \leq 2$. Assume that $\sigma(L)$ is not a $T(\varepsilon)$–space. Then $\{p\}' \land \{q\}'$ is neither empty nor singleton for different prime elements $p, q$ of $L$. Thus there are different prime elements $k, t$ of $L$ such that $k, t \in \{p\}' \land \{q\}'$. Thus we get $p < k, p < t$ and $q < k, q < t$. If $p, q$ are minimal prime elements of $L$, then $k, t \in \rho_>(p, q)$, which is a contradiction. If $p, q$ are not
minimal prime elements of $L$, then there exist minimal prime elements $u, v$ of $L$ such that $u < p$ and $v < q$. Then $u < p < k$, $u < p < t$ and $v < p < k$, $v < q < t$. Hence $k, t$ are maximal elements of $L$ since $\dim(L) \leq 2$. It is clear that $p < k \wedge t$ and $q < k \wedge t$. Indeed, if $p = k \wedge t$ or $q = k \wedge t$, then $k \leq p$ or $t \leq p$ or $k \leq q$ or $t \leq q$ as $p, q$ are prime elements of $L$. This is contradict with $k, t$ where are maximal elements of $L$. So we get $p < k \wedge t$ and $q < k \wedge t$. Hence $p, q \in \rho_{\subseteq}(k, t)$, a contradiction.

**Definition 6** Let $p \in \sigma(L)$ be not maximal. Then $L$ is called $u$–lattice if $a \lor b = 1_L$ for every elements $a, b$ in $\{p\}'$.

**Proposition 4** Let $\dim(L)$ be finite and $L$ be a $u$–lattice. Then $\sigma(L)$ is a $T_{UD}$–space.

**Proof.** Let $p \in \sigma(L)$ and $\beta$ be the set of minimal prime elements of $\{p\}'$. Assume that $x \in \{p\}'$. Then $p < x$. We have that there is a $q \in \beta$ such that $p < q \leq x$. So $x \in V(q)$. Thus $x \in \bigcup_{q \in \beta} V(q)$. Conversely, let $x \in \bigcup_{q \in \beta} V(q)$. Then $x \in V(q)$ for some $q \in \beta$. So $q \leq x$ and since $p < q$, then $p < x$. Hence $x \in \{p\}'$. Therefore $\{p\}' = \bigcup_{q \in \beta} V(q)$. Also $V(q_1) \cap V(q_2) = \emptyset$ for minimal prime elements $q_1, q_2$ with $p < q_1$ and $p < q_2$. Indeed, if $V(q_1) \cap V(q_2) \neq \emptyset$, then there is a $t \in \sigma(L)$ such that $t \in V(q_1) \cap V(q_2)$. Thus $q_1 \leq t$ and $q_2 \leq t$. Therefore, $q_1 \lor q_2 \leq t$. Hence $t = 1_L$, a contradiction.

**Definition 7** A lattice $L$ is called $d$–lattice if every prime non maximal element is different to the meet of the prime elements properly containing it.

**Proposition 5** $\sigma(L)$ is a $T_D$–space if and only if $L$ is a $d$–lattice.

**Proof.** ($\Rightarrow$) : Let $\sigma(L)$ be a $T_D$–space. If $L$ is not a $d$–lattice, then there exists a prime non maximal element $p$ such that $p$ is the meet of the prime elements properly containing it, that is, $p = \bigwedge_{p < q} q$. Since $\sigma(L)$ is a $T_D$–space, then $\{p\}'$ is closed set. Then $\{p\}' = V(a)$ for $a \in L$. Since $a \leq q$ for every $q \in \{p\}'$, $a \leq \bigwedge_{q \in \{p\}'} q = p$. Thus $p \in V(a)$, a contradiction.

($\Leftarrow$) : Let $L$ be a $d$–lattice. If $p \in \sigma(L)$ is a maximal, then it is clear. Let $p \in \sigma(L)$ be a non maximal element. Since $L$ is a $d$–lattice, then $p \neq \bigwedge_{p < q} q$. But $V(\bigwedge_{p < q} q) \subseteq V(p)$ since $p \leq \bigwedge_{p < q} q$. Since $p \notin V(\bigwedge_{p < q} q)$, then $V(\bigwedge_{p < q} q) \subseteq \{p\}'$. Now, let $k \in \{p\}'$. Then $p < k$ and so $\bigwedge_{p < q} q \leq k$, that is, $k \in V(\bigwedge_{p < q} q)$.

**Definition 8** A lattice $L$ is called $y$–lattice if any two different minimal prime element are contained in at most one maximal element.
Proposition 6 Let $L$ be a $C$–lattice. $\sigma(L)$ is a $T_Y$–space if and only if $L$ is a $y$–lattice and $\dim(L) \leq 1$.

Proof. $(\Rightarrow)$ Let $\sigma(L)$ be a $T_Y$–space. Assume that $L$ is not a $y$–lattice. Let $p,q$ be minimal prime elements of $L$. Then there are two different maximal elements $m,n$ of $L$ containing $p,q$, that is, $p < m$, $p < n$ and $q < m$, $q < n$. Then $m,n \in V(p) = \{p\}$ and $m,n \in V(q) = \{q\}$. Thus $m,n \in \{p\} \cap \{q\}$, a contradiction. Suppose that $\dim(L) \geq 2$. Then there exist prime elements $p,q,k$ such that $p < q < k$. So $q,k \in V(p) \cap V(q)$, that is, $q,k \in \{p\} \cap \{q\}$, a contradiction.

$(\Leftarrow)$ Let $L$ be a $y$–lattice and $\dim(L) \leq 1$. Assume that $\sigma(L)$ is not a $T_Y$–space. Then $\{p\} \cap \{q\}$ is neither empty nor singleton for two different minimal prime elements $p,q$. Thus we get $k,t \in \{p\} \cap \{q\}$. Then $p \leq k$, $p \leq t$ and $q \leq k$, $q \leq t$. Also $p \neq k$ and $p \neq t$. Indeed, if $p = k$ or $p = t$, then $p \in V(q) = \{q\}$. Thus $q < p$. We get that $p$ is maximal since $\dim(L) \leq 1$. Thus $V(p) = \{p\}$ and so $\{p\} \cap \{q\} = \{p\}$, a contradiction. In a similar way, $q \neq k$ and $q \neq t$. Hence $p < k$, $p < t$ and $q < k$, $q < t$. Since $\dim(L) \leq 1$, then $k$ and $t$ are maximal elements, a contradiction.

Proposition 7 Let $L$ be a $C$–lattice. $\sigma(L)$ is a $T_{YS}$–space if and only if $L$ is an $m$–lattice and $\dim(L) \leq 1$.

Proof. $(\Rightarrow)$ Let $\sigma(L)$ be a $T_{YS}$–space. Suppose that $L$ is not an $m$–lattice. Let $p \in \sigma(L)$. Then there are two different minimal prime elements $u,v$ such that $u < p$ and $v < p$. Thus $p \in V(u) = \{u\}$ and $p \in V(v) = \{v\}$ and so $p \in \{u\} \cap \{v\}$, a contradiction. Assume that $\dim(L) \geq 2$. Then there exist prime elements $p,q,k$ such that $p < q < k$. Thus $k \in V(p) = \{p\}$ and $k \in V(q) = \{q\}$. Therefore, $k \in \{p\} \cap \{q\}$, a contradiction.

$(\Leftarrow)$ Let $L$ be an $m$–lattice and $\dim(L) \leq 1$. Let $p,q \in \sigma(L)$ with $p \neq q$. Assume that $\{p\} \cap \{q\} \neq \emptyset$ and $\{p\} \cap \{q\} \neq \{p\}$. Let $k \in \{p\} \cap \{q\}$. Then $k \neq p$ and $p < k$, $q \leq k$. Since $\dim(L) \leq 1$, then $p$ is minimal prime element. Then $q = k$ because $L$ is an $m$–lattice. Hence $\{p\} \cap \{q\} = \{q\}$.

3. Separation axioms between $T_0$ and $T_1$ on $\sigma(M)$

In this section, we will investigate the like of our studying on lattices for lattice modules.

Definition 9 An $L$–module $M$ is said to be $PM$–lattice module if every prime element is contained a unique maximal element.

Proposition 8 Let $M$ be an $L$–module. If $\sigma(M)$ is a $T(\beta)$–space, then the followings are hold:

a) $M$ is a $PM$–lattice module.

b) $\dim(M) \leq 1$. 
Proof. Let \( \sigma(M) \) be a \( T(\beta) \)-space.

a) Assume that \( M \) is not a \( PM \)-lattice module. Then there exists a prime element \( P \) such that \( P < N \) and \( P < K \) with \( N \) and \( K \) two different maximal elements. Since \( (P : 1_M) \leq (N : 1_M) \), \( (P : 1_M) \leq (K : 1_M) \) and \( P \neq N, P \neq K \), then \( N, K \in \{P\}' \), a contradiction.

b) If \( \dim(M) \geq 2 \), then there are prime elements \( P, N, K \) such that \( P < N < K \). So \( (P : 1_M) \leq (N : 1_M) \leq (K : 1_M) \). Therefore \( N, K \in \{P\}' \), which is a false.

Definition 10 An \( L \)-module \( M \) is called \( \mu \)-lattice module if every prime element contains a unique minimal prime element.

Proposition 9 Let \( M \) be an \( L \)-module. Let \( \sigma(M) \) be a \( T(\beta') \)-space. Then the following statements are hold:

a) \( M \) is a \( \mu \)-lattice module.

b) \( \dim(M) \leq 1 \).

Proof. Let \( \sigma(M) \) be a \( T(\beta') \)-space.

a) If \( M \) is not a \( \mu \)-lattice module, then there is a prime element \( P \) such that \( K < P \) and \( T < P \) with \( K \) and \( T \) two different minimal prime elements. Then \( (K : 1_M) \leq (P : 1_M) \) and \( (T : 1_M) \leq (P : 1_M) \). Hence, \( P \in V(K) \) and \( P \in V(T) \). Then \( K, T \in \{P\} = \{\overline{P}\}\{P\} \), which is a contradiction.

b) If \( \dim(M) \geq 2 \), then there are prime elements \( P, N, K \) such that \( P < N < K \). Hence \( (P : 1_M) \leq (N : 1_M) \leq (K : 1_M) \). Thus \( P, N \in \{K\} = \{\overline{K}\}\{K\} \), a contradiction.

Definition 11 Let \( M \) be an \( L \)-module. Let \( N, K \) be maximal elements of \( M \) and \( U, V \) be minimal prime elements of \( M \). Then

\[
\rho(N, K) = \{P \in \sigma(M) \mid (P : 1_M) \leq (N \land K : 1_M) \text{ and } P \neq N \land K\}
\]

\[
\delta(U, V) = \{P \in \sigma(M) \mid P \in \{U\}' \text{ and } P \in \{V\}'\}.
\]

Definition 12 An \( L \)-module \( M \) is a \( \varepsilon \)-lattice module if the set \( \rho(N, K) \) and \( \delta(U, V) \) are empty or singleton for any different maximal elements \( N, K \) and different minimal prime elements \( U, V \).

Let \( M \) be an \( L \)-module. We call \( M \) is a multiplication lattice \( L \)-module if for every element \( N \) of \( M \) there exists an element \( a \) of \( L \) such that \( N = a1_M \). In this case, we know that \( N = (N : 1_M)1_M \). (For more detail, see (10)).

Proposition 10 Let \( M \) be a multiplication lattice \( L \)-module. Then \( M \) is a \( \varepsilon \)-module and \( \dim(M) \leq 2 \) if and only if \( \sigma(M) \) is a \( T(\varepsilon) \)-space.
Proof. Let $M$ be an $\varepsilon$–module and $\dim(M) \leq 2$. Suppose that $\sigma(M)$ is not a $T(\varepsilon)$–space. Then $\{P\}' \cap \{Q\}'$ is neither empty nor singleton for different prime elements $P, Q$ of $M$. Thus there are different prime elements $N, K$ of $M$ such that $N, K \in \{P\}' \cap \{Q\}'$. Thus we get $(P : 1_M) \leq (N : 1_M), (P : 1_M) \leq (K : 1_M), P \neq N, K$ and $(Q : 1_M) \leq (N : 1_M), (Q : 1_M) \leq (K : 1_M), Q \neq N, K$.

Since $M$ is a multiplication lattice $L$–module, we obtain $P < N, P < K$ and $Q < N, Q < K$. If $P, Q$ are minimal prime elements, then $N, K \in \delta(P, Q)$, which is a contradiction. If $P, Q$ are not minimal prime elements, then there are different minimal prime elements $U, V$ such that $U < P$ and $V < Q$. Then we have $U < P < N, U < P < K$ and $V < Q < N, V < Q < K$. Thus $N, K$ are maximal elements since $\dim(M) \leq 2$. It is clear that $P < N \land K$ and $Q < N \land K$.

Indeed, if $P = N \land K$ or $Q = N \land K$, $N \leq P$ or $K \leq P$ or $N \leq Q$ or $K \leq Q$ as $P, Q$ are prime elements of $M$. This is contradict with $N, K$ where are maximal elements of $M$. So we get $P < N \land K$ and $Q < N \land K$. Hence $P, Q \in \rho(N, K)$, a contradiction.

Conversely, let $\sigma(M)$ be a $T(\varepsilon)$–space. If $M$ is not a $\varepsilon$–lattice module, then the set $\rho(N, K)$ or $\delta(U, V)$ is neither empty nor singleton for any different maximal elements $N, K$ and different minimal prime elements $U, V$. If $\rho(N, K)$ is neither empty nor singleton, then there are $P, Q \in \sigma(M)$ such that $P, Q \in \rho(N, K)$, that is, $(P : 1_M) \leq (N \land K : 1_M), P \neq N \land K$ and $(Q : 1_M) \leq (N \land K : 1_M), Q \neq N \land K$. Since $M$ is a multiplication lattice $L$–module, then $P < N \land K \leq N, P < N \land K \leq K$ and $Q < N \land K \leq N, Q < N \land K \leq K$. Thus $N \in \{P\}'$, $K \in \{P\}'$ and $N \in \{Q\}'$, $K \in \{Q\}'$. Thus, $N, K \in \{P\}' \cap \{Q\}'$, a contradiction. If $\delta(U, V)$ is neither empty nor singleton, then there are $S, T \in \sigma(M)$ such that $S, T \in \delta(U, V)$. Thus $S, T \in \{U\}'$ and $S, T \in \{V\}'$. Consequently, $S, T \in \{U\}' \cap \{V\}'$, a contradiction. If $\dim(M) \geq 3$, then there are prime elements $P, Q, S, T$ such that $P < Q < S < T$. So $(P : 1_M) \leq (Q : 1_M) \leq (S : 1_M) \leq (T : 1_M)$. Thus, $S, T \in \{P\}' \cap \{Q\}'$, which is a contradiction.

Definition 13 An $L$–module $M$ is a $D$–lattice module if every prime non-maximal element is not equal to the meet of the prime elements properly containing it.

Proposition 11 Let $M$ be an $L$–module. If $\sigma(M)$ is a $T_D$–space, then $M$ is a $D$–lattice module. The converse is also true when $M$ is a multiplication lattice $L$–module.

Proof. Assume that $\sigma(M)$ is a $T_D$–space. Let $P \in \sigma(M)$ be a non-maximal element. Assume that $P = \bigwedge_{Q < P \in \sigma(M)} Q$. Since $\{P\}'$ is a closed set, there is an element $K$ of $M$ such that $\{P\}' = V(K)$. Then $(K : 1_M) \leq (Q : 1_M)$ for every $Q \in \{P\}' \Rightarrow (K : 1_M) \leq \bigwedge_{Q \in \{P\}'} (Q : 1_M) \Rightarrow (K : 1_M) \leq (\bigwedge_{Q \in \{P\}'} Q : 1_M) \leq (\bigwedge_{Q \in \sigma(M)} Q : 1_M) = (P : 1_M) \Rightarrow (K : 1_M) \leq (P : 1_M)$. Thus $P \in V(K) \Rightarrow P \in \{P\}'$. 


which is a contradiction. Hence we get \( P \neq \bigwedge_{Q \in \sigma(M)} Q \).

Conversely, let \( M \) be a multiplication lattice \( L \)–module. If \( P \in \sigma(M) \) is maximal element, then \( V(P) = \{P\} \). Then \( \{P\}' = \emptyset \), that is, \( \{P\}' \) is a closed set. If \( P \) is prime non-maximal element, then \( P \) is not equal to the meet of the prime elements properly containing it, that is, \( P \neq \bigwedge_{Q \in \sigma(M)} P < Q \).

\( \bigwedge_{Q \in \sigma(M)} P < Q \). 

Conversely, let \( M \) be a multiplication lattice \( L \)–module. If \( P \in \sigma(M) \) is maximal element, then \( V(P) = \{P\} \). Then \( \{P\}' = \emptyset \), that is, \( \{P\}' \) is a closed set. If \( P \) is prime non-maximal element, then \( P \) is not equal to the meet of the prime elements properly containing it, that is, \( P \neq \bigwedge_{Q \in \sigma(M)} P < Q \).

Since \( P < \bigwedge_{Q \in \sigma(M)} P < Q \), then \( V(\bigwedge_{Q \in \sigma(M)} P < Q) \subseteq V(P) \). Then \( P \in V(\bigwedge_{Q \in \sigma(M)} P < Q) \) but \( P \notin V(\bigwedge_{Q \in \sigma(M)} P < Q) \). Indeed, if \( P \in V(\bigwedge_{Q \in \sigma(M)} P < Q) \), then \( (\bigwedge_{Q \in \sigma(M)} P < Q) : 1_M \leq (P : 1_M) \) and since \( M \) is a multiplication lattice \( L \)–module, then \( \bigwedge_{Q \in \sigma(M)} P < Q \leq P \), it is a contradiction. Hence \( V(\bigwedge_{Q \in \sigma(M)} P < Q) \subseteq \{P\}' \).

Conversely, let \( K \in \{P\}' \). Then \( (P : 1_M) \leq (K : 1_M) \) and \( P \neq K \). Since \( M \) is a multiplication lattice \( L \)–module, then \( P < K \). Thus \( \bigwedge_{Q \in \sigma(M)} P < Q \leq K \) and so \( \{P\}' \subseteq V(\bigwedge_{Q \in \sigma(M)} P < Q) \). Consequently, \( \{P\}' = V(\bigwedge_{Q \in \sigma(M)} P < Q) \), that is, \( \{P\}' \) is a closed set.

**Proposition 12** Let \( \sigma(M) \) be a \( T_{DD} \)–space. Then the following statements are hold:

a) \( M \) is a \( \mu \)–lattice module.

b) \( \dim(M) \leq 1 \).

**Proof.** Let \( \sigma(M) \) be a \( T_{DD} \)–space.

a) Assume that \( M \) is not a \( \mu \)–lattice module. Then there is a prime element \( P \) of \( M \) such that \( U < P \) and \( T < P \) with \( U \) and \( T \) two different minimal prime elements. So \( (U : 1_M) \leq (P : 1_M) \) and \( (T : 1_M) \leq (P : 1_M) \). Hence \( P \in \{U\}' \cap \{T\}' \), which is a contradiction.

b) Suppose that \( \dim(M) \geq 2 \). Then there are prime elements \( P,N,K \) such that \( P < N < K \). Hence \( (P : 1_M) \leq (N : 1_M) \leq (K : 1_M) \). Thus \( K \in \{P\}' \cap \{N\}' \), a contradiction.

**Definition 14** An \( L \)–module \( M \) is a \( Y \)–lattice module if any two distinct minimal prime elements are contained in at most one maximal element.
Proposition 13 Let $M$ be an $L$–module. If $\sigma(M)$ is a $T_Y$–space, then $M$ is a $Y$–lattice module and $\dim(M) \leq 1$. The converse is also true when $M$ is a multiplication lattice $L$–module.

Proof. Let $\sigma(M)$ be a $T_Y$–space. Suppose that $M$ is not a $Y$–lattice module. Let $P, Q$ be two different minimal prime elements. Then there are two different maximal elements $N, K$ such that $P < N, P < K$ and $Q < N, Q < K$. Then $(P : 1_M) \leq (N : 1_M)$, $(P : 1_M) \leq (K : 1_M)$ and $(Q : 1_M) \leq (N : 1_M)$, $(Q : 1_M) \leq (K : 1_M)$. Thus $N, K \in V(P) = \{P\}$ and $N, K \in V(Q) = \{Q\}$. Hence $N, K \in \{P\} \cap \{Q\}$, this is a contradiction. Let $\dim(M) \geq 2$. Then there are prime elements $P, Q, T$ such that $P < Q < T$. Thus $(P : 1_M) \leq (Q : 1_M) \leq (T : 1_M)$. So $Q, T \in V(Q)$ and $Q, T \in V(P)$. Then $Q, T \in \{P\} \cap \{Q\}$, which is a false.

Conversely, let $M$ be a multiplication lattice $L$–module. Let $M$ be a $Y$–lattice module and $\dim(M) \leq 1$. Assume that $\sigma(M)$ is not a $T_Y$–space. Then $\{P\} \cap \{Q\} \neq \emptyset$ and $\{P\} \cap \{Q\}$ is not singleton for two different prime elements $P, Q$. So there are two prime elements $N, K$ such that $N, K \in \{P\} \cap \{Q\}$ and so $(P : 1_M) \leq (N : 1_M), (P : 1_M) \leq (K : 1_M)$ and $(Q : 1_M) \leq (N : 1_M), (Q : 1_M) \leq (K : 1_M)$. Since $M$ is a multiplication lattice $L$–module, $P \leq N, P \leq K$ and $Q \leq N, Q \leq K$. Also, $P \neq N$ and $P \neq K$. Indeed, if $P = N$ or $P = K$, then $P$ is maximal as $\dim(M) \leq 1$. Hence $V(P) = \{P\} \neq \{P\}$ and so $\{P\} \cap \{Q\} = \{P\}$, a contradiction. In a similar way, $Q \neq N$ and $Q \neq K$. Then we get $P < N, P < K$ and $Q < N, Q < K$, so $N$ and $K$ are maximal elements, since $\dim(M) \leq 1$. But this contradicts with $M$ which is a $Y$–lattice module.

Proposition 14 Let $M$ be an $L$–module. If $\sigma(M)$ is a $T_{YS}$–space, then $M$ is a $\mu$–lattice module and $\dim(M) \leq 1$. The converse is also true when $M$ is a multiplication lattice $L$–module.

Proof. Let $\sigma(M)$ be a $T_{YS}$–space. Suppose that $M$ is not a $\mu$–lattice module. Then there exists a prime element $P$ containing two different minimal prime elements $Q, T$. So $P \in V(Q) \cap V(T)$, that is, $P \in \{Q\} \cap \{T\}$. This is a contradiction. Suppose that $\dim(M) \geq 2$. Then there are prime elements $P, Q, T$ such that $P < Q < T$. Hence $(P : 1_M) \leq (Q : 1_M) \leq (T : 1_M)$. Thus $T \in V(P) \cap V(Q) \Rightarrow T \in \{P\} \cap \{Q\}$, which is a false.

Conversely, assume that $M$ is a multiplication lattice $L$–module. Let $M$ be a $\mu$–lattice module and $\dim(M) \leq 1$. Let $P, Q$ be different prime elements of $M$. Assume that $\{P\} \cap \{Q\} \neq \emptyset$ and $\{P\} \cap \{Q\} \neq \{P\}$. Let $K \in \{P\} \cap \{Q\}$. Then $K \neq P$, also $(P : 1_M) \leq (K : 1_M)$ and $(Q : 1_M) \leq (K : 1_M)$. Since $M$ is a multiplication lattice $L$–module, then $P < K$ and $Q \leq K$. As $\dim(M) \leq 1$, then $P$ is a minimal prime element and since $M$ is a $\mu$–lattice module, $Q = K$. Thus $\{P\} \cap \{Q\} = \{Q\}$.

References


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