SEPARATION AXIOMS BETWEEN T_0 AND T_1 ON LATTICES AND LATTICE MODULES

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Abstract. In this study, we characterize the lattices and lattice modules whose prime spectrum satisfy some of the separation axioms between T_0 and T_1. This characterizations are the notion of $pm-$lattice, $m-$lattice, $\varepsilon-$lattice, e.t.c. Similarly, this characterizations are the notion of $PM-$lattice module, $\mu-$lattice module, $\varepsilon-$lattice module, etc.

Keywords: separation axioms, prime element, prime spectrum.

1. Introduction

A multiplicative lattice L is a complete lattice in which there is defined a commutative, associative multiplication which distributes over arbitrary joins and has compact greatest element 1_L (least element 0_L) as a multiplicative identity (zero). Multiplicative lattices have been studied extensively by E.W.Johnson, C.Jayaram, the current authors, and others, see, for example, [8]-[17]. An element a of a multiplicative lattice L is called compact if $a \leq \vee b_\alpha$ implies $a \leq b_{\alpha_1} \vee b_{\alpha_2} \vee ... \vee b_{\alpha_n}$ for some subset $\{\alpha_1, \alpha_2, ..., \alpha_n\}$. Throughout this paper, L denotes multiplicative lattice and L_* denotes the set of all compact elements of L. An element $a \in L$ is said to be proper if $a < 1_L$. An element $p < 1_L$ in L is said to be prime if $ab \leq p$ implies $a \leq p$ or $b \leq p$. An element $m < 1_L$ in L is said to be maximal if $m < x \leq 1_L$ implies $x = 1_L$. By a minimal prime element over an element n of L (or a prime element minimal over n), we mean a prime element which is minimal in the collection of all prime elements p with $n \leq p$. Minimal prime elements over 0_L are simply called the minimal prime elements. We define the dimension of L, denoted with $\dim(L)$, to be the maximal positive integer k, if such exists, such that there exists a chain of prime different elements of L as $p_0 < p_1 <\cdots< p_k$.

By a $C-$lattice we mean a (not necessarily modular) complete multiplicative lattice, with least element 0_L and compact greatest element 1_L (a multiplicative identity), which is generated under joins by multiplicatively closed subset C of compact elements. We note that in $C-$lattice, a finite product of compact elements is again compact. Since 1_L is compact element, then maximal elements exist in L. Further, if L is a $C-$lattice, then there is a minimal prime element of L (See, [15]). If $\dim(L) = 0$, then L is said to be zero-dimensional $C-$lattice. Note that L is a zero-dimensional lattice if and only if every prime element of L is maximal.

We denote by $\sigma(L)$ the set of prime elements of L. We put $V(a) = \{p \in \sigma(L) \mid a \leq p\}$ for any $a \in L$. Then we introduce a topology on $\sigma(L)$ such that $\{V(a) \mid a \in L\}$ is the family of all closed sets. With the above definition the following axioms are hold:

(i) $V(0_L) = \sigma(L)$ and $V(1_L) = \emptyset$.

(ii) $\bigcap\limits_{\alpha} V(a_\alpha) = V(\bigvee\limits_{\alpha} a_\alpha)$ for any index set $\triangle$.

(iii) $V(a) \cup V(b) = V(a \wedge b) = V(ab)$ (See, [16]).

We define $\zeta(L) = \{V(a) \mid a \text{ is an element of } L\}$ as it is family of all closed sets that satisfies (i), (ii) and (iii) conditions for L. Then there is a topology on $\sigma(L)$, say τ, and τ is called the Zariski topology. Any open set in $\sigma(L)$ is denoted $\sigma(L)-V(a)$ such that $a \in L$ since $V(a)$ is closed set in $\sigma(L)$. Let $D_a = \sigma(L)-V(a)$ for $a \in L$ (See, [7]).

Let M be a complete lattice. Recall that M is a lattice module over the multiplicative lattice L, or simply an L-module in case there is a multiplication between elements of L and M, denoted by lB for $l \in L$ and $B \in M$, which satisfies the following properties:

(i) $(lb)\,B = l\,(bB)$;

(ii) $\left(\bigvee\limits_{\alpha} l_\alpha\right)\left(\bigvee\limits_{\beta} B_\beta\right) = \bigvee\limits_{\alpha,\beta} l_\alpha B_\beta$;

(iii) $1_L B = B$;

(iv) $0_L B = 0_M$;

for all l, l_α, b in L and for all B, B_β in M.

Let M be an L-module. The greatest element of M will be denoted by 1_M. An element $N \in M$ is said to be proper if $N < 1_M$. An element $N < 1_M$ in M is said to be prime if $aX \leq N$ implies $X \leq N$ or $a1_M \leq N$ for $a \in L$, $X \in M$. If N is a prime element of M, then $(N : 1_M)$ is a prime element of L (See [10]). An element $N < 1_M$ in M is said to be maximal if $N < X \leq 1_M$ implies $X = 1_M$. If 1_M is compact element of M, then M has a maximal element by (See, [6]). If N and K belong to M, $(N : K)$ is the join of all $a \in L$ such that $aK \leq N$. By a minimal prime element over an element N of M (or a prime element minimal over N), we mean a prime element which is minimal in the collection of all prime elements P with $N \leq P$. Minimal prime elements over 0_M are simply called the minimal prime elements. We define the dimension of M, denoted with $\dim(M)$, to be the maximal positive integer k, if such exists, such that there exists a chain of prime distinguished elements of L as $P_0 < P_1 < \cdots < P_k$.

Let $\sigma(M)$ be the set of all prime elements of M. Next we define $V(N) = \{P \in \sigma(M) \mid (N : 1_M) \leq (P : 1_M)\}$. Then the following axioms are hold:

(i) $V(0_M) = \sigma(M)$ and $V(1_M) = \emptyset$.

(ii) $\bigcap_{\alpha \in \triangle} V(N_\alpha) = V(\bigvee_{\alpha \in \triangle} (N_\alpha : 1_M)1_M)$ for any index set $\triangle$.

(iii) $V(N) \cup V(K) = V(N \wedge K)$ (See, [7]).

We define $\zeta(M) = \{V(N) \mid N \text{ is an element of } M\}$ as it is family of all closed sets that satisfies (i), (ii) and (iii) conditions for M. Then there is a topology on $\sigma(M)$, say τ, and τ is called the Zariski topology. Any open set in $\sigma(M)$ is denoted by $\sigma(L) - V(N)$ such that $N \in M$ since $V(N)$ is closed set in $\sigma(M)$. We define $X_a = \sigma(M) - V(a1_M)$ for each $a \in L$ and it is clear that every X_a is an open set in $\sigma(M)$ (See, [7]).

For a topological space (X, τ) and $x \in X$, define the kernel of x, $\widehat{\{x\}}$, as the set $\{y \in X \mid x \in \overline{\{y\}}\}$. The set $\widehat{\{x\}} \backslash \{x\}$ is called shell of x and it will be denoted in this work by $\widetilde{\{x\}}$. The closure of a subset A of a topological space (X, τ) is the intersection of the members of the family of all closed sets containing A. The set A is called a point closure if $A = \overline{\{p\}}$ for some $p \in X$. A point x is an accumulation point of a subset A of a topological space (X, τ) if every neighborhood of x contains points of A other than x. Then it is true that each neighborhood of a point x intersects A if and only if x is either a point of A or an accumulation point of A. The set of all accumulation points of a set A is called the derived set of A and it will be denoted by A'. The closure of any set is the union of the set and the derived set of it (For more detail, see [5]).

With the Zariski topology on $\sigma(L)$, since $\overline{\{p\}} = V(p)$ for $p \in \sigma(L)$, it is clear that $\widetilde{\{p\}} = \widehat{\{p\}} \backslash \{p\} = \{q \in \sigma(L) \mid p \in V(q) \text{ and } p \neq q\}$ and $\{p\}' = V(p) \backslash \{p\} = \{q \in \sigma(L) \mid p < q\}$. For this topology, $\{p\}$ is a closed set if and only if p is a maximal element of L (See, [7]).

With the Zariski topology on $\sigma(M)$, the above definitions and $\overline{\{P\}} = V(P) = \{Q \in \sigma(M) \mid (P : 1_M) \leq (Q : 1_M)\}$ for $P \in \sigma(M)$, it is clear that $\widetilde{\{P\}} = \widehat{\{P\}} \backslash \{P\} = \{Q \in \sigma(M) \mid P \in V(Q) \; and \; P \neq Q\}$ and $\{P\}' = V(P) \backslash \{P\} = \{Q \in \sigma(M) | \; (P : 1_M) \leq (Q : 1_M) \; and \; P \neq Q\}$ (see,[7]), Proposition 25).

An $L-$module M is called a multiplication lattice module if for every element $N \in M$ there exists an element $a \in L$ such that $N = a1_M$. In this case, $N = (N : 1_M)1_M$ (see [10] for more detail). If the set $\{P\}$ is a closed in $\sigma(M)$ then P is a maximal element of M. The converse is also true, when M is a multiplication lattice $L-$module(See [7]).

Throughout this paper, we assume that 1_L and 1_M are compact element in L and M, respectively.

2. Separation axioms between T_0 and T_1 on $\sigma(L)$

Definition 1 A topological space (X, τ) is:

1. $T(\beta)$ if for any $x \in X$, $\{x\}'$ is empty or singleton.
2. $T(\beta')$ if for any $x \in X$, $\widetilde{\{x\}}$ is empty or singleton.
3. $T(\varepsilon)$ if for any $x, y \in X$, $x \neq y$, $\{x\}' \cap \{y\}'$ is empty or singleton.
4. T_{UD} if for any $x \in X$, $\{x\}'$ is the union of disjoint closed sets.
5. T_D if for any $x \in X$, $\{x\}'$ is a closed set.
6. T_{DD} if it is T_D and for any $x, y \in X$, $x \neq y$, $\{x\}' \cap \{y\}' = \emptyset$.
7. T_Y if for any $x, y \in X$, $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ is empty or singleton.
8. T_{YS} for any $x, y \in X$, with $x \neq y$, $\overline{\{x\}} \cap \overline{\{y\}}$ is either empty or $\{x\}$ or $\{y\}$.

Definition 2 L is said to be $pm-$lattice if every prime element is contained a unique maximal element.

Proposition 1 *$\sigma(L)$ is a $T(\beta)-$space if and only if L is a $pm-$lattice and* $\dim(L) \leq 1$.

Proof. $(\Rightarrow)$: Let $\sigma(L)$ be a $T(\beta)-$space. Assume that L is not a $pm-$lattice. Then there is a prime element p such that $p < k$ and $p < t$ with k and t two different maximal elements. So $k, t \in V(p)$ and since $p \neq k$ and $p \neq t$, then $k, t \in \{p\}'$, a contradiction.

Assume that $\dim(L) \geq 2$. Then there exist prime elements p, q, k such that $p < q < k$. So $q, k \in \{p\}'$, a contradiction.

$(\Leftarrow)$: Let L be a $pm-$lattice and $\dim(L) \leq 1$. Suppose that $\sigma(L)$ is not a $T(\beta)-$space. Then $\{p\}' \neq \emptyset$ and $\{p\}'$ is not singleton for some $p \in \sigma(L)$. Thus there are two different prime elements k, t with $k, t \in \{p\}'$. So $p < k$ and $p < t$. Since $\dim(L) \leq 1$, then k and t are maximal elements, a contradiction. ∎

Definition 3 A lattice L is said to be $m-$lattice if every prime element contains a unique minimal prime element of L.

Proposition 2 *Let L be a $C-$lattice. Then $\sigma(L)$ is a $T(\beta')-$space if and only if L is an $m-$lattice and* $\dim(L) \leq 1$.

Proof. $(\Rightarrow)$: Let $\sigma(L)$ be a $T(\beta')-$space. Assume that L is not an $m-$lattice. Then there is a prime element p such that $u < p$ and $v < p$ with u and v two different minimal prime elements. Hence $p \in V(u)$ and $p \in V(v)$, that is, $p \in \overline{\{u\}}$ and $p \in \overline{\{u\}}$. Since $p \neq u$ and $p \neq v$, then $u, v \in \widetilde{\{p\}} = \widehat{\{p\}} \backslash \{p\}$, a contradiction.

Assume that $\dim(L) \geq 2$. Then there exist prime elements p, q, k such that $p < q < k$. Thus $k \in V(p)$ and $k \in V(q)$, that is, $k \in \overline{\{p\}}$ and $k \in \overline{\{q\}}$. Therefore, $p, q \in \widetilde{\{k\}}$, a contradiction.

$(\Leftarrow)$: Let L be a $m-$lattice and $\dim(L) \leq 1$. Suppose that $\sigma(L)$ is not a $T(\beta')-$space. Then $\widetilde{\{p\}} \neq \emptyset$ and $\widetilde{\{p\}}$ is not singleton for some $p \in \sigma(L)$. Thus there are two different prime elements k, t with $k, t \in \widetilde{\{p\}}$. So $k < p$ and $t < p$. Since $\dim(L) \leq 1$, then k and t are minimal prime elements, a contradiction. ∎

Definition 4 Let m, n be maximal elements of L and k, t be minimal prime elements of L. Then $\rho_{\subset}(m, n) = \{p \in \sigma(L) \mid p < m \wedge n\}$ and $\rho_{\supset}(k, t) = \{p \in \sigma(L) \mid k < p \text{ and } t < p\}$.

Definition 5 A lattice L is called $\varepsilon-$lattice if $\rho_{\subset}(m, n)$ and $\rho_{\supset}(k, t)$ are empty or singleton for any different maximal elements m, n of L and different minimal prime elements k, t of L.

Proposition 3 *Let L be a $C-$lattice. Then $\sigma(L)$ is a $T(\varepsilon)-$space if and only if L is a $\varepsilon-$lattice and* $\dim(L) \leq 2$.

Proof. $(\Rightarrow)$: Let $\sigma(L)$ be a $T(\varepsilon)-$space. If L is not a $\varepsilon-$lattice, then $\rho_{\subset}(m, n)$ or $\rho_{\supset}(k, t)$ is neither empty nor singleton for any different maximal elements m, n of L and different minimal prime elements k, t of L. Assume that $\rho_{\subset}(m, n)$ is neither empty nor singleton. Then there are different prime elements p, q such that $p, q \in \rho_{\subset}(m, n)$. Thus $p < m \wedge n$ and $q < m \wedge n$, that is, $p < m$ and $p < n$, $q < m$ and $q < n$. Then $m, n \in \{p\}'$ and $m, n \in \{q\}'$. Hence $m, n \in \{p\}' \cap \{q\}'$, a contradiction. Suppose that $\rho_{\supset}(k, t)$ is neither empty nor singleton. Then there exist different prime elements u, v such that $u, v \in \rho_{\supset}(k, t)$. Hence $k < u$, $t < u$ and $k < v$, $t < v$. Therefore, $u, v \in \{k\}'$ and $u, v \in \{t\}'$, a contradiction. If $\dim(L) \geq 3$, then there are $p, q, k, t \in \sigma(L)$ with $p < q < k < t$. So $k, t \in \{p\}' \cap \{q\}'$, which is a contradiction.

$(\Leftarrow)$: Let L be a $\varepsilon-$lattice and $\dim(L) \leq 2$. Assume that $\sigma(L)$ is not a $T(\varepsilon)-$space. Then $\{p\}' \cap \{q\}'$ is neither empty nor singleton for different prime elements p, q of L. Thus there are different prime elements k, t of L such that $k, t \in \{p\}' \cap \{q\}'$. Thus we get $p < k$, $p < t$ and $q < k$, $q < t$. If p, q are minimal prime elements of L, then $k, t \in \rho_{\supset}(p, q)$, which is a contradiction. If p, q are not minimal prime elements of L, then there exist minimal prime elements u, v of L such that $u < p$ and $v < q$. Then $u < p < k$, $u < p < t$ and $v < p < k$, $v < q < t$. Hence k, t are maximal elements of L since $\dim(L) \leq 2$. It is clear that $p < k \wedge t$ and $q < k \wedge t$. Indeed, if $p = k \wedge t$ or $q = k \wedge t$, then $k \leq p$ or $t \leq p$ or $k \leq q$ or $t \leq q$ as p, q are prime elements of L. This is contradict with k, t where are maximal elements of L. So we get $p < k \wedge t$ and $q < k \wedge t$. Hence $p, q \in \rho_{\subset}(k, t)$, a contradiction. ∎

Definition 6 Let $p \in \sigma(L)$ be not maximal. Then L is called $u-$lattice if $a \vee b = 1_L$ for every elements a, b in $\{p\}'$.

Proposition 4 *Let* $\dim(L)$ *be finite and* L *be a* $u-$*lattice. Then* $\sigma(L)$ *is a* $T_{UD}-$*space.*

Proof. Let $p \in \sigma(L)$ and β be the set of minimal prime elements of $\{p\}'$. Assume that $x \in \{p\}'$. Then $p < x$. We have that there is a $q \in \beta$ such that $p < q \leq x$. So $x \in V(q)$. Thus $x \in \bigcup_{q\in\beta} V(q)$. Conversely, let $x \in \bigcup_{q\in\beta} V(q)$. Then $x \in V(q)$ for some $q \in \beta$. So $q \leq x$ and since $p < q$, then $p < x$. Hence $x \in \{p\}'$. Therefore $\{p\}' = \bigcup_{q\in\beta} V(q)$. Also $V(q_1) \cap V(q_2) = \emptyset$ for minimal prime elements q_1, q_2 with $p < q_1$ and $p < q_2$. Indeed, if $V(q_1) \cap V(q_2) \neq \emptyset$, then there is a $t \in \sigma(L)$ such that $t \in V(q_1) \cap V(q_2)$. Thus $q_1 \leq t$ and $q_2 \leq t$. Therefore, $q_1 \vee q_2 \leq t$. Hence $t = 1_L$, a contradiction. ∎

Definition 7 A lattice L is called $d-$*lattice* if every prime non maximal element is different to the meet of the prime elements properly containing it.

Proposition 5 $\sigma(L)$ *is a* T_D-*space if and only if* L *is a* $d-$*lattice.*

Proof. $(\Rightarrow)$: Let $\sigma(L)$ be a T_D-space. If L is not a $d-$lattice, then there exists a prime non maximal element p such that p is the meet of the prime elements properly containing it, that is, $p = \bigwedge_{p<q} q$. Since $\sigma(L)$ is a T_D-space, then $\{p\}'$ is closed set. Then $\{p\}' = V(a)$ for $a \in L$. Since $a \leq q$ for every $q \in \{p\}'$, $a \leq \bigwedge_{q\in\{p\}'} q = p$. Thus $p \in V(a)$, a contradiction.

$(\Leftarrow)$: Let L be a $d-$lattice. If $p \in \sigma(L)$ is a maximal, then it is clear. Let $p \in \sigma(L)$ be a non maximal element. Since L is a $d-$lattice, then $p \neq \bigwedge_{p<q} q$. But $V(\bigwedge_{p<q} q) \subseteq V(p)$ since $p \leq \bigwedge_{p<q} q$. Since $p \notin V(\bigwedge_{p<q} q)$, then $V(\bigwedge_{p<q} q) \subseteq \{p\}'$. Now, let $k \in \{p\}'$. Then $p < k$ and so $\bigwedge_{p<q} q \leq k$, that is, $k \in V(\bigwedge_{p<q} q)$. ∎

Definition 8 A lattice L is called $y-$lattice if any two different minimal prime element are contained in at most one maximal element.

Proposition 6 *Let L be a $C-$lattice. $\sigma(L)$ is a T_Y-space if and only if L is a $y-$lattice and* $\dim(L) \leq 1$.

Proof. $(\Rightarrow)$: Let $\sigma(L)$ be a T_Y-space. Assume that L is not a $y-$lattice. Let p, q be minimal prime elements of L. Then there are two different maximal elements m, n of L containing p, q, that is, $p < m$, $p < n$ and $q < m$, $q < n$. Then $m, n \in V(p) = \overline{\{p\}}$ and $m, n \in V(q) = \overline{\{q\}}$. Thus $m, n \in \overline{\{p\}} \cap \overline{\{q\}}$, a contradiction. Suppose that $\dim(L) \geq 2$. Then there exist prime elements p, q, k such that $p < q < k$. So $q, k \in V(p) \cap V(q)$, that is, $q, k \in \overline{\{p\}} \cap \overline{\{q\}}$, a contradiction.

$(\Leftarrow)$: Let L be a $y-$lattice and $\dim(L) \leq 1$. Assume that $\sigma(L)$ is not a T_Y-space. Then $\overline{\{p\}} \cap \overline{\{q\}}$ is neither empty nor singleton for two different minimal prime elements p, q. Thus we get $k, t \in \overline{\{p\}} \cap \overline{\{q\}}$. Then $p \leq k$, $p \leq t$ and $q \leq k$, $q \leq t$. Also $p \neq k$ and $p \neq t$. Indeed, if $p = k$ or $p = t$, then $p \in V(q) = \overline{\{q\}}$. Thus $q < p$. We get that p is maximal since $\dim(L) \leq 1$.Thus $V(p) = \{p\}$ and so $\overline{\{p\}} \cap \overline{\{q\}} = \{p\}$, a contradiction. In a similar way, $q \neq k$ and $q \neq t$. Hence $p < k$, $p < t$ and $q < k$, $q < t$. Since $\dim(L) \leq 1$, then k and t are maximal elements, a contradiction. ∎

Proposition 7 *Let L be a $C-$lattice. $\sigma(L)$ is a $T_{YS}-$space if and only if L is an $m-$lattice and* $\dim(L) \leq 1$.

Proof. $(\Rightarrow)$: Let $\sigma(L)$ be a $T_{YS}-$space. Suppose that L is not an $m-$lattice. Let $p \in \sigma(L)$. Then there are two different minimal prime elements u, v such that $u < p$ and $v < p$. Thus $p \in V(u) = \overline{\{u\}}$ and $p \in V(v) = \overline{\{v\}}$ and so $p \in \overline{\{u\}} \cap \overline{\{v\}}$, a contradiction. Assume that $\dim(L) \geq 2$. Then there exist prime elements p, q, k such that $p < q < k$. Thus $k \in V(p) = \overline{\{p\}}$ and $k \in V(q) = \overline{\{q\}}$. Therefore, $k \in \overline{\{p\}} \cap \overline{\{q\}}$, a contradiction.

$(\Leftarrow)$: Let L be an $m-$lattice and $\dim(L) \leq 1$. Let $p, q \in \sigma(L)$ with $p \neq q$. Assume that $\overline{\{p\}} \cap \overline{\{q\}} \neq \emptyset$ and $\overline{\{p\}} \cap \overline{\{q\}} \neq \{p\}$. Let $k \in \overline{\{p\}} \cap \overline{\{q\}}$. Then $k \neq p$ and $p < k$, $q \leq k$. Since $\dim(L) \leq 1$, then p is minimal prime element. Then $q = k$ because L is an $m-$lattice. Hence $\overline{\{p\}} \cap \overline{\{q\}} = \{q\}$. ∎

3. Separation axioms between T_0 and T_1 on $\sigma(M)$

In this section, we will investigate the like of our studying on lattices for lattice modules.

Definition 9 An $L-$module M is said to be $PM-$lattice module if every prime element is contained a unique maximal element.

Proposition 8 *Let M be an $L-$module. If $\sigma(M)$ is a $T(\beta)-$space, then the followings are hold:*

a) *M is a $PM-$lattice module.*

b) $\dim(M) \leq 1$.

Proof. Let $\sigma(M)$ be a $T(\beta)-$space.

a) Assume that M is not a $PM-$lattice module. Then there exists a prime element P such that $P < N$ and $P < K$ with N and K two different maximal elements. Since $(P : 1_M) \leq (N : 1_M)$, $(P : 1_M) \leq (K : 1_M)$ and $P \neq N$, $P \neq K$, then $N, K \in \{P\}'$, a contradiction.

b) If $\dim(M) \geq 2$, then there are prime elements P, N, K such that $P < N < K$. So $(P : 1_M) \leq (N : 1_M) \leq (K : 1_M)$. Therefore $N, K \in \{P\}'$, which is a false. ∎

Definition 10 An $L-$module M is called $\mu-$lattice module if every prime element contains a unique minimal prime element.

Proposition 9 *Let M be an $L-$module. Let $\sigma(M)$ be a $T(\beta')-$space. Then the following statements are hold:*

a) *M is a $\mu-$lattice module.*

b) $\dim(M) \leq 1$.

Proof. Let $\sigma(M)$ be a $T(\beta')-$space.

a) If M is not a $\mu-$lattice module, then there is a prime element P such that $K < P$ and $T < P$ with K and T two different minimal prime elements. Then $(K : 1_M) \leq (P : 1_M)$ and $(T : 1_M) \leq (P : 1_M)$. Hence, $P \in V(K)$ and $P \in V(T)$. Then $K, T \in \widetilde{\{P\}} = \widehat{\{P\}} \backslash \{P\}$, which is a contradiction.

b) If $\dim(M) \geq 2$, then there are prime elements P, N, K such that $P < N < K$. Hence $(P : 1_M) \leq (N : 1_M) \leq (K : 1_M)$. Then $K \in V(N)$ and $K \in V(P)$. Thus, $P, N \in \widetilde{\{K\}} = \widehat{\{K\}} \backslash \{K\}$, a contradiction. ∎

Definition 11 Let M be an $L-$module. Let N, K be maximal elements of M and U, V be minimal prime elements of M. Then

$$\rho(N, K) = \{P \in \sigma(M) \mid (P : 1_M) \leq (N \wedge K : 1_M) \text{ and } P \neq N \wedge K\} \text{ and}$$

$$\delta(U, V) = \{P \in \sigma(M) \mid P \in \{U\}' \text{ and } P \in \{V\}'\}.$$

Definition 12 An $L-$module M is a $\varepsilon-$lattice module if the set $\rho(N, K)$ and $\delta(U, V)$ are empty or singleton for any different maximal elements N, K and different minimal prime elements U, V.

Let M be an $L-$module. We call M is a multiplication lattice $L-$module if for every element N of M there exists an element a of L such that $N = a1_M$. In this case, we know that $N = (N : 1_M)1_M$ (For more detail, see (10)).

Proposition 10 *Let M be a multiplication lattice $L-$module. Then M is a $\varepsilon-$module and* $\dim(M) \leq 2$ *if and only if $\sigma(M)$ is a $T(\varepsilon)-$space.*

Proof. Let M be an $\varepsilon-$module and $\dim(M) \leq 2$. Suppose that $\sigma(M)$ is not a $T(\varepsilon)-$space. Then $\{P\}' \cap \{Q\}'$ is neither empty nor singleton for different prime elements P, Q of M. Thus there are different prime elements N, K of M such that $N, K \in \{P\}' \cap \{Q\}'$. Thus we get $(P : 1_M) \leq (N : 1_M)$, $(P : 1_M) \leq (K : 1_M)$, $P \neq N, K$ and $(Q : 1_M) \leq (N : 1_M)$, $(Q : 1_M) \leq (K : 1_M)$, $Q \neq N, K$. Since M is a multiplication lattice $L-$module, we obtain $P < N, P < K$ and $Q < N, Q < K$. If P, Q are minimal prime elements, then $N, K \in \delta(P, Q)$, which is a contradiction. If P, Q are not minimal prime elements, then there are different minimal prime elements U, V such that $U < P$ and $V < Q$. Then we have $U < P < N, U < P < K$ and $V < Q < N, V < Q < K$. Thus N, K are maximal elements since $\dim(M) \leq 2$. It is clear that $P < N \wedge K$ and $Q < N \wedge K$. Indeed, if $P = N \wedge K$ or $Q = N \wedge K$, $N \leq P$ or $K \leq P$ or $N \leq Q$ or $K \leq Q$ as P, Q are prime elements of M. This is contradict with N, K where are maximal elements of M. So we get $P < N \wedge K$ and $Q < N \wedge K$. Hence $P, Q \in \rho(N, K)$, a contradiction.

Conversely, let $\sigma(M)$ be a $T(\varepsilon)-$space. If M is not a $\varepsilon-$lattice module, then the set $\rho(N, K)$ or $\delta(U, V)$ is neither empty nor singleton for any different maximal elements N, K and different minimal prime elements U, V. If $\rho(N, K)$ is neither empty nor singleton, then there are $P, Q \in \sigma(M)$ such that $P, Q \in \rho(N, K)$, that is, $(P : 1_M) \leq (N \wedge K : 1_M)$, $P \neq N \wedge K$ and $(Q : 1_M) \leq (N \wedge K : 1_M)$, $Q \neq N \wedge K$. Since M is a multiplication lattice $L-$module, then $P < N \wedge K \leq N$, $P < N \wedge K \leq K$ and $Q < N \wedge K \leq N$, $Q < N \wedge K \leq K$. Thus $N \in \{P\}'$, $K \in \{P\}'$and $N \in \{Q\}'$, $K \in \{Q\}'$. Thus, $N, K \in \{P\}' \cap \{Q\}'$, a contradiction. If $\delta(U, V)$ is neither empty nor singleton, then there are $S, T \in \sigma(M)$ such that $S, T \in \delta(U, V)$. Thus $S, T \in \{U\}'$ and $S, T \in \{V\}'$. Consequently, $S, T \in \{U\}' \cap \{V\}'$, a contradiction. If $\dim(M) \geq 3$, then there are prime elements P, Q, S, T such that $P < Q < S < T$. So $(P : 1_M) \leq (Q : 1_M) \leq (S : 1_M) \leq (T : 1_M)$. Thus, $S, T \in \{P\}' \cap \{Q\}'$, which is a contradiction. ■

Definition 13 An $L-$module M is a $D-$lattice module if every prime non-maximal element is not equal to the meet of the prime elements properly containing it.

Proposition 11 *Let M be an $L-$module. If $\sigma(M)$ is a T_D-space, then M is a $D-$lattice module. The converse is also true when M is a multiplication lattice $L-$module.*

Proof. Assume that $\sigma(M)$ is a T_D-space. Let $P \in \sigma(M)$ be a non-maximal element. Assume that $P = \bigwedge_{\substack{Q \in \sigma(M) \\ P < Q}} Q$. Since $\{P\}'$ is a closed set, there is an element K of M such that $\{P\}' = V(K)$. Then $(K : 1_M) \leq (Q : 1_M)$ for every $Q \in \{P\}' \Rightarrow (K : 1_M) \leq \bigwedge_{Q \in \{P\}'} (Q : 1_M) \Rightarrow (K : 1_M) \leq (\bigwedge_{Q \in \{P\}'} Q : 1_M) \leq (\bigwedge_{\substack{Q \in \sigma(M) \\ P < Q}} Q : 1_M) = (P : 1_M) \Rightarrow (K : 1_M) \leq (P : 1_M)$. Thus $P \in V(K) \Rightarrow P \in \{P\}'$, which is a contradiction. Hence we get $P \neq \bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q$.

Conversely, let M be a multiplication lattice $L-$module. If $P \in \sigma(M)$ is maximal element, then $V(P) = \{P\}$. Then $\{P\}' = \emptyset$, that is, $\{P\}'$ is a closed set. If P is prime non-maximal element, then P is not equal to the meet of the prime elements properly containing it, that is, $P \neq \bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q$. Since $P < \bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q$, then $V(\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q) \subseteq V(P)$. Then $P \in V(P)$ but $P \notin V(\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q)$. Indeed, if $P \in V(\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q)$, then $(\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q : 1_M) \leq (P : 1_M)$ and since M is a multiplication lattice $L-$module, then $\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q \leq P$, it is a contradiction. Hence $V(\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q) \subseteq \{P\}'$. Conversely, let $K \in \{P\}'$. Then $(P : 1_M) \leq (K : 1_M)$ and $P \neq K$. Since M is a multiplication lattice $L-$module, then $P < K$. Thus $\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q \leq K$ and so $(\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q : 1_M) \leq (K : 1_M)$. Then $K \in V(\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q)$ and so $\{P\}' \subseteq V(\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q)$. Consequently, $\{P\}' = V(\bigwedge_{\substack{Q\in\sigma(M)\\ P<Q}} Q)$, that is, $\{P\}'$ is a closed set. ■

Proposition 12 *Let $\sigma(M)$ be a $T_{DD}-$space. Then the following statements are hold:*

a) *M is a $\mu-$lattice module.*

b) $\dim(M) \leq 1$.

Proof. Let $\sigma(M)$ be a $T_{DD}-$space.

a) Assume that M is not a $\mu-$lattice module. Then there is a prime element P of M such that $U < P$ and $T < P$ with U and T two different minimal prime elements. So $(U : 1_M) \leq (P : 1_M)$ and $(T : 1_M) \leq (P : 1_M)$. Hence $P \in \{U\}' \cap \{T\}'$, which is a contradiction.

b) Suppose that $\dim(M) \geq 2$. Then there are prime elements P, N, K such that $P < N < K$. Hence $(P : 1_M) \leq (N : 1_M) \leq (K : 1_M)$. Thus $K \in \{P\}' \cap \{N\}'$, a contradiction. ■

Definition 14 An $L-$module M is a $Y-$lattice module if any two distinct minimal prime elements are contained in at most one maximal element.

Proposition 13 *Let M be an $L-$module. If $\sigma(M)$ is a T_Y-space, then M is a $Y-$lattice module and* $\dim(M) \leq 1$*. The converse is also true when M is a multiplication lattice $L-$module.*

Proof. Let $\sigma(M)$ be a T_Y-space. Suppose that M is not a $Y-$lattice module. Let P, Q be two different minimal prime elements. Then there are two different maximal elements N, K such that $P < N, P < K$ and $Q < N, Q < K$. Then $(P : 1_M) \leq (N : 1_M)$, $(P : 1_M) \leq (K : 1_M)$ and $(Q : 1_M) \leq (N : 1_M)$, $(Q : 1_M) \leq (K : 1_M)$. Thus $N, K \in V(P) = \overline{\{P\}}$ and $N, K \in V(Q) = \overline{\{Q\}}$. Hence $N, K \in \overline{\{P\}} \cap \overline{\{Q\}}$, this is a contradiction. Let $\dim(M) \geq 2$. Then there are prime elements P, Q, T such that $P < Q < T$. Thus $(P : 1_M) \leq (Q : 1_M) \leq (T : 1_M)$. So $Q, T \in V(Q)$ and $Q, T \in V(P)$. Then $Q, T \in \overline{\{P\}} \cap \overline{\{Q\}}$, which is a false.

Conversely, let M be a multiplication lattice $L-$module. Let M be a $Y-$lattice module and $\dim(M) \leq 1$. Assume that $\sigma(M)$ is not a T_Y-space. Then $\overline{\{P\}} \cap \overline{\{Q\}} \neq \emptyset$ and $\overline{\{P\}} \cap \overline{\{Q\}}$ is not singleton for two different prime elements P, Q. So there are two prime elements N, K such that $N, K \in \overline{\{P\}} \cap \overline{\{Q\}}$ and so $(P : 1_M) \leq (N : 1_M), (P : 1_M) \leq (K : 1_M)$ and $(Q : 1_M) \leq (N : 1_M), (Q : 1_M) \leq (K : 1_M)$. Since M is a multiplication lattice $L-$module, $P \leq N, P \leq K$ and $Q \leq N, Q \leq K$. Also, $P \neq N$ and $P \neq K$. Indeed, if $P = N$ or $P = K$, then P is maximal as $\dim(M) \leq 1$. Hence $V(P) = \{P\} = \overline{\{P\}}$ and so $\overline{\{P\}} \cap \overline{\{Q\}} = \{P\}$, a contradiction. In a similar way, $Q \neq N$ and $Q \neq K$. Then we get $P < N$, $P < K$ and $Q < N$, $Q < K$, so N and K are maximal elements, since $\dim(M) \leq 1$. But this contradicts with M which is a $Y-$lattice module. ∎

Proposition 14 *Let M be an $L-$module. If $\sigma(M)$ is a $T_{YS}-$space, then M is a $\mu-$lattice module and* $\dim(M) \leq 1$*. The converse is also true when M is a multiplication lattice $L-$module.*

Proof. Let $\sigma(M)$ be a $T_{YS}-$space. Suppose that M is not a $\mu-$lattice module. Then there exists a prime element P containing two different minimal prime elements Q, T. So $P \in V(Q) \cap V(T)$, that is, $P \in \overline{\{Q\}} \cap \overline{\{T\}}$. This is a contradiction. Suppose that $\dim(M) \geq 2$. Then there are prime elements P, Q, T such that $P < Q < T$. Hence $(P : 1_M) \leq (Q : 1_M) \leq (T : 1_M)$. Thus $T \in V(P) \cap V(Q) \Rightarrow T \in \overline{\{P\}} \cap \overline{\{Q\}}$, which is a false.

Conversely, assume that M is a multiplication lattice $L-$module. Let M be a $\mu-$lattice module and $\dim(M) \leq 1$. Let P, Q be different prime elements of M. Assume that $\overline{\{P\}} \cap \overline{\{Q\}} \neq \emptyset$ and $\overline{\{P\}} \cap \overline{\{Q\}} \neq \{P\}$. Let $K \in \overline{\{P\}} \cap \overline{\{Q\}}$. Then $K \neq P$, also $(P : 1_M) \leq (K : 1_M)$ and $(Q : 1_M) \leq (K : 1_M)$. Since M is a multiplication lattice $L-$module, then $P < K$ and $Q \leq K$. As $\dim(M) \leq 1$, then P is a minimal prime element and since M is a $\mu-$lattice module, $Q = K$. Thus $\overline{\{P\}} \cap \overline{\{Q\}} = \{Q\}$. ∎

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