ON SEMI p-COVER-AVOIDING SUBGROUPS AND $\mathscr{H}$-SUBGROUPS OF FINITE GROUPS

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Abstract. In recent years, some authors gave many valuable results on the structure of a finite group, provided its some subgroups have semi p-cover-avoiding property. At the same time, some interesting conclusions about finite group were also obtained under the assumption that some subgroups of G belong to $\mathscr{H}(G)$. All these results are important in the research of finite groups. Here the authors discussed the connection between the semi p-cover-avoiding subgroups and the $\mathscr{H}$-subgroups on the structure of a finite group, and again obtained some more interesting results about finite groups.

Keywords: semi p-cover-avoiding subgroups, $\mathscr{H}$-subgroups, p-nilpotent, p-supersolvable.

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1. Introduction

It is well known that some properties of some subgroups of finite groups play important roles in the study of the structure of finite groups, especially various generalizations of normality were defined and used to make restrictions on various kinds of subgroups of a finite group, many useful results of finite group have been obtained under the assumption that some certain subgroups of G of prime power orders are well situated in G.

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A cover-avoiding subgroup was introduced in [1] in 1962, and recently, semi p-cover-avoiding subgroup was defined in [2]. It is worth to mention that a semi p-cover-avoiding subgroup is also known as a partial cover-avoiding subgroup, for example in [18]. Such kinds of subgroups have many interesting properties. For example, Gillam [3] and Tomkinson [4] have found some kind of subgroups of a finite sovable group G having the cover-avoiding properties. Fan, Guo and Shum [2] proved that the C-normal subgroup and the near normal subgroup are all semi cover-avoiding subgroups, and gave many meaningful results using the semi p-cover-avoiding subgroups, especially about the supersolvability and p-nilpotency of finite groups.

The concept of $\mathscr{H}$-subgroup was defined by Bianchi et al. in [5]. It is easy to prove by the definition that, the Sylow p-subgroups, the normal subgroups and the self-normalizing subgroups of an arbitrary group are all $\mathscr{H}$-subgroups. Many facts have proved that the $\mathscr{H}$-property of some subgroups can give a good insight into the structure of a group. For example, M. Asaad [6] studied the p-nilpotency and supersolvability of finite group under the assumption that some subgroups of G of prime power orders having $\mathscr{H}$-property.

Although " semi p-cover-avoiding subgroups " and " $\mathscr{H}$-subgroups " both can characterize the supersolvability and nilpotency of groups, there is no implications between them. In this paper, we will give two examples to explain. We study the p-supersolvability and p-nilpotency of a finite groups by assuming that a special kinds of subgroups of G are semi cover-avoiding subgroups or $\mathscr{H}$-subgroups. We got some more interesting results about the p-supersolvability and p-nilpotency of finite groups. Some well-known results in [6], [7] and [8] would be the direct corollaries of our results.

Throughout this paper, all groups considered are finite groups. The terminology and notations employed are standard and referred to Robinson [9]. For example, we denote by $F_p(G)$ the $p-$nilpotent radical, also known as the p-Fitting subgroup of a group G, which is equal to the product of all p-nilpotent normal subgroups of G, i.e. , $F_p(G) = O_{pp'}(G)$. $F(G)$ the Fitting subgroup of a group G (the product of all nilpotent normal subgroups of G).

2. Basic definitions and preliminaries

In this section, we first state some preliminaries in the literature which will be used in the sequel.

Let M and N be normal subgroups of a group G with $N \leq M$. Then the quotient group M/N is called a normal factor of G. Furthermore, if $p \mid |M/N|$, then M/N is called a p-singular. A subgroup H of G is said to cover M/N if $HM = HN$. And in the case of $H \cap M = H \cap N$, H is said to avoid M/N.

Definition 2.1 Let H be a subgroup of a group G. H is said to be semi p-cover-avoiding in G if there exists a chief series such that either H covers or avoids its every p-singular chief factor.

Definition 2.2 Let H be a subgroup of a group G. H is called an $\mathscr{H}$-subgroup of G if the following condition is satisfied:

$$N_G(H) \cap H^g \leq H, for\ all\ g \in G.$$

The set of all $\mathscr{H}$-subgroups of a group G will be denoted by $\mathscr{H}(G)$.

Note that there is no implications between the two concepts, our studies are meaningful:

Example 2.1 (a semi p-cover-avoiding subgroup but not a $\mathscr{H}$-subgroup) Let $G = S_4$ and $H = \langle(12)\rangle$. Easy to know $1 \trianglelefteq K_4 \trianglelefteq A_4 \trianglelefteq G$ is the unique chief series of G and H is semi 2-cover-avoiding in G. But there exists an element $g = (13)(24)$ in G such that $N_G(H) \cap H^g = \{(1), (34)\} \nleq H$, hence H is not in $\mathscr{H}(G)$.

Example 2.2 (a $\mathscr{H}$-subgroup but not a semi p-cover-avoiding subgroup) Let $G = A_n (n \geq 5)$. Then G is a simple group and has only one chief series $1 \trianglelefteq G$. Take S be any Sylow p-subgroup of G. Then clearly S belongs to $\mathscr{H}(G)$, but not semi p-cover-avoiding in G.

Lemma 2.3 [2, Lemma 2] *Let H be a semi p-cover-avoiding subgroup of G, and N be a normal subgroup of G. Then HN/N is semi p-cover-avoiding in G/N if one of the following holds:*

(1) *$N \subseteq H$;*

(2) *$(|H|, |N|) = 1$, where $(-,-)$ denotes the greatest common divisor.*

Similar to [8, Lemma 2.5], we have:

Lemma 2.4 *Let H be a subgroup of a group G. If H is semi p-cover-avoiding in G, then H is semi p-cover-avoiding in K for every subgroup K of G with containing H.*

Lemma 2.5 [5, Lemma 2(1)] *If $L \leq H \leq G$ and $L \trianglelefteq G$, then $H \in \mathscr{H}(G)$ if and only if $H/L \in \mathscr{H}(G/L)$.*

Lemma 2.6 [5, Theorem 6(2)] *Let G be a group and $H \in \mathscr{H}(G)$. If H is subnormal in G, then $H \trianglelefteq G$.*

Lemma 2.7 [5, Lemma 7(2)] *Let G be a group and H, K be subgroups of G such that $H \leq K$. If $H \in \mathscr{H}(G)$, then $H \in \mathscr{H}(K)$.*

Lemma 2.8 [10, Lemma 6] *If $L \trianglelefteq G$, P is a p-subgroup of G which belongs to $\mathscr{H}(G)$, and $(|L|, |P|) = 1$, then $PL/L \in \mathscr{H}(G/L)$.*

Lemma 2.9 [11, Lemma 2.9] *Let P be an elementary abelian p-subgroup of G of order p^n, where $n \geq 2$. Then the following two statements are equivalent:*

(i) *The subgroups of order p in P are normal in G;*

(ii) *The maximal subgroups of P are normal in G.*

Lemma 2.10 [12, Theorem 3.1] *Let p be an odd prime divisor of the order of G with $(|G|, p-1) = 1$, $P \in Syl_p(G)$. If $N_G(P)$ is p-nilpotent, then G is p-nilpotent.*

Lemma 2.11 [13, Lemma 2.6] *Let G be a group and p a prime divisor of $|G|$ with $(|G|, p-1) = 1$. Suppose M is a subgroup of G with $|G : M| = p$, then M is normal in G.*

Lemma 2.12 [14, Lemma 2.6] *Let N be a solvable normal subgroup of a group G with $N \neq 1$. If every minimal normal subgroup of G which is contained in N is not contained in $\Phi(G)$, then the Fitting subgroup $F(N)$ of N is the direct product of minimal normal subgroups of G which are contained in N.*

Lemma 2.13 [7, Lemma 2.6] *Let G be a p-solvable group. Suppose that G has a chief series:*

$$1 \leq \cdots \leq \Phi(G) = H_0 \leq H_1 \cdots \leq H_s = F_p(G) \leq \cdots \leq G$$

such that H_i/H_{i-1} are cyclic groups of order p or p'-groups for all $1 \leq i \leq s$, then G is p-supersolvable, where $F_p(G)$ is the p-Fitting subgroup of G.

Lemma 2.14 *Let G be a p-solvable group and p a prime divisor of order G. If $F_p(G) = N_1 \times \cdots \times N_s$, where $N_i (i = 1, \cdots, s)$ are minimal normal subgroups of G of order p, then G is p-supersolvable.*

Proof. Let $K_0 = 1$ and $K_j = N_1 \times \cdots \times N_j (j = 1, \cdots, s)$. Then the following series is a chief series of G:

$$1 = \Phi(G) = K_0 \leq K_1 \leq \cdots \leq K_s = F_p(G) \leq \cdots \leq G$$

Since $|K_j/K_{j-1}| = p (j = 1, \cdots, s)$. By Lemma 2.13, G is p-supersolvable. ∎

Lemma 2.15 [*9*, Theorem 9.3.1] *If G is a $\pi-$separable group, then $C_G(O_{\pi'\pi}(G)/O_{\pi'}(G)) \leqslant O_{\pi'\pi}(G)$.*

3. On p-nilpotency of finite groups

Theorem 3.1 *Let G be a p-solvable group and p a prime divisor of $|G|$ such that $(|G|, p-1) = 1$. If every maximal subgroup of Sylow p-subgroup P of G is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-nilpotent.*

Proof. Assume the theorem is false and let G be a counterexample of minimal order. Choose a minimal normal subgroup N of G, where N is a p-group or p'-group. We have the following conditions:

(1) N is a p-group and the unique minimal normal subgroup of G, $\Phi(G) = 1$, and G/N is p-nilpotent.

Consider the quotient group G/N. If N is a p'-group, then the maximal subgroup of the Sylow p-subgroup of G/N is the form P_0N/N, where P_0 is a maximal subgroup of P. By Lemma 2.3 and Lemma 2.8, the hypotheses are inherited by G/N. Thus G/N is p-nilpotent by the choice of G and it follows that G is p-nilpotent, a contradiction. Hence N is a p-group.

If $N = P$, then G/N is a p'-group and of course G/N is p-nilpotent. If $N < P$, the maximal subgroup of the Sylow p-subgroup of G/N is the form P_0/N, where P_0 is a maximal subgroup of P. Then by Lemma 2.3 and Lemma 2.5, G/N satisfies the hypotheses of the theorem. By the minimality of G, G/N is p-nilpotent.

Therefore, N is the unique minimal normal subgroup of G and $\Phi(G) = 1$, since the class of p-nilpotent groups forms a saturated formation.

(2) $|N| = p$.

Let P be a Sylow p-subgroup of G. If $N_G(P) < G$, then $N_G(P)$ is p-nilpotent by the minimality of G. Then it follows from Lemma 2.10 that G is p-nilpotent, a contradiction. This forces that $N_G(P) = G$ and then $P \trianglelefteq G$. By (1), $\Phi(P) = 1$ and hence P is elementary abelian.

By (1), $N \leq P$. Let $|P| = p^n$. If $n = 1$, we have done. Suppose $n \geq 2$. Let $\mathbb{F}_p$ be the field with p-elements and view P as a vector space of dimension n over $\mathbb{F}_p$. Obviously, the subgroups of order p in P are 1-dimensional subspaces and the maximal subgroups of P are $(n-1)$-dimensional subspaces. Given a subgroup $X = \langle x \rangle$ in N of order p. Consider the basis $B = \{x = x_1, x_2, ..., x_n\}$ of P as vector space. Denote M_i the $(n-1)$-dimensional subspace generated by $B \backslash \{x_i\} (i = 2, ..., n)$. Then it is easy to see that $X = \langle x \rangle = \bigcap_{i=2}^{n} M_i$.

By the hypotheses, $M_i (i = 2, ..., n)$ either is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$. Without loss of generality, we assume that $M_2, ..., M_t$ are semi p-cover-avoiding in G; $M_{t+1}, \cdots, M_n$ belong to $\mathscr{H}(G)$.

Since N is the unique minimal normal subgroup of G, $M_iN{=}M_i$ or $M_i \cap N{=}1$ for $i = 2, ..., t$. But $x \in M_i \cap N$, hence $M_iN = M_i$ $(i = 2, ..., t)$. Then $N \leq M_i$ $(i = 2, ..., t)$. On the other hand, $M_j \trianglelefteq P \trianglelefteq G$ for $j = t+1, ..., n$. Then by Lemma 2.6, $M_j \trianglelefteq G$, where $j = t+1, ..., n$. Hence $N \cap M_j \trianglelefteq G$ for $j = t+1, ..., n$. By the minimality of N, $N \cap M_j = 1$ *or* $N (j = t+1, ..., n)$. But $x \in M_j \cap N$, hence $N \cap M_j = N$ $(j = t+1, ..., n)$. Then $N \leq M_j$ $(j = t+1, ..., n)$.

Now the above paragraph implies that $N \leq \bigcap_{i=2}^{n} M_i = \langle x \rangle$, hence $N = \langle x \rangle$ and it follows then $|N| = p$.

(3) The final contradiction.

By (1), G/N is p-nilpotent. Let H/N be a normal p-complement of G/N. Then $H = KN$, where K is a $p'-$group and $N \in Syl_p(H)$. Since $|N| = p$, $|H : K| = p$. It follows then $K \trianglelefteq H$ by Lemma 2.15. So K is a normal Hall p'-subgroup of H and hence $K \trianglelefteq G$. This implies that K is a normal p-complement of G and then G is p-nilpotent, the final contradiction. ∎

If we consider the smallest prime divisor of the order of G, we have the following results, actually, it is covered by Theorem 3.1.

Corollary 3.2 *Let G be a $p-$solvable group and p the smallest prime divisor of the order of G. If every maximal subgroup of Sylow p-subgroups of G is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-nilpotent.*

Remark 3.3 By Theorem 3.1 and Corollary 3.2, the results [8, Theorem 3.2] and [6, Corollary 1.2] are immediate.

Actually, we can generalizes various results, for example, replace the condition "p is the smallest prime divisor of $|G|$" with "p is a prime divisor of $|G|$ such that $(|G|, p-1) = 1$" in [6, Corollary 1.2] and [8, Theorem 3.2].

Remark 3.4 The assumption that " $(|G|, p-1) = 1$ " or " p is the smallest prime " in the above Theorems and Corollaries could not be removed.

Example: let $G = S_3$ be a the symmetric group of degree 3 and choose $p = 3((|G|, 3-1) = 2 \neq 1)$. It is clear that the order of Sylow 3-subgroup P_3 of G is 3 and the maximal subgroup of P_3 is 1. Clearly, 1 is not only semi p-cover-avoiding in G but also belongs to $\mathscr{H}(G)$. However, G is not 3-nilpotent. ∎

Remark 3.5 The authors do not know if the hypothesis " G is p-solvable" can be removed or not. But It looks ok to be removed. If it is really removed then the minimal counter example may be a simple group of even order by assumption $(|G|, p-1) =$ and Theorem of Solvability of Finite Groups of Odd order, so we can use the Classification Theorem of Finite simple Groups to check if such kinds of simple counter examples go to contradictions. Surely it may be a long paper and we don't know if it can be realized.

4. On p-supersolvablity of finite groups

Theorem 4.1 *Let G be a p-solvable group and $P \in Syl_p(F_p(G))$, where p is a prime divisor of $|G|$. If every maximal subgroup of P is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-supersolvable.*

Proof. Let G be a counterexample of minimal order.

Let $\mathscr{M}(P) = \{P_1, ..., P_m\}$ be the set of all the maximal subgroups of P. By the hypotheses, $P_i (i = 1, ..., m)$ either is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$. Without loss of generality, let $1 \leq k \leq m$ such that $P_1, ..., P_k$ are semi p-cover-avoiding in G and $P_{k+1}, ..., P_m$ belong to $\mathscr{H}(G)$. Then

(1) $O_{p'}(G) = 1$

Consider the quotient group $G/O_{p'}(G)$. The fact $F_p(G/O_{p'}(G)) = F_p(G)/O_{p'}(G)$ implies that $F_p(G/O_{p'}(G)) = PO_{p'}(G)/O_{p'}(G)$, therefore $F_p(G/O_{p'}(G))$ is a p-group. For any maximal subgroup $H/O_{p'}(G)$ of $F_p(G/O_{p'}(G)) = PO_{p'}(G)\ /O_{p'}(G)$, there exists a maximal subgroup P_0 of P such that $H/O_{p'}(G) = P_0O_{p'}(G)/\ O_{p'}(G)$. Hence $P_iO_{p'}(G)/O_{p'}(G)(i = 1, ..., m)$ are all maximal subgroups of $F_p(G/O_{p'}(G))$. By Lemma 2.3 and Lemma 2.8, $P_iO_{p'}(G)/O_{p'}(G)(i = 1, ..., k)$ are semi p-cover-avoiding in G and $P_iO_{p'}(G)/O_{p'}(G)(i = k + 1, ..., m)$ belong to $\mathscr{H}(G/O_{p'}(G))$. Hence $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. The minimality of G implies that $O_{p'}(G) = 1$.

(2) $P = N_1 \times ... \times N_s$, where $N_1, ..., N_s$ are minimal normal subgroups of G.

Consider $G/\Phi(G)$. Since $G/\Phi(G)$ is a p-solvable group, $F_p(G/\Phi(G)) > 1$. Since $O_{p'}(G) = 1$, we have that $F_p(G) = P$. Clearly, $F_p(G/\Phi(G)) = F_p(G)/\Phi(G)$. By Lemma 2.3 and Lemma 2.5, $P_1/\Phi(G), ..., P_k/\Phi(G)$ are semi p-cover-avoiding in $G/\Phi(G)$ and $P_{k+1}/\Phi(G), ..., P_m/\Phi(G)$ belong to $\mathscr{H}(G/\Phi(G))$. It follows that $G/\Phi(G)$ satisfies the hypotheses of the theorem. The minimality of G implies that $\Phi(G) = 1$ and then $\Phi(P) = 1$. It follows then P is an elementary abelian p-group.

Let N be a minimal normal subgroup of G. Since G is p-solvable and $O_{p'}(G)$=1, $N \leq F_p(G) = O_p(G) = P$. By Lemma 2.14, the minimality of G implies that $|P| > p$. If $k = m$, then G is already p-supersolvable by [7, $Theorem$ 3.1]. So now we assume $k < m$. Then $P_{k+1}, ..., P_m$ are normal in G by Lemma 2.6. Hence $N < F_p(G) = O_p(G) = P$.

By Lemma 2.12, there exist minimal normal subgroups $N_1, ..., N_s$ of G which are contained in $F(F_p(G)) = F_p(G) = O_p(G) = P$ such that$P = N_1 \times ... \times N_s$, $s > 1$.

(3) $|N_i| = p$, $i = 1, ..., s$.

We just need to prove $|N_1| = p$. Since $\Phi(G) = 1$, there exists a maximal subgroup M of G such that $G = N_1M$ and $M \cap N_1 = 1$. Let $R = N_2 \times ... \times N_s$. It is easy to see that $R \leq M$. Let P_0 be a maximal subgroup of P such that $R \leq P_0$. Then $|N_1/N_1 \cap P_0| = p$. By the hypotheses, P_0 is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$.

If P_0 belongs to $\mathscr{H}(G)$, then by Lemma 2.6, $P_0 \trianglelefteq G$. Then $P_0 \cap N_1 = 1$ or $P_0 \cap N_1 = N_1$. If $P_0 \cap N_1 = N_1$, then $N_1 \leq P_0 = G_p^* \cap P \leq G_p^*$, a contradiction. Hence $P_0 \cap N_1 = 1$. It follows from $P = P_0N_1$ that $|N_1| = p$.

If P_0 is semi p-cover-avoiding in G, then it is clear that $|N_1| = p$ when P is cyclic. Now assume that P is not cyclic. Then P_0/R is semi p-cover-avoiding in G/R by Lemma 2.3. So there is a chief series of G/R:

$$1 = G_0/R < G_1/R < G_2/R < ... < G_s/R = G/R$$

such that P_0/R covers or avoids every p-singular chief factor of it. It is easy to see that P_0 covers or avoids every p-singular chief factor of the following chief series of G:

$$1 < N_2 < N_2N_3 < ... < N_2N_3...N_s = R = G_0 < G_1 < ... < G_s = G$$

Since $P \nleqslant G_0 \cdot P_0$ but $P \leqslant G_s \cdot P_0 = G$, there is a chief factor G_i/G_{i-1} such that $P \leqslant G_i \cdot P_0$ but $P \nleqslant G_{i-1} \cdot P_0$. Also since P is a p-group, it is clear that G_i/G_{i-1} is a p-singular chief factor. Noticing that P_0 covers or avoids G_i/G_{i-1} and $G_i \cdot P_0 \neq G_{i-1} \cap P_0$, we have $G_i \cap P_0 = G_{i-1} \cap P_0$. Since $P = P \cap G_i P_0 = P_0(G_i \cap P)$, $G_i \cap P > G_0 = R$. It follows from $G_i \cap P \trianglelefteq G$ that $G_i \cap P = P$ and therefore $P_0 = G_i \cap P_0 = G_{i-1} \cap P_0$. On the other hand, $P > P \cap G_{i-1}P_0 = P_0(G_{i-1} \cap P)$. Thus $P \cap G_{i-1} = G_0 = M$ since $P \cap G_{i-1} \trianglelefteq G$, so $G_{i-1} \cap P_0 = R$. Hence $P_0 = R$ and therefore $|N_1| = p$.

Similarly, we can get $|N_i| = p$ for $i > 1$.

(4) The final contradiction.

By (3), we have got that $F_p(G) = N_1 \times ... \times N_s$, where $N_i (i = 1, ..., s)$ are minimal normal subgroups of G of order p. Then G is p-supersolvable by Lemma 2.14, the final contradiction. ∎

From Theorem 4.1, we immediate get the following corollaries:

Corollary 4.2 [7, Theorem 3.1] *Let p be a prime divisor of the order of a p-solvable group G and P a Sylow p-subgroups of the p-Fitting subgroup of G. If P is cyclic or every maximal subgroup of P is semi p- cover-avoiding in G, then G is p-supersolvable.*

Corollary 4.3 [15, Theorem 3.1] *Let G be a p-solvable group and P a Sylow p-subgroup of the p-Fitting subgroup $F_p(G)$ of G, where p is a prime dividing the order of G. If every maximal subgroup of P belongs to $\mathscr{H}(G)$, then G is p-supersolvable.*

Corollary 4.4 *Let H be a p-solvable normal subgroup of a group G such that G/H is p-supersolvable. If every maximal subgroup of Sylow p-subgroups of $F_p(H)$ is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-supersolvable.*

Proof. Assume that the corollary is incorrect and let G be a counterexample with the smallest order.

Consider the quotient group $G/O_{p'}(H)$. By using the same arguments as in the proof of Theorem 4.1, we can see that $G/O_{p'}(H)$ satisfies the hypotheses of the corollary. The minimality of G implies that $O_{p'}(H) = 1$.

Similar to the proof of Theorem 4.1, we may assume that $\Phi(H) = 1$. Thus, $F(F_p(H)) = F_p(H) = O_p(H)$ is the direct product of minimal normal subgroups of G contained in H by Lemma 2.12. Let $F_p(H) = N_1 \times ... \times N_s$, where $N_i (i = 1, ..., s)$ are minimal normal subgroups of G contained in H. By the same arguments as in Theorem 4.1, we have that $N_i (i = 1, ..., s)$ are all cyclic groups of order p. Then we can assume that $F_p(H) = \langle a_1 \rangle \times \langle a_2 \rangle \times \cdots \times \langle a_s \rangle$, where $\langle a_i \rangle (i = 1, ..., s)$ are normal in G with prime order p.

Since $G/C_G(\langle a_i\rangle) \lesssim Aut(\langle a_i\rangle)$ and $\langle a_i\rangle$ is cyclic of order p, $G/C_G(\langle a_i\rangle)$ is cyclic and of course p-supersolvable for each i. It follows then $G/(\cap_{i=1}^{s} C_G(\langle a_i\rangle))$ is also p-supersolvable. Then $G/C_G(F_p(H))$ is p-supersolvable since $C_G(F_p(H)) = \cap_{i=1}^{s} C_G(\langle a_i\rangle)$. Now we see that $G/C_H(F_p(H)) = G/(H \cap C_G(F_p(H)))$ is p-supersolvable. Clearly $F_p(H) \leq C_H(F_p(H))$. On the other hand, $C_H(F_p(H)) = C_H(O_p(H)) \leq O_p(H) = F_p(H)$ by Lemma 2.15 since H is p-solvable and $O_{p'}(H)$=1. Hence $F_p(H) = C_H(F_p(H))$ and so $G/F_p(H)$ is p-supersolvable. Since $F_p(H) = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_s\rangle$ is a direct product of cyclic normal subgroups of G, G is p-supersolvable, the final contradiction. ∎

Theorem 4.5 *Let $\mathcal{F}$ be a saturated formation containing the class of all supersolvable groups $\mathcal{U}$ and let H be a solvable normal subgroup of a group G such that $G/H \in \mathcal{F}$. If, for every prime p dividing the order of $F(H)$, every maximal subgroup of Sylow p-subgroups of $F(H)$ is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then $G \in \mathcal{F}$.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. Then we have:

(1) $\Phi(H) = 1$.

Otherwise, let r be a prime divisor of $|\Phi(H)|$ and take $R \in Syl_r(\Phi(H))$. Easy to see $R \trianglelefteq G$ and $(G/R)/(H/R) \cong G/H \in \mathcal{F}$. Clearly, $R \leq F(H)$ and $F(H/R) = F(H)/R$.

It is easy to see that G/R always satisfies the hypotheses of the theorem respect to the normal subgroup H/R. The minimality of G implies that $G/R \in \mathcal{F}$. Since $G/\Phi(G) \cong (G/R)/(\Phi(G)/R) \in \mathcal{F}$ and $\mathcal{F}$ is a saturated formation, we see that $G \in \mathcal{F}$, a contradiction.

(2) $F(H) = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_n\rangle$, where $\langle a_i\rangle (i = 1, ..., n)$ are normal in G with prime order. Furthermore, $G/F(H) \in \mathcal{F}$.

Let P be a Sylow p-subgroup of $F(H)$. Then $P \trianglelefteq G$. By $\Phi(H) = 1$ and Lemma 2.12, $P = N_1 \times N_2 \times \cdots \times N_s$, where N_i are all minimal normal in G. As in Theorem 4.1, we can see that every N_i is a cyclic group of order p. Then $F(H) = \langle a_1\rangle \times \langle a_2\rangle \times \cdots \times \langle a_n\rangle$, where $\langle a_i\rangle (i = 1, ..., n)$ are normal subgroups of prime orders in G.

Since $G/C_G(\langle a_i\rangle)$ is isomorphic to a subgroup of $Aut(\langle a_i\rangle)$, $G/C_G(\langle a_i\rangle)$ is cyclic and therefore it lies in $\mathcal{U}$ for each i. Thus $G/(\cap_{i=1}^{n} C_G(\langle a_i\rangle)) \in \mathcal{U}$. Since $C_G(F(H)) = \cap_{i=1}^{n} C_G(\langle a_i\rangle)$, $G/C_G(F(H)) \in \mathcal{U} \subseteq \mathcal{F}$. Then $G/C_H(F(H)) = G/(H \cap C_G(F(H))) \in \mathcal{F}$. Noticing that $F(H)$ is abelian, we have $F(H) \leq C_H(F(H))$. On the other hand, $C_H(F(H)) \leq F(H)$ since H is solvable. Hence $F(H) = C_H(F(H))$ and so $G/F(H) \in \mathcal{F}$.

(3) If N is a minimal normal subgroup of G which is contained in H, then $G/N \in \mathcal{F}$.

Suppose that N is a minimal normal subgroup of G which is contained in H. Since H is solvable, N is an elementary abelian p-group for some prime p and so $N \leq F(H)$. Now consider the solvable normal subgroup $F(H)/N$ in G/N. It is clear that $(G/N)/(F(H)/N) \simeq G/F(H) \in \mathcal{F}$ and $F(F(H)/N) = F(H)/N$.

So it is easy to see that G/N satisfies the hypotheses of our theorem respect to the solvable normal subgroup $F(H)/N$. By the minimality of G, we have that $G/N \in \mathcal{F}$.

(4) The final contradiction.

Since G is a counterexample of minimal order and take into account (2), we see that $F(H) = N$ is a unique minimal normal subgroup of G which is contained in H and $|F(H)| = |N| = p$. Suppose that K is a minimal normal subgroup of G such that $K \neq N$. Then $NK/K \simeq N$ is a solvable normal subgroup of G/K and $F(NK/K) = NK/K$ is a cyclic group. Thus G/K satisfies the hypotheses of our theorem for the normal subgroup KN/K. So by the minimality of G, we have $G/K \in \mathcal{F}$. It follows then $G \in \mathcal{F}$ since $K \neq N$, a contradiction. Hence $F(H) = N$ is the unique minimal normal subgroup of G.

If $N \leqslant \Phi(G)$, then $G/\Phi(G) \in \mathcal{F}$ by $G/N \in \mathcal{F}$, so that $G \in \mathcal{F}$, a contradiction. Otherwise, there exists a maximal subgroup M of G such that $N \nleq M$. Then $G = MN$ and $M \cap N = 1$. If $N < C_G(N)$, then $1 < C_G(N) \cap M$. It is clear that $C_G(N) \cap M \trianglelefteq G$. The unique minimal normality of N implies that $N \leq C_G(N) \cap M \leq M$, a contradiction. Hence $C_G(N) = N$, which yields that $G/N = G/C_G(N)$ is a cyclic group of order dividing $p-1$ and therefore $G \in \mathcal{U} \subseteq \mathcal{F}$ by noticing that N is of prime order, the final contradiction. ∎

Corollary 4.6 *Let H be a solvable normal subgroup of a group G such that G/H is supersolvable. If, for every prime p dividing the order of $F(H)$, every maximal subgroup of Sylow p-subgroups of $F(H)$ is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is supersolvable, where $F(H)$ is the Fitting subgroup of H.* ∎

Finally, we get a sufficient condition of p-nilpotency by using Theorem 4.1 and limiting p be the smallest prime dividing the order of a group G.

Theorem 4.7 *Let p be the smallest prime divisor of the order of a p-solvable group G. If every maximal subgroup of Sylow p-subgroups of $F_p(G)$ is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-nilpotent.*

Proof. Assume that the theorem is false and let G be a counterexample with the minimal order. Then, by the arguments used in the proof of Theorem 4.1, it is easy to see that $O_{p'}(G) = 1$, $\Phi(G) = 1$ and $F_p(G) = N_1 \times ... \times N_s$, where $N_i(i = 1, ..., s)$ are minimal normal subgroups of G of order p.

For any N_i, $G/C_G(N_i)$ is a subgroup of a cyclic group of order $p - 1$. The minimality of p implies that $C_G(N_i) = G$, hence $G = C_G(O_p(G))$. On the other hand, by Lemma 2.15, $C_G(O_p(G)) \leq O_p(G)$. It follows that $G = O_p(G)$ is p-nilpotent, a contradiction. The proof of the theorem is complete. ∎

From Theorem 4.7, we immediately get the following results:

Corollary 4.8 [7, Theorem 3.3] *Let p be the smallest prime dividing the order of a p-solvable group G. If Sylow p-subgroup of $F_p(G)$ are cyclic or every maximal subgroup of Sylow p-subgroups of $F_p(G)$ is semi p- cover-avoiding in G, then G is p-nilpotent.*

Corollary 4.9 [15, Theorem 3.3] *Let p be the smallest prime divisor of the order of a p-solvable group G. If every maximal subgroup of Sylow p-subgroups of $F_p(G)$ belongs to $\mathscr{H}(G)$, then G is p-nilpotent.*

Corollary 4.10 *Let p be the smallest prime divisor of the order of a group G and H be a p-solvable normal subgroup of G such that G/H is p-nilpotent. If every maximal subgroup of Sylow p-subgroups of $F_p(H)$ is semi p-cover-avoiding in G or belongs to $\mathscr{H}(G)$, then G is p-nilpotent, where $F_p(H)$ is the p-Fitting subgroup of H.*

Proof. Since the proof is similar to the ones of Corollary 4.4 and Theorem 4.7, we omit the proof. ∎

Remark 4.11 The assumption that p is the smallest prime divisor of the order of G in Theorem 4.7 and Corollary 4.8, 4.9, 4.10 could not be removed.

Example: let $G = S_4$, the symmetric group of degree 4 and $p = 3$. It is clear that $F_p(G) = A_4$ and the maximal subgroup 1 of Sylow 3-subgroups of $F_p(G)$ is not only semi p-cover-avoiding in G but also belongs to $\mathscr{H}(G)$. However, G is not 3-nilpotent. ∎

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