THE QUOTIENT ULTRA-GROUPS

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Abstract. The main purpose of this paper is to construct the quotient ultra-groups, which are based on congruences. By these means we can present fundamental theorem and Lagrange theorem for the ultra-groups. Also, we show not necessary that the order of each element of the ultra-group divided by the order of the ultra-group.

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1. Introduction

The notion of congruence was first introduced by Karl Fredrich Gauss in the beginning of the nineteenth century. Congruence is a special type of equivalence relation which plays a vital role in the study of quotient structures of different algebraic structures (see [6]). In this paper we study the quotient structure of ultra-group by using the notion of congruence in ultra-group and study some interesting properties ultra-groups homomorphism. The notion of ultra-group over a group was introduced in [4]. In fact an ultra-group is an algebraic structure whose underlying set is depend on a group and its subgroup. Moreover, an abelian ultra-group is equivalence to quasigroup and finite loop (see [5]). We present now some basic definitions and results about ultra-groups. Let H be a subgroup in group G. A subset M of group G is called a (right) transversal to H in G if $G = \bigcup_{m\in M} Hm$. Therefore, the pair (H, M) is a (right) transversal if and only if

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the subset M of the group G obtained by selecting one and only one member from each right coset of G modulo subgroup H of G. Throughout the paper, a right (left) transversal will be assumed to contain the identity of the group (see [1]). For the group G, which satisfies the above conditions, we have $MH \subseteq G = HM$, and subset M of G is called (right unitary) complementary set with respect to subgroup H. This notion was introduced by Kurosh in [3] which is the base of the concept of an ultra-group. Therefore, for every element $mh \in MH$ there exists unique $h' \in H$ and $m' \in M$ such that $mh = h'm'$. We denote h' and m' by ${}^{m}h$ and m^h, respectively. Similarly, for any elements $m_1, m_2 \in M$, there exists unique elements $[m_1, m_2] \in M$ and $(m_1, m_2) \in H$ such that $m_1 m_2 = (m_1, m_2)[m_1, m_2]$. Similarly for every element $a \in M$, there exists a^{-1} belonging to G. As $G = HM$, there is $a^{(-1)} \in H$ and $a^{[-1]} \in M$ such that $a^{-1} = a^{(-1)}a^{[-1]}$. Now, we are ready to define an ultra-group.

Definition 1.1 A (*right*) *ultra-group* ${}_HM$ is a complementary set of H over group G with a binary operation $\alpha :_H M \times_H M \to_H M$ and unary operation $\beta_h :_H M \to_H M$ defined by $\alpha((m_1, m_2)) := [m_1, m_2]$ and $\beta_h(m) := m^h$ for all $h \in H$.

It is easy to see that every group is an ultra-group. The following example is an ultra-group which is not a group.

Example 1.2 Let $G = S_3$. Then $H = \{(1), (13)\}$ is a subgroup of G. By $H \setminus G = \{\{(1), (13)\}, \{(23), (123)\}, \{(12), (132)\}\}$ we have four possibility for set M. For example let $M = \{(1), (12), (23)\}$, then M is an ultra-group with the following maps :

α	(1)	(23)	(12)
(1)	(1)	(23)	(12)
(23)	(23)	(1)	(23)
(12)	(12)	(12)	(1)

β	(1)	(13)
(1)	(1)	(1)
(23)	(23)	(12)
(12)	(12)	(23)

Example 1.3 Let $G = C^*$ and $H = U$, where C^* be the set of complex numbers without zero and $U = \{z \in C^* : |z| = 1\}$. The set C^* under Multiplication is an infinite abelian group and $U \leq C^*$. The set $U \backslash C^*$ consists of concentric circles. So, an obvious (left or right) ultra-group for U in C^* is positive real numbers R^+, which is even a subgroup of $U \backslash C^*$. Another (left or right) ultra-group for U in C^* is $R^- = \{x : x < 0\}$, which is not a subgroup of C^*.

Notice that there are many other choices of ultra-groups are available.

Example 1.4 Let G be the symmetric group of order 4. This group contains three subgroups H_i of order 8 where $i = 1, 2, 3$:

$$H_1 = \{(1), (12)(34), (13)(24), (14)(23), (12), (34), (1324), (1423)\};$$
$$H_2 = \{(1), (12)(34), (13)(24), (14)(23), (13), (24), (1234), (1432)\};$$
$$H_3 = \{(1), (12)(34), (13)(24), (14)(23), (14), (23), (1243), (1342)\}.$$

Every H_i/S_4 has three elements for $i = 1, 2, 3$. It is not hard to show that any subgroup H_i has 64 (right) complementary sets M. Therefore ultra-group of order 3 in S_4 are 192.

A (*left*) *ultra-group* M_H is defined similarly via (left unitary) complementary set. From now on, unless specified otherwise, ultra-group means right ultra group. Although every element in right ultra-groups do not have right inverse, but they have left inverse with respect to the first binary operation α. The first binary operation of an ultra-group M has the right cancellation. We observe that we have not left cancellation for right ultra-groups. A subset $S \subseteq {}_HM$ which contains e, is called a *subultra-group* of H over G, if S is closed under the operations α and β_h in the Definition 1.1. It is obvious that $\{e\}$ is a trivial subultra-group for all ultra-groups ${}_HM$ where e is the identity element of H. A subultra-group N of ${}_HM$ is called *normal* if $[a,[N,b]] = [N,[a,b]]$, for all $a, b \in {}_HM$. In addition $[N,S]$ is a subultra-group of ${}_HM$, where S is a subultra-group of ${}_HM$. Moreover, $[N,S]$ is a normal subultra-group of ${}_HM$ if S is also normal subultra-group of ${}_HM$. Suppose A, B are two subsets of the ultra-group ${}_HM$. Moreover we use the notation $[A,B]$ for the set of all $[a,b]$, where $a \in A$ and $b \in B$. If B is a singleton $\{b\}$, then we denote $[A,B]$ by $[A,b]$.

Definition 1.5 Suppose ${}_{H_i}M_i$ is ultra-group of H_i over group G_i, $i = 1,2$. A function $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ is an ultra-group homomorphism provided that for all $m, m_1, m_2 \in {}_{H_1}M_1$ and $h \in H_1$.

(i) $f([m_1,m_2]) = [f(m_1), f(m_2)]$,

(ii) $(f(m))^{\varphi(h)} = f(m^h)$,
where φ is a group homomorphism between two subgroups H_1 and H_2.

If f is a surjective and injective ultra-group homomorphism, we call it isomorphism and denote it by ${}_{H_1}M_1 \cong {}_{H_2}M_2$.

Now, some properties of the ultra-group homomorphism as many algebras reviews.

Theorem 1.6 *Let $f : {}_{H_1}M_1 \longrightarrow {}_{H_2}M_2$ be an ultra-group homomorphism. Then*

(i) $f(e_{{}_{H_1}M_1}) = e_{H_2M_2}$,

(ii) $f(a^{[-1]}) = (f(a))^{[-1]}$ *for all* $a \in {}_{H_1}M_1$.

Proof. The assertion follows immediately from the first binary operation α of an ultra-group has the right cancellation and Definition 1.5. ∎

Definition 1.7 Let $f :_{H_1} M_1 \to {}_{H_2}M_2$ be ultra-groups homomorphism. Then $\mathrm{Ker}(f)$ is defined by

$$\{(m_1,m_2) \in_{H_1} M_1 \times {}_{H_1}M_1 : f(m_1) = f(m_2)\}.$$

In other words, if $(m_1,m_2) \in \mathrm{Ker}(f)$ then $f([m_1^{(m_2)^{(-1)}}, m_2^{[-1]}]) = f(e)$. Therefore $([m_1^{(m_2)^{(-1)}}, m_2^{[-1]}], e) \in \mathrm{Ker}(f)$. We can consider $\mathrm{Ker}(f)$ as a subset of the ${}_HM$ defined the following

$$\mathrm{Ker}(f) = \{m \in {}_{H_1}M_1 : f(m) = e\}.$$

Congruence is a special type of equivalence relation which plays a vital role in the study of quotient structures of different algebraic structures (see [2]).

2. Main result

The structure of an ultra-group ${}_HM$ over the group G is revealed by its subultra-groups and homomorphisms into other ultra-groups.

Lemma 2.1 *Let S be a subultra-group of ultra-group ${}_HM$ over the group G and $a, b \in {}_HM$. Then the following conditions are equivalent.*

(i) $a \in [S, b]$,

(ii) $[S, a] = [S, b]$,

(iii) $[a^{(b^{(-1)})}, b^{[-1]}] \in S$.

Proof. It follows that from the Definition of subultra-group. ∎

By Lemma 2.1 we have $[S, a] = [S, b]$ or $[S, a] \cap [S, b] = \emptyset$, which implies ${}_HM = \cup_{a \in {}_HM} [S, a]$.

Lemma 2.2 *For the ultra-group homomorphism f between two ultra groups ${}_{H_1}M_1$ and ${}_{H_2}M_2$ with kernel K, we deduce that $f(m_1) = f(m_2)$ is equivalent to the fact $m_1 = [k, m_2]$ for some $k \in K$.*

Proof. Suppose that $f(m_1) = f(m_2)$, so we have $[(f(m_1))^{(f(m_2))^{(-1)}}, (f(m_2))^{[-1]}]$ $= f([m_1^{m_2^{(-1)}}, m_2^{[-1]}]) = e$. This means that $[m_1^{m_2^{(-1)}}, m_2^{[-1]}] \in K$, thus $m_1 = [k, m_2]$ for some $k \in K$. Conversely $f(m_1) = f([k, m_2]) = [f(k), f(m_2)] = f(m_2)$. ∎

Theorem 2.3 (see [4]) *Suppose the same notations in the Definition* 1.7*. Then* Ker(f) *is a congruence on ${}_{H_1}M_1$ and it is a subultra-group of ${}_{H_1}M_1$.*

In the following establish a connection between congruence on an ultra-group and a normal subultra-group.

Lemma 2.4 *Let S be a subultra-group of ${}_HM$ and θ be a relation on ${}_HM$ defined by $a\theta b$ if and only if there is $s \in S$ such that $a = [s, b]$. Then θ is an equivalence relation.*

Proof. The assertion follows immediately from Lemma 2.1. ∎

From now on, we use the notation θ for the equivalence relation which is satisfied in the Lemma 2.4. For instance, we denote the equivalence class of e with respect to the equivalence relation of θ in the Lemma 2.4 by $[e]_\theta = S$ is a subultra-group of ${}_HM$.

Theorem 2.5 *If S is a subultra-group of ultra-group ${}_HM$ over the group G, then the equivalence relation θ is a congruence if and only if S is a normal subultra-group ${}_HM$.*

Proof. Let θ be a congruence. For every $b \in {}_HM$ by definition of θ we have $[s,b]\ \theta\ b$ for every $s \in S$. By θ is a congruence we have $[a,[s,b]]\theta[a,b]$. Therefore $[a,[s,b]] = [s^{'},[a,b]]$ for some $s^{'} \in S$. Thus S is a normal subultra-group of ${}_HM$. For converse, let $a\theta b$ and $a'\theta b'$, where $a,b,a',b' \in {}_HM$. Then by the definition of normal subultra-group we have $[a,a'] = [[n,b],[n',b]] = [n'',[b,b']]$. This means $[a,a']\theta[b,b']$, for $n,n',n'' \in N$. For compatibility of the second operation of the ultra-group, assume $a\theta b$. Thus $a^h = [n',b^h]$ and the definition of θ implies $a^h\theta b^h$, for all $h \in H$. ∎

A right coset of subultra-group S in ultra-group ${}_HM$ is a subset of the form $[S,a] = \{[s,a] : s \in S\}$. By two right cosets are either equal or disjoint. The set of right cosets of subultra-group S in ultra-group ${}_HM$ is denoted M/θ.

Theorem 2.6 *All right cosets of subultra-group S of the ultra-group ${}_HM$ over group G have the same cardinality.*

Proof. Define a mapping $f : [S,a] \to [S,b]$ by $f(x) = [[x^{a^{(-1)}}, a^{[-1]}], b]$. f is a mapping one-to-one and onto. ∎

This observation is the key to the very important theorem Lagrange theorem for ultra-groups.

Theorem 2.7 (Lagrange's theorem) *If S is a subultra-group of ${}_HM$, then the cardinality of S divides the cardinality of ${}_HM$.*

Proof. The distinct right cosets of S are mutually disjoint by Theorem 2.1 and each has the same size by Theorem 2.6. Since the union of the right cosets is ${}_HM$, the cardinality of ${}_HM$ is the cardinality of S times the number of distinct right cosets of S. ∎

Definition 2.8 An element a of an ultra-group ${}_HM$ over group G is an idempotent element, if $\alpha(a,a) = a$ where α is its first binary operation over it.

Corollary 2.9 *The only idempotent element in ultra-group ${}_HM$ is $e_{{}_HM}$.*

Proof. Suppose that a is idempotent element in ultra-group ${}_HM$. By $a.a = (a,a)[a,a]$ and $a = (a,a)$ we have $a \in {}_HM \cap H$. ∎

Corollary 2.10 *Every subultra-group of order 2 of ultra-group ${}_HM$ is a group.*

Proof. This follows immediately from $[a,a] = e$, where a of ultra-group ${}_HM$ over group G. ∎

Example 2.11 Suppose $M = \{e, a, a^5b, a^3, a^4, a^2b\}$ is an ultra-group over subgroup $H = \{e, ab\}$ arise from group

$$G = \{a^ib^j \mid a^6 = b^2 = e,\ a^ib = ba^{6-i},\ i = 0,\dots,5,\ j = 0,1\}.$$

Then, the only subultra-groups of ultra-group M are

$$N_1 = \{e\},\ N_2 = \{e,a^3\},\ N_3 = \{e,a^5b\},\ N_4 = \{e,a^2b\},\ N_5 = \{e,a^4,a^5b\}\ N_6 = G.$$

Although we have an associativity property for the groups, but this property is not valid for the binary operation α of the ultra-groups. Therefore, we convent $\alpha(a,a,a) = \alpha(\alpha(a,a),a) = \alpha^3(a)$ where a is the element of the ultra-group M and α is its first binary operation over it.

Definition 2.12 The order of $a \in_H M$ is the smallest positive power k such that $\alpha^k(a) = e$.

In group theory, the order of any element of a finite group divides the order of the group. In this case, with following example we show that this is not established in ultra-groups.

Example 2.13 Let $M = \{e, ab, a^2b\}$ be an ultra-group of order 3 arise from subgroup $H = \langle a^3, b\rangle = \{e, a^3, a^6, b, a^3b, a^6b\}$ over group D_9. Then the order of element ab is 2 with following table :

α	e	a^2b	ab
e	e	a^2b	ab
a^2b	a^2b	e	a^2b
ab	ab	ab	$e.$

3. Quotient ultra-groups and homomorphism theorems

As respects, in some algebraic structure can be defined the quotient algebra with congruence, we also define the quotient ultra-group.

Definition 3.1 Let M be an ultra-group over subgroup H_1 of group G and θ be a congruence over M. The set $M/\theta = \{[m]_\theta; m \in M\}$ with the operations α_θ and β_{θ_h},

(i) $\alpha_\theta([m]_\theta, [m']_\theta) = [\alpha(m, m')]_\theta$,

(ii) $\beta_{\theta_h}([m]_\theta) = [\beta_h(m)]_\theta$,

is an ultra-group of H_2 over the group G, where $H_1 \leq H_2 \leq G$. This ultra-group is called a *quotient ultra-group.*

Any ultra-group homomorphism $f :_{H_1} M_1 \rightarrow_{H_2} M_2$ induces an isomorphism from a quotient of ${}_{H_1}M_1$ to a subgroup of ${}_{H_2}M_2$. More precisely, we have the following.

Theorem 3.2 *(First isomorphism theorem for ultra-groups) Let f be a surjective ultra-group homomorphism between two ultra-groups ${}_{H_1}M_1$ and ${}_{H_2}M_2$ and θ a congruence over ${}_{H_1}M_1$ such that $\theta \subseteq \mathrm{Ker}(f)$. If $\pi :_{H_1} M_1 \longrightarrow_{H_1} M_1/\theta$ is the natural ultra-group homomorphism, then there exists a homomorphism g so that $g\pi = f$.*

Proof. It is enough to define the map $g :_{H_1} M_1/\theta \longrightarrow {}_{H_2}M_2$ by $g([m]_\theta) = f(m)$. Since $[m_1]_\theta = [m_2]_\theta \Leftrightarrow m_1 \ \theta \ m_2$, by the hypothesis $\theta \subseteq \mathrm{K}er(f)$ we have $f(m_1) = f(m_2)$ which implies g is well-defined. The map g preserves the first operation α_θ and the unary operation β_{θ_h}. Thus we can conclude that g is an ultra homomorphism. The map g preserve the first operation α_θ as follows, $g(\alpha_\theta([m_1]_\theta, [m_2]_\theta)) = g([[m_1, m_2]]_\theta) = [f(m_1), f(m_2)] = [g([m_1]_\theta), g([m_2]_\theta)]$. For the second operation β_{θ_h} we have, $g(\beta_{\theta_h}([m]_\theta)) = g([m^h]_\theta) = f(m^h) = (f(m))^{\varphi(h)} = \beta_{\varphi(h)}(g([m]_\theta))$. ∎

Theorem 3.3 *Let γ be a congruence on ${}_HM$ and θ be a relation on ${}_HM$ defined by $a\theta b$ if and only if there is $s \in S$ such that $a = [s, b]$ where $S = [e]_\gamma$. Then γ and θ are equivalent.*

Proof. By congruence definition and right cancellation property we conclude,

$$\begin{array}{rcl} a \ \gamma \ b & \Longleftrightarrow & a^{b^{(-1)}} \ \gamma \ b^{b^{(-1)}} \Longleftrightarrow [a^{b^{(-1)}}, b^{[-1]}] \ \gamma \ [b^{b^{(-1)}}, b^{[-1]}] \\ & \Longleftrightarrow & [a^{b^{(-1)}}, b^{[-1]}] \ \gamma \ e \Longleftrightarrow [a^{b^{(-1)}}, b^{[-1]}] \in S \\ & \Longleftrightarrow & a \ \theta \ b. \end{array}$$

∎

Theorem 3.4 (see [4]) *If γ is a congruence on the ultra-group ${}_HM$ and $[e]_\gamma = N$, then ${}_HM/\theta \cong_H M/N$.*

If γ, θ are congruences on ${}_HM$ and $\theta \subseteq \gamma$, then clearly $\gamma/\theta = \{(a/\theta, b/\theta) \in ({}_HM/\theta)^2 : (a, b) \in \gamma\}$ is a congruence on ${}_HM/\theta$. Recall that two sets S and T have the same cardinality if and only if there is a one-to-one mapping from one set onto the other.

Theorem 3.5 (Second isomorphism theorem of ultra-groups) *If N', N are normal subultra-groups of ultra-group ${}_HM$ such that $N \subseteq N'$, then $\frac{\frac{{}_HM}{N}}{\frac{N'}{N}} \cong \frac{{}_HM}{N'}$.*

Proof. By the argument before the theorem N'/N is a normal subultra-group of ${}_HM/N$. The map $\psi :_H M/N/N'/N \longrightarrow M/N'$ is a homomorphism with $N'/N \subseteq \mathrm{K}er(\psi)$ so the result follows by the first isomorphism theorem. ∎

The third isomorphism theorem [2, Theorem 2.6.18] which is valid for any algebra can be translate for ultra-groups. Although, we can prove the third isomorphism theorem for ultra-groups by the same method of the proof of the first isomorphism theorem. We need Lemma 3.6 in order to mimic the proof of Theorem 2.6.18 in [2].

Lemma 3.6 *Let B be a subultra-group of ${}_HM$ and θ a congruence on ${}_HM$. Then*

(i) $B^2 \cap \theta = (B \cap N)^2$,

(ii) $B^\theta = [N, B]$,

where N is $[e]_\theta$.

Proof. (i) With some basic properties of the congruence in hand, we have: $B^2 \cap \theta = \{(b_1, b_2) : b_1, b_2 \in B \text{ and } b_1\ \theta\ b_2\} = \{(b_1, b_2) : b_1, b_2 \in N\} = B^2 \cap N^2$.

(ii) By [2, Definition 2.6.16] we deduce $B^\theta = \{a \in_H M : \exists\ b \in B, a\ \theta\ b\} = \{a \in {}_HM : \exists\ b \in B, a = [n, b] \text{ for some } n \in N\} \subseteq [N, B]$. The rest is clear. ∎

Theorem 3.7 (Third isomorphism theorem) *If B is a subultra-group of ${}_HM$ and N is a normal subultra-group of ${}_HM$, then* $\frac{B}{B\cap N} \cong \frac{[N,B]}{N}$.

Proof. Since $[e]_\theta = B \cap N$ the proof is straightforward. ∎

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