THE INFLUENCE ON THE ASYMPTOTICS OF THE RANDOM WALKS CAUSED BY THE VARIATION OF THE INCREMENTS

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Abstract. Let $\{X_n: n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common mean $\mu \in (-\infty, 0)$ and $\{S_n : n \geq 0\}$ be the random walk generated by $\{X_n: n \geq 1\}$. For any $\varepsilon \in (0, -\mu)$, let $\{S_n^{(\pm\varepsilon)} : n \geq 0\}$ be the random walks generated by $\{X_n \pm \varepsilon: n \geq 1\}$. This paper considers the limits of $P\left(\sup\limits_{n\geq 0} S_n^{(\pm\varepsilon)} > x\right) / P\left(\sup\limits_{n\geq 0} S_n > x\right)$ as $x \to \infty$.
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1. Introduction

Let $\{X_n: n \geq 1\}$ be a sequence of independent and identically distributed random variables with a common (non-degenerate) distribution F on $\mathbf{R} = (-\infty, \infty)$. Let $\{S_n: n \geq 0\}$ be the random walk generated by $\{X_n: n \geq 1\}$ (or by F), where $S_0 = 0$ and $S_n = \sum\limits_{i=1}^{n} X_i$ for all $n \geq 1$. Let $\tau_+^s = \inf\{n \geq 1 : S_n > 0\}$ and $\tau_-^w = \inf\{n \geq 1 : S_n \leq 0\}$, then we call $S_{\tau_+^s}$ the first strictly ascending ladder height and $S_{\tau_-^w}$ the first weakly descending ladder height. Denote the distributions of $S_{\tau_+^s}$, $S_{\tau_-^w}$ by F_+^s, F_-^w, respectively. Put

$$A = \sum_{n=1}^{\infty} n^{-1} P(S_n \leq 0) \text{ and } B = \sum_{n=1}^{\infty} n^{-1} P(S_n > 0).$$

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Since $A+B=\infty$, only three cases may occur:

(i) $A<\infty$ (and $B=\infty$); i.e., $F_+^s(\infty)=1$ and $F_-^w(0^+)=1-e^{-A}$;

(ii) $B<\infty$ (and $A=\infty$); i.e., $F_+^s(\infty)=1-e^{-B}$ and $F_-^w(0^+)=1$;

(iii) $A=B=\infty$; i.e., $F_+^s(\infty)=F_-^w(0^+)=1$.

If $\int_{-\infty}^{\infty}|x|\mathrm{d}F(x)<\infty$, then we define $\mu=\int_{-\infty}^{\infty}x\mathrm{d}F(x)$. Then it is well-known that the cases (i)-(iii) are implied by $\mu>0$, $\mu<0$ and $\mu=0$, respectively. The converse implications are not true. See, for instance, Chapter VIII of Asmussen (2003).

We now state our motivation concretely. In the case $\mu\in(-\infty,0)$, the random walk $\{S_n:\ n\geq 0\}$ drifts to $-\infty$, and its supremum M is finite almost surely. And for any $\varepsilon\in(0,-\mu)$, let $\{S_n^{(\pm\varepsilon)}:\ n\geq 0\}$ be the random walks generated by $X_n\pm\varepsilon, n\geq 1$. Then $\{S_n^{(\pm\varepsilon)}:\ n\geq 0\}$ also drift to $-\infty$ and their suprema $M^{(\pm\varepsilon)}$ are finite almost surely. Denote by W and $W^{(\pm\varepsilon)}$ the distributions of M and $M^{(\pm\varepsilon)}$, respectively. What we are concerned about is, whether the limits $\lim\limits_{x\to\infty}\frac{\overline{W^{(\pm\varepsilon)}}(x)}{\overline{W}(x)}$ exist? If the limits exist and are denoted by $h(\pm\varepsilon)$, then as ε tends to 0, whether $h(\pm\varepsilon)$ tend to 1? We discuss these two problems in three cases in which the asymptotics of the suprema can be derived, namely, the Cramér case, the intermediate case and the subexponential case, see, for instance, Veraverbeke (1977).

We now introduce some basic notations and theories on random walks. Let $\tau_+^w=\inf\{n\geq 1: S_n\geq 0\}$ and $\tau_-^s=\inf\{n\geq 1: S_n<0\}$, then we call $S_{\tau_+^w}$ the first weakly ascending ladder height and $S_{\tau_-^s}$ the first strictly descending ladder height. Denote the distributions of $S_{\tau_+^w}$, $S_{\tau_-^s}$ by F_+^w, F_-^s, respectively. Then it is well-known that

$$F_+^w=\zeta\delta_0+(1-\zeta)F_+^s$$

and

$$F_-^w=\zeta\delta_0+(1-\zeta)F_-^s,$$

where $\zeta=P(S_{\tau_-^w}=0)=P(\tau_-^w\neq\tau_-^s)<1$ and δ_0 is the distribution degenerated at zero, see, for instance, Proposition 1.1 of Section VIII of Asmussen (2003). By Theorem 2 of XII.7 of Feller (1971), we know that $F_+^w(\infty)<1$ if and only if $F_+^s(\infty)<1$, and when this happens, $F_+^w(\infty)=1-\exp\{-\sum\limits_{n=1}^{\infty}n^{-1}P(S_n\geq 0)\}$. By simple calculation, one has $\zeta=1-\exp\{-\sum\limits_{n=1}^{\infty}n^{-1}P(S_n=0)\}$.

Let f, f_+^s, f_-^w be the Fourier-Stieltjes transforms of X_1, $S_{\tau_+^s}$, $S_{\tau_-^w}$, that is,

$$\begin{aligned} f(t) &= \int_{-\infty}^{\infty}e^{itx}\mathrm{d}F(x), t\in\mathbf{C},\\ f_+^s(t) &= \int_{0^+}^{\infty}e^{itx}\mathrm{d}F_+^s(x), t\in\mathbf{C}, \end{aligned}$$

and

$$f_-^w(t) \quad = \quad \int_{-\infty}^{0^+} e^{itx} \mathrm{d}F_-^w(x), t \in \mathbf{C},$$

where $\mathbf{C}$ denotes the complex plane. The well-known Wiener-Hopf factorization says

$$1 - f(t) = (1 - f_+^s(t))(1 - f_-^w(t)), t \in \mathbf{C}.$$

Further, we have the following representations:

$$(1.1) \qquad 1 - f_+^s(t) = \exp\bigg\{ - \sum_{n=1}^{\infty} n^{-1} \int_{0^+}^{\infty} e^{itx} \mathrm{d}F^{*n}(x) \bigg\}, \quad Im(t) \geq 0,$$

$$(1.2) \qquad 1 - f_-^w(t) = \exp\bigg\{ - \sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0^+} e^{itx} \mathrm{d}F^{*n}(x) \bigg\}, \quad Im(t) \leq 0,$$

where $Im(t)$ is the imaginary part of t and F^{*n} is the n-fold convolution of F, with the convention that $F^{*1} = F$ and F^{*0} is the distribution degenerated at 0. See, for instance, Veraverbeke (1977, Pages 27-28). Moreover, denote by f_+^w, f_-^s the Fourier-Stieltjes transforms of $S_{\tau_+^w}$, $S_{\tau_-^s}$, similar to (1.1) and (1.2), we have

$$1 - f_+^w(t) = \exp\bigg\{ - \sum_{n=1}^{\infty} n^{-1} \int_{0^-}^{\infty} e^{itx} \mathrm{d}F^{*n}(x) \bigg\}, \quad Im(t) \geq 0,$$

$$(1.3) \qquad 1 - f_-^s(t) = \exp\bigg\{ - \sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0-} e^{itx} \mathrm{d}F^{*n}(x) \bigg\}, \quad Im(t) \leq 0,$$

see, for instance, (3.9) and (3.17) of Section XVIII.3 of Feller (1971).

In particular, for any $t \in \mathbf{R}$, we denote $f(-it)$, $f_+^s(-it)$, $f_-^w(-it)$ by $\widehat{F}(t)$, $\widehat{F_+^s}(t)$, $\widehat{F_-^w}(t)$, respectively. Denote $\gamma_F = \sup\{t \in \mathbf{R} : \widehat{F}(t) < \infty\}$ and $\beta_F = \inf\{t \in \mathbf{R} : \widehat{F}(t) < \infty\}$, then by the Wiener-Hopf factorization, it is obvious that $\gamma_F = \sup\{t \in \mathbf{R} : \widehat{F_+^s}(t) < \infty\}$ and $\beta_F = \inf\{t \in \mathbf{R} : \widehat{F_-^w}(t) < \infty\}$. We may find that γ_F has an important influence on the asymptotics of the supremum of a random walk. For convenience, we write $\gamma = \gamma_F$.

For any $\varepsilon \in (0, -\mu)$, we denote the first ladder epochs of the random walk $\{S_n^{(+\varepsilon)} : \ n \geq 0\}$ by

$$\begin{aligned} \tau_+^{s,(+\varepsilon)} &= \inf\{n \geq 1 : S_n^{(+\varepsilon)} > 0\}, \\ \tau_-^{w,(+\varepsilon)} &= \inf\{n \geq 1 : S_n^{(+\varepsilon)} \leq 0\}, \\ \tau_+^{w,(+\varepsilon)} &= \inf\{n \geq 1 : S_n^{(+\varepsilon)} \geq 0\}, \\ \tau_-^{s,(+\varepsilon)} &= \inf\{n \geq 1 : S_n^{(+\varepsilon)} < 0\}, \end{aligned}$$

denote the corresponding ladder heights by $S_{\tau_+^{s,(+\varepsilon)}}$, $S_{\tau_-^{w,(+\varepsilon)}}$, $S_{\tau_+^{w,(+\varepsilon)}}$, $S_{\tau_-^{s,(+\varepsilon)}}$, and denote the distributions of the ladder heights by $F_+^{s,(+\varepsilon)}$, $F_-^{w,(+\varepsilon)}$, $F_+^{w,(+\varepsilon)}$, $F_-^{s,(+\varepsilon)}$, respectively. Similarly, let $\tau_+^{s,(-\varepsilon)}$, $\tau_-^{w,(-\varepsilon)}$, $\tau_+^{w,(-\varepsilon)}$, $\tau_-^{s,(-\varepsilon)}$ be the ladder epochs of the random walk $\{S_n^{(-\varepsilon)} : n \geq 0\}$, $S_{\tau_+^{s,(-\varepsilon)}}$, $S_{\tau_-^{w,(-\varepsilon)}}$, $S_{\tau_+^{w,(-\varepsilon)}}$, $S_{\tau_-^{s,(-\varepsilon)}}$ be corresponding ladder heights, and $F_+^{s,(-\varepsilon)}$, $F_-^{w,(-\varepsilon)}$ $F_+^{w,(-\varepsilon)}$, $F_-^{s,(-\varepsilon)}$ be the distributions of corresponding ladder heights.

Hereafter, unless otherwise stated, a limit is taken as $x \to \infty$. For two positive functions $a(\cdot)$ and $b(\cdot)$, we write $a(x) = O(b(x))$ if $\limsup a(x)/b(x) < \infty$, $a(x) = o(b(x))$ if $\lim a(x)/b(x) = 0$ and $a(x) \sim b(x)$ if $\lim a(x)/b(x) = 1$.

We say that a distribution V belongs to the exponential distribution class with index $\alpha \geq 0$, denoted by $V \in \mathcal{L}(\alpha)$, if $\overline{V}(x) > 0$ for all x and $\overline{V}(x - t) \sim e^{\alpha t}\overline{V}(x)$ for any $t \in \boldsymbol{R}$. It is obvious that if $V \in \mathcal{L}(\alpha)$ for some $\alpha \geq 0$, then $\alpha = \gamma_V$.

We say that a proper distribution V belongs to the convolution equivalent distribution class with some index $\alpha \geq 0$, denoted by $V \in \mathcal{S}(\alpha)$, if $V \in \mathcal{L}(\alpha)$, $\widehat{V}(\alpha) < \infty$, and $\overline{V^{*2}}(x) \sim 2\widehat{V}(\alpha)\overline{V}(x)$.

In particular, we call $\mathcal{L}(0)$ the long-tailed distribution class and $\mathcal{S}(0)$ the subexponential distribution class, denoted by $\mathcal{L}$ and $\mathcal{S}$ respectively. In addition, if V is supported on $[0, \infty)$ and $\alpha = 0$, the premise condition $V \in \mathcal{L}$ can be omitted in the definition of the subexponential distribution class.

The rest of this paper is organized as follows. The main results of this paper, as well as some existing results on the asymptotics of the suprema of random walks, are presented in Section 2. And in Section 3, the proofs of our main results are presented.

2. Some related theories and the main results

For a random variable X or its distribution F with $\int_0^\infty \overline{F}(y)\mathrm{d}y < \infty$, we define its integrated tail distribution by

$$F^I(x) = \begin{cases} \int_0^x \overline{F}(t)dt, & \text{if } x > 0; \\ 0, & \text{if } x \leq 0. \end{cases}$$

Theorem 2 of Veraverbeke (1977) presented the following results.

Theorem 2.A. *Suppose $B < \infty$.* (A) *Suppose $\gamma > 0$.*

(i) *If $\widehat{F}(\gamma) > 1$, then*

$$\overline{W}(x) \sim \frac{1 - F_+^s(\infty)}{\kappa} \cdot \frac{1 - \widehat{F_-^w}(\kappa)}{\widehat{F}'(\kappa)} e^{-\kappa x}, \tag{2.1}$$

where $\kappa \in (0, \gamma)$ such that $\widehat{F}(\kappa) = 1$.

(ii) *If $\widehat{F}(\gamma) = 1$ and $\widehat{F}'(\gamma) < \infty$, then*

$$\overline{W}(x) \sim \frac{1 - F_+^s(\infty)}{\gamma} \cdot \frac{1 - \widehat{F_-^w}(\gamma)}{\widehat{F}'(\gamma)} e^{-\gamma x}. \tag{2.2}$$

If $\widehat{F}(\gamma) = 1$ *and* $\widehat{F}'(\gamma) = \infty$, *then*

$$\overline{W}(x) = o(e^{-\gamma x}). \tag{2.3}$$

(iii) *If* $\widehat{F}(\gamma) < 1$, *then* (2.3) *holds. Further, if* $F \in \mathcal{S}(\gamma)$, *then* $W \in \mathcal{S}(\gamma)$ *and*

$$\overline{W}(x) \sim \frac{1 - F_+^s(\infty)}{(1 - \widehat{F}(\gamma))(1 - \widehat{F_+^s}(\gamma))} \overline{F}(x); \tag{2.4}$$

(B) *Suppose* $\gamma = 0$. *If* $\mu < 0$ *and* $F^I \in \mathcal{S}$, *then* $W \in \mathcal{S}$ *and*

$$\overline{W}(x) \sim -\frac{1}{\mu} \int_x^\infty \overline{F}(t)\mathrm{d}t. \tag{2.5}$$

We may find that in the above theorem, the variations of the increments affect the asymptotics of the supremum. Below are our main results.

Theorem 2.1. *Suppose* $\mu < 0$.

(A) *Suppose* $\gamma > 0$.

(i) *If* $\widehat{F}(\gamma) \geq 1$, *then for any* $\varepsilon \in (0, -\mu)$,

$$\overline{W}(x) = o\left(\overline{W^{(+\varepsilon)}}(x)\right). \tag{2.6}$$

In particular, if $\widehat{F}(\gamma) > 1$, *then*

$$\overline{W^{(-\varepsilon)}}(x) = o(\overline{W}(x)).$$

(ii) *Suppose that* $\widehat{F}(\gamma) < 1$. *If* $F \in \mathcal{S}(\gamma)$, *then for any* $\varepsilon \in (0, -\frac{\ln \widehat{F}(\gamma)}{\gamma})$, *the limits* $\lim \frac{\overline{W^{(\pm\varepsilon)}}(x)}{\overline{W}(x)}$ *exist and*

$$\lim_{\varepsilon \downarrow 0} \lim \frac{\overline{W^{(\pm\varepsilon)}}(x)}{\overline{W}(x)} = 1. \tag{2.7}$$

(B) *Suppose* $\gamma = 0$. *If* $F^I \in \mathcal{S}$, *then for any* $\varepsilon \in (0, -\mu)$,

$$\overline{W^{(\pm\varepsilon)}}(x) \sim \frac{\mu}{\mu \pm \varepsilon} \overline{W}(x). \tag{2.8}$$

3. Proof of Theorem 2.1

The following two lemmas play important roles in the proof of Theorem 2.1. By the dominanted convergence theorem, the first one is obvious.

Lemma 3.1. *If $\mu < 0$, then as $\varepsilon \downarrow 0$,*

$$\begin{array}{rcl} F_+^{s,(+\varepsilon)}(\infty) & \longrightarrow & F_+^{w}(\infty), \\ F_+^{s,(-\varepsilon)}(\infty) & \longrightarrow & F_+^{s}(\infty), \\ F_+^{w,(+\varepsilon)}(\infty) & \longrightarrow & F_+^{w}(\infty), \\ F_+^{w,(-\varepsilon)}(\infty) & \longrightarrow & F_+^{s}(\infty). \end{array}$$

Lemma 3.2. *Suppose that $\mu < 0$ and $\gamma > 0$. For any $\alpha \in [0, \gamma]$, if $\widehat{F}(\alpha) < 1$, then as $\varepsilon \downarrow 0$,*

$$\begin{array}{rcl} \widehat{F_-^{w,(+\varepsilon)}}(\alpha) & \longrightarrow & \widehat{F_-^{s}}(\alpha), \\ \widehat{F_-^{w,(-\varepsilon)}}(\alpha) & \longrightarrow & \widehat{F_-^{w}}(\alpha), \\ \widehat{F_-^{s,(+\varepsilon)}}(\alpha) & \longrightarrow & \widehat{F_-^{s}}(\alpha), \\ \widehat{F_-^{s,(-\varepsilon)}}(\alpha) & \longrightarrow & \widehat{F_-^{w}}(\alpha). \end{array}$$

Proof. We only prove the first conclusion, and the others can be proved similarly.

If $\widehat{F}(\alpha) < 1$, then there exist two positive constants ε_0 and δ such that for all $0 < \varepsilon < \varepsilon_0$, $Ee^{\alpha(X_1+\varepsilon)} < e^{-\delta}$, then for all $n \geq 1$,

$$\int_{-\infty}^{\infty} e^{\alpha x} \mathrm{d}P(S_n \leq x) e^{n\alpha\varepsilon} < e^{-\delta n}.$$

So by the dominated convergence theorem, when $\varepsilon \downarrow 0$, we have

$$\sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0^+} e^{\alpha(x+n\varepsilon)} \mathrm{d}P(S_n \leq x) \to -\log\{1 - \widehat{F_-^{w}}(\alpha)\}, \tag{3.1}$$

and

$$\sum_{n=1}^{\infty} n^{-1} \int_{-n\varepsilon}^{0^+} e^{\alpha(x+n\varepsilon)} \mathrm{d}P(S_n \leq x) \to \sum_{n=1}^{\infty} n^{-1}[F^{*n}(0^+) - F^{*n}(0^-)]. \tag{3.2}$$

By (3.1) and (3.2), when $\varepsilon \downarrow 0$, we have

$$\begin{aligned} 1 - \widehat{F_-^{w,(+\varepsilon)}}(\alpha) &= \exp\left\{ -\sum_{n=1}^{\infty} n^{-1} \int_{-\infty}^{0^+} e^{\alpha x} \mathrm{d}P(S_n \leq x - n\varepsilon) \right\} \\ &= \exp\left\{ -\sum_{n=1}^{\infty} n^{-1} \left(\int_{-\infty}^{0^+} - \int_{-n\varepsilon}^{0^+} \right) e^{\alpha(x+n\varepsilon)} \mathrm{d}P(S_n \leq x) \right\} \\ &\to \left(1 - \widehat{F_-^{w}}(\alpha)\right) \exp\left\{ \sum_{n=1}^{\infty} n^{-1}[F^{*n}(0^+) - F^{*n}(0^-)] \right\} \\ &= 1 - \widehat{F_-^{s}}(\alpha), \end{aligned}$$

where (1.2) and (1.3) were used in the last step. This completes the proof. ∎

Proof of Theorem 2.1. We only prove the results for $\overline{W^{(+\varepsilon)}}$, the results for $\overline{W^{(-\varepsilon)}}$ may be proved similarly.

(A) Suppose that $\gamma > 0$.

(i) If $\widehat{F}(\gamma) > 1$, then by Theorem 2.A (A)(i), there exists $\kappa \in (0, \gamma)$ satisfying $Ee^{\kappa X_1} = 1$ such that (2.1) holds. Moreover, for any $\varepsilon \in (0, -\mu)$, the random walk $\{S_n^{(+\varepsilon)} : n \geq 0\}$ also drifts to $-\infty$ and $E^{\kappa(X_1+\varepsilon)} > 1$. Again by Theorem 2.A (A)(i), there exists $\kappa' \in (0, \kappa)$ satisfying $Ee^{\kappa'(X_1+\varepsilon)} = 1$ such that

$$\overline{W^{(+\varepsilon)}}(x) \sim K(\varepsilon)e^{-\kappa' x}. \tag{3.3}$$

where $K(\varepsilon) = \frac{1-F_+^{s,(+\varepsilon)}(\infty)}{\kappa'} \cdot \frac{1-\widehat{F_-^{w,(+\varepsilon)}}(\kappa')}{e^{\kappa'\varepsilon}(\widehat{F}'(\kappa')+\varepsilon\widehat{F}(\kappa'))}$. Thus (2.6) follows from (2.1) and (3.3).

If $\widehat{F}(\gamma) = 1$, then for any $\varepsilon \in (0, -\mu)$, $Ee^{\gamma(X_1+\varepsilon)} > 1$. Then according to Theorem 2.A (A)(i), there exists $\kappa' \in (0, \gamma)$ satisfying $Ee^{\kappa'(X_1+\varepsilon)} = 1$ such that (3.3) holds. And by (2.2) and (2.3),

$$\overline{W}(x) = O(e^{-\gamma x}). \tag{3.4}$$

Thus (2.6) follows by (3.3) and(3.4).

(ii) If $\widehat{F}(\gamma) < 1$ and $F \in \mathcal{S}(\gamma)$, then, for any $\varepsilon \in (0, -\frac{\ln \widehat{F}(\gamma)}{\gamma})$, $Ee^{\gamma(X_1+\varepsilon)} < 1$ and $E(X_1+\varepsilon) < 0$ (see Remark 3.1 below). In addition, the distribution of $X_1+\varepsilon$ belongs to the class $\mathcal{S}(\gamma)$. By Theorem 2.A (A)(iii),

$$\overline{W^{(+\varepsilon)}}(x) \sim \frac{(1-F_+^{s,(+\varepsilon)}(\infty))(1-\widehat{F_-^{w,(+\varepsilon)}}(\gamma))}{(1-e^{\gamma\varepsilon}\widehat{F}(\gamma))^2} e^{\gamma\varepsilon}\overline{F}(x). \tag{3.5}$$

By (2.4) and (3.5)

$$\lim \frac{\overline{W^{(+\varepsilon)}}(x)}{\overline{W}(x)} = \frac{(1-F_+^{s,(+\varepsilon)}(\infty))(1-\widehat{F_-^{w,(+\varepsilon)}}(\gamma))(1-\widehat{F}(\gamma))^2}{(1-F_+^{s}(\infty))(1-\widehat{F_-^{w}}(\gamma))(1-e^{\gamma\varepsilon}\widehat{F}(\gamma))^2} e^{\gamma\varepsilon},$$

which tends to 1 as $\varepsilon \downarrow 0$ by Lemmas 3.1 and 3.2. Thus we get (2.7) immediately.

(B) If $F^I \in \mathcal{S}$, then the integrated tail distribution of $X_1 + \varepsilon$ belongs to the subexponential distribution class. Then by Theorem 2.A (B),

$$\overline{W^{(+\varepsilon)}}(x) \sim -\frac{1}{\mu+\varepsilon}\int_x^\infty \overline{F}(t)dt. \tag{3.6}$$

Thus (2.8) follows immediately by (2.5) and (3.6). □

Remark 3.1. For any given $\alpha \in (0, \infty)$, the function $f_\alpha(x) = e^{\alpha x}, x \in \mathbf{R}$ is a convex function. So for any $\alpha \in (0, \gamma]$, one has $Ef_\alpha(X) > f_\alpha(EX)$, that is, $\widehat{F}(\alpha) > e^{\alpha\mu}$, which implies

$$-\frac{\ln \widehat{F}(\alpha)}{\alpha} < -\mu.$$

So in case (ii), the condition $\varepsilon \in \left(0, -\frac{\ln \widehat{F}(\gamma)}{\gamma}\right)$ implies $E(X_1 + \varepsilon) < 0$.

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