GENERALIZED CUBIC SOFT SETS AND THEIR APPLICATIONS TO ORDERED ABEL-GRASSMANN'S GROUPOIDS

Asad Ali
School of Mathematics and Statistics
Beijing Institute of Technology
Beijing, 100081
P.R. China
e-mail: asad_maths@hotmail.com

Madad Khan
Department of Mathematics
COMSATS Institute of Information Technology
Abbottabad
Pakistan
e-mail: madadmath@yahoo.com

Fu-Gui Shi
School of Mathematics and Statistics
Beijing Institute of Technology
Beijing, 100081
P.R.China
e-mail: fuguishi@bit.edu.cn

Abstract. In this paper, we introduced the concept of generalized cubic soft left (resp., right, bi-) ideals to study the structural properties of ordered $\mathcal{AG}$-groupoids. We characterized intra-regular ordered $\mathcal{AG}$-groupoids using the properties of generalized cubic soft ideals.

Keywords: ordered $\mathcal{AG}$-groupoids, cubic soft sets, left invertive, medial law, paramedial law and generalized cubic soft ideals.

Introduction

In order to deal with many complicated problems in the fields of engineering, economics, social science, medical science involving uncertainties, classical methods are found to be inadequate in recent times. Molodstov [15] pointed out that the important existing theories viz. probability theory, fuzzy set theory, intuitionistic fuzzy set theory, rough set theory etc, which can be considered as mathematical tools for dealing with uncertainties, have their own difficulties. He further pointed out that the reason for these difficulties, is possibly the inadequacy parametrization tool of the theory. Molodtsov in

1999 introduced the fundamental concept of a soft set which provides a natural framework for generalizing several basic notions of algebra. Many related concepts with soft sets, especially soft sets operations, have also undergone tremendous studies. Maji et al. [16] presented some definitions on soft sets and based on the analysis of several operations on soft sets Ali et al. [7] introduced several new algebraic operations on soft sets. Cagman and Enginoglu [3] developed the uni-int decision making method in virtue of soft sets Feng et al. [5] investigated soft semirings by using soft set theory. Aktas and Cagman [1] defined the notion of soft groups and derived some related properties. This initiated an important research direction concerning algebraic properties of soft sets in miscellaneous kinds of algebras such as BCK/BCI-algebras, d-algebras, semirings, rings, Lie algebras and K-algebras.Feng and Li [6] ascertained the relationship among five different types of soft subsets and considered the free soft algebras associated with soft product operations. It has been shown that soft sets have some nonclassical algebraic properties which are distinct from those of crisp sets or fuzzy sets. Zhan et al. [19], worked on soft ideals of BL-algebras. Atagun and Sezgin [2] defined the concept of soft subrings and ideals of a ring, soft subfields of a field and soft sub modules of a module and studied the related properties with respect soft set operations. Jun et al. [10] gave the concept of soft BCK/BCI-algebras. Further, Jun discussed the applications of soft sets in ideal theory of BCK/BCI-algebras and in d-algebras respectively. The theory of soft sets have many applications in different fields of learning such as medical sciences, engineering, economics, computer sciences, artificial intelligence and mathematics etc.

Recently, combining cubic sets and soft sets the first author with Al-roqi [13] introduced the notions of (external, internal) cubic soft sets, P-cubic (resp., R-cubic) soft subsets, R-union (resp., R-intersection, P-union and P-intersection) of cubic soft sets and the complement of a cubic soft set. They investigated several related properties and applied the notion of cubic soft sets to BCK/BCI-algebras. The theory of soft sets introduced in [9], attracted several mathematicians. This theory have been done by many researchers in algebra. In this paper, we applied the theory of cubic soft sets to the ideal theory of $\mathcal{AG}$-groupoid and introduced the concept of generalized cubic soft sets to study the structural properties of non-associative algebraic structure.

The idea of generalization of a commutative semigroup (which we call the left almost semigroup) was introduced by M. A Kazim and M. Naseeruddin in 1972 [11].

A groupoid $(\mathcal{G},\cdot)$ is called an $\mathcal{AG}$-groupoid (Abel-Grassmann's Groupoid) if it satisfies the left invertive law that is $(ab)c=(cb)a$, $\forall\, a,b,c\in\mathcal{G}$. In an $\mathcal{AG}$-groupoid medial law $(ab)(cd)=(ac)(bd)$, holds $\forall\, a,b,c,d\in\mathcal{G}$ [11]. An $\mathcal{AG}$-groupoid $\mathcal{G}$ with left identity satisfies the paramedial law $(ab)(cd)=(db)(ca)$, $\forall\, a,b,c,d\in\mathcal{G}$. If an $\mathcal{AG}$-groupoid $\mathcal{G}$ which contains left identity. Then, $a(bc)=b(ac),\forall a,b,c\in\mathcal{G}$ holds.

In this paper, first we remind some basic definitions about ordered $\mathcal{AG}$-groupoids, cubic sets, $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic sets, $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic sets intersection, product and cubic characteristic function. Then, cubic soft sets, $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft sets $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft ordered $\mathcal{AG}$-groupoid, $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft left (resp., right, bi-) ideals in ordered $\mathcal{AG}$-groupoid are defined and studied with respect to cubic soft sets operations and obtained some interesting new results. Moreover, we give some characterizations of intra-regular ordered $\mathcal{AG}$-groupoids using the properties of $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft ideals.

Preliminaries notes

In this section, we define some basic definitions that are required for next sections.

An ordered $\mathcal{AG}$-groupoid ($\mathfrak{po}$-$\mathcal{AG}$-groupoid) is a structure $(\mathcal{G}, \cdot, \leq)$ in which the following conditions hold [12]:

(i) $(\mathcal{G}, \cdot)$ *is an* $\mathcal{AG}$*-groupoid.*

(ii) $(\mathcal{G}, \leq)$ *is a poset.*

(iii) $\forall\, a, b, x \in \mathcal{G},\ a \leq b \Rightarrow ax \leq bx\ (xa \leq xb)$.

Example 1 Let $\mathcal{G} = \{a, b, c\}$ be an ordered $\mathcal{AG}$-groupoid with the following multiplication table and two different orders below:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

(1) $$\leq := \{(a, a), (b, b), (c, c), (c, a), (c, b)\}$$

(2) $$\leq := \{(a, a), (b, b), (c, c), (a, c), (a, b)\}$$

An ordered $\mathcal{AG}$-groupoid is the generalization of an ordered semigroup. If an ordered $\mathcal{AG}$-groupoid has a right identity, then it becomes an ordered semigroup.

Let A be a non-empty subset of an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$, then

$$(A] = \{t \in \mathcal{G} \mid t \leq a, \text{ for some } a \in A\}.$$

For $A = \{a\}$, we usually written as $(a]$.

Now, we recall the concept of interval valued fuzzy sets.

By an interval number, we mean a closed subinterval $\widetilde{x} = [x^-, x^+]$ of the closed interval $I = [0, 1]$,where $0 \leqslant x^- \leqslant x^+ \leqslant 1$. Let $D[0, 1]$ denote the family of all closed subintervals of $[0, 1]$, i.e.,

$$D[0, 1] = \{\widetilde{x} = [x^-, x^+] : x^- \leqslant x^+, \text{ for all } x^-, x^+ \in [0, 1]\},$$

where the elements in $D[0, 1]$ are called the interval numbers on $[0, 1]$, $\widetilde{0} = [0, 0]$ and $\widetilde{1} = [1, 1]$. For any two elements in $D\,[0, 1]$, the redefined minimum and redefined maximum, respectively, denoted by $r\min$ and $r\max$, and the symbols "$\preceq$","$\succeq$","$=$" are defined. We consider two elements $\widetilde{x} = [x^-, x^+]$ and $\widetilde{y} = [y^-, y^+]$ in $D[0, 1]$. Then

$$\begin{aligned}
\widetilde{x} &\succeq \widetilde{y} \text{ if and only if } x^- \geq y^- \text{ and } x^+ \geq y^+ \\
\widetilde{x} &\preceq \widetilde{y} \text{ if and only if } x^- \leqslant y^- \text{ and } x^+ \leqslant y^+ \\
\widetilde{x} &= \widetilde{y} \text{ if and only if } x^- = y^- \text{ and } x^+ = y^+ \\
r\min\{\widetilde{x}, \widetilde{y}\} &= \min\{x^-, y^-\}, \min\{x^+, y^+\} \\
r\max\{\widetilde{x}, \widetilde{y}\} &= \max\{x^-, y^-\}, \max\{x^+, y^+\}.
\end{aligned}$$

Let $\widetilde{x}_i \in D[0,1]$, where $i \in \Lambda$. We define

$$r\inf_{i\in\Lambda}\widetilde{x}_i = \left[\inf x_i^-, \inf x_i^+\right]_{i\in\Lambda} \quad \text{and} \quad r\sup_{i\in\Lambda}\widetilde{x}_i = \left[\sup x_i^-, \sup x_i^+\right]_{i\in\Lambda}.$$

Let A be a non-empty set. A cubic set $\widetilde{\mu}_X$ on A is defined as

$$\widetilde{\mu}_X = \left\{\left\langle a, [\mu_X^-(a), \widetilde{\mu}_X^+(a)]\right\rangle : a \in A\right\},$$

where $\mu_X^-(a) \leq \mu_X^+(a)$, for all $a \in A$. Then the ordinary fuzzy sets $\mu_X^-(a) : A \to [0,1]$ and $\mu_X^+(a) : A \to [0,1]$ are called a lower fuzzy sets and upper fuzzy sets of $\widetilde{\mu}_X$ respectively. Let $\widetilde{\mu}_X(a) = [\mu_X^-(a), \widetilde{\mu}_X^+(a)]$ then

$$X = \{\langle a, \widetilde{\mu}_X(a)\rangle : a \in A\}, \text{ where } \widetilde{\mu}_X : A \to D[0,1].$$

For rest of the study, let us use the symbols $\lambda = (\widetilde{\gamma}_1, \gamma_2)$ and $\mu = (\widetilde{\delta}_1, \delta_2)$ to avoid symbols complications.

$(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic sets

Jun et al. [8] introduced the concept of cubic sets defined on a non-empty set A as objects having the form

$$\Xi = \left\{\left\langle a, \widetilde{\Psi}_\Xi(a), \phi_\Xi(a)\right\rangle : a \in A\right\},$$

which is briefly denoted by $\Xi = \left\langle \widetilde{\Psi}_\Xi, \phi_\Xi \right\rangle$, where the function $\widetilde{\Psi}_\Xi : A \to D[0,1]$ and $\phi_\Xi : A \to [0,1]$.

Let $\Xi = \left\langle \widetilde{\Psi}_\Xi, \phi_\Xi \right\rangle$ and $F = \left\langle \widetilde{\Psi}_F, \phi_F \right\rangle$ be two cubic sets of an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$. Then

$$\begin{aligned} \Xi \cap F &= \left\{\left\langle a, r\min\left\{\widetilde{\Psi}_\Xi(a), \widetilde{\Psi}_F(a)\right\}, \max\{\phi_\Xi(a), \phi_F(a)\}\right\rangle : a \in \mathcal{G}\right\}, \\ \Xi \odot F &= \left\{\left\langle \left(\widetilde{\Psi}_\Xi \circ \widetilde{\Psi}_\phi\right)(a), (\phi_\Xi \circ \phi_F)(a)\right\rangle : a \in \mathcal{G}\right\}, \end{aligned}$$

where

$$\begin{aligned} \left(\widetilde{\Psi}_\Xi \circ \widetilde{\Psi}_F\right)(a) &= \begin{cases} r\sup\limits_{a\leq bc}\{r\min\left\{\widetilde{\Psi}_\Xi(b), \widetilde{\Psi}_F(c)\right\} & \text{if } a \leq bc \\ [0,0] & \text{otherwise} \end{cases} \\ (\phi_\Xi \circ \phi_F)(a) &= \begin{cases} r\min\limits_{a\leq bc}\{\max\{\phi_\Xi(b), \phi_F(c)\}\} & \text{if } a \leq bc \\ 1 & \text{otherwise} \end{cases} \end{aligned}$$

Let $\mathcal{C}(\mathcal{G})$ denote the family of all cubic sets in $\mathcal{G}$. Then, it becomes an ordered $\mathcal{AG}$-groupoid. Let $\widetilde{\alpha} \in D(0,1]$ and $\beta \in [0,1)$ be such that $\widetilde{0} \prec \widetilde{\alpha}$ and $\beta < 1$. Then by cubic point (CP) we mean $x_{(\widetilde{\alpha},\beta)}(y) = \langle x_{\widetilde{\alpha}}(y), x_\beta(y)\rangle$, where

$$x_{\widetilde{\alpha}}(y) = \begin{cases} \widetilde{\alpha} & \text{if } x \leq y \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

$$x_\beta(y) = \begin{cases} \beta & \text{if } x \leq y \\ 1 & \text{otherwise.} \end{cases}$$

For any cubic set $\Xi = \left\langle \widetilde{\Psi}_\Xi, \phi_\Xi \right\rangle$ and for a cubic point $x_{(\widetilde{\alpha},\beta)}$, with the condition that $[\alpha, \beta] + [\alpha, \beta] = [2\alpha, 2\beta]$ such that $0 \leq 2\alpha < 1$ and $2\beta \leq 1$, we have

(i) $x_{(\widetilde{\alpha},\beta)} \in_\lambda \Xi$ if $\widetilde{\Psi}_\Xi(x) \succeq \widetilde{\alpha} \succ \widetilde{\gamma}_1$ and $\phi_\Xi(x) \leq \beta < \gamma_2$.

(ii) $x_{(\widetilde{\alpha},\beta)} q_\mu \Xi$ if $\widetilde{\Psi}_\Xi(x) + \widetilde{\alpha} \succ 2\widetilde{\delta}_1$ and $\phi_\Xi(x) + \beta < 2\delta_2$.

(iii) $x_{(\widetilde{\alpha},\beta)} \in_\lambda \vee q_\mu \Xi$ if $x_{(\widetilde{\alpha},\beta)} \in_\lambda \Xi$ or $x_{(\widetilde{\alpha},\beta)} q_\mu \Xi$.

(iv) $x_{(\widetilde{\alpha},\beta)} \in_\lambda \wedge q_\mu \Xi$ if $x_{(\widetilde{\alpha},\beta)} \in_\lambda \Xi$ and $x_{(\widetilde{\alpha},\beta)} q_\mu \Xi$.

Definition 1 Let $\mathcal{G}$ be an ordered $\mathcal{AG}$-groupoid. Then, the cubic characteristic function

$$\mathcal{X}^\mu_{\lambda\Sigma} = \left\langle \widetilde{\Psi}_{\mathcal{X}^\mu_{\lambda\Sigma}}, \phi_{\mathcal{X}^\mu_{\lambda\Sigma}} \right\rangle$$

of $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle$ is defined as

$$\widetilde{\Psi}_{\mathcal{X}^\mu_{\lambda\Sigma}}(a) \succeq \begin{cases} \widetilde{\delta}_1 = [1,1] & \text{if } a \in \Sigma \\ \widetilde{\gamma}_1 = [0,0] & \text{if } a \notin \Sigma \end{cases} \quad \text{and} \quad \phi_{\mathcal{X}^\mu_{\lambda\Sigma}}(a) \leq \begin{cases} \delta_2 = 0 & \text{if } a \in \Sigma \\ \gamma_2 = 1 & \text{if } a \notin \Sigma \end{cases}$$

where $\widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.

Now, we define a new relation on $\mathcal{C}(\mathcal{G})$ denoted by $\subseteq \vee q_{(\lambda,\mu)}$follow:

Let $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle, \Omega = \left\langle \widetilde{\Psi}_\Omega, \phi_\Omega \right\rangle \in \mathcal{C}(\mathcal{G})$, by $\Sigma \subseteq \vee q_{(\lambda,\mu)} \Omega$ we mean that $a_{(\widetilde{\alpha},\beta)} \in_\lambda \Sigma$ implies that $a_{(\widetilde{\alpha},\beta)} \in_\lambda \vee q_\mu \Omega$ for all $a \in \mathcal{G}$. Moreover, Σ and Ω are said to be (λ, μ)-equal if $\Sigma \subseteq \vee q_{(\lambda,\mu)} \Omega$ and $\Omega \subseteq \vee q_{(\lambda,\mu)} \Sigma$. The above definitions can be found in [14].

Lemma 1 [14] *Let* $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle, \Omega = \left\langle \widetilde{\Psi}_\Omega, \phi_\Omega \right\rangle$ *be the cubic subsets of* $\mathcal{C}(\mathcal{G})$. *Then,* $\Sigma \subseteq \vee q_{(\lambda,\mu)} \Omega$ *if and only if*

$$r\max\left\{\widetilde{\Psi}_\Omega(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_\Sigma(a), \widetilde{\delta}_1\right\} \text{ and } \min\{\phi_\Omega(a), \gamma_2\} \leq \max\{\phi_\Sigma(a), \delta_2\}$$

for all $a \in \mathcal{G}$.

Lemma 2 [14] *Let* $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle$, $\Omega = \left\langle \widetilde{\Psi}_\Omega, \phi_\Omega \right\rangle$, $\Pi = \left\langle \widetilde{\Psi}_\Pi, \phi_\Pi \right\rangle \in \mathcal{C}(\mathcal{G})$. *If* $\Sigma \subseteq \vee q_{(\lambda,\mu)} \Omega$ *and* $\Omega \subseteq \vee q_{(\lambda,\mu)} \Pi$, *then,* $\Sigma \subseteq \vee q_{(\lambda,\mu)} \Pi$.

From Lemmas 1 and 2 we say that " $=_{(\lambda,\mu)}$ " is an equivalence relation on $\mathcal{C}(\mathcal{G})$.

Lemma 3 *Let* $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle$*and* $\Omega = \left\langle \widetilde{\Psi}_\Omega, \phi_\Omega \right\rangle \in \mathcal{C}(\mathcal{G})$. *Then, we have*

(i) $\Sigma \subseteq \Omega$ *if and only if* $\mathcal{X}^\mu_{\lambda\Sigma} \subseteq \vee q_{(\lambda,\mu)} \mathcal{X}^\mu_{\lambda\Omega}$.

(ii) $\mathcal{X}^\mu_{\lambda\Sigma} \cap \mathcal{X}^\mu_{\lambda\Omega} =_{(\lambda,\mu)} \mathcal{X}^\mu_{\lambda(\Sigma\cap\Omega)}$.

(iii) $\mathcal{X}^\mu_{\lambda\Sigma} \odot \mathcal{X}^\mu_{\lambda\Omega} =_{(\lambda,\mu)} \mathcal{X}^\mu_{\lambda(\Sigma\Omega]}$.

Proof. It is simple. ∎

Cubic soft sets

Molodtsov et al. [15], Maji et al. [16] and Muhiuddin et al. [13], introduced soft set theory which provided new definitions and various results on soft set theory.

We introduced the concept of an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft set which is actually the generalization of soft set.

Definition 2 [13] Let U be an initial universal set and E be set of parameters under consideration. Let $\mathcal{C}^U$ denotes the set all cubic sets of U. Let $A \subseteq E$. A pair $\langle \alpha, A \rangle$ is called cubic soft set over U, where α is a mapping given by $\alpha : A \to \mathcal{C}^U$.

Note that the the pair $\langle \alpha, A \rangle$ can be expressed as the following set:

$$\langle \alpha, A \rangle := \{\alpha(e) : e \in A\}, \text{ where } \alpha(e) = \left\langle \widetilde{\Psi}_{\alpha(e)}, \phi_{\alpha(e)} \right\rangle.$$

In general, for every $e \in A, \alpha(e)$ is a cubic set of U and it is called cubic value set of parameter e. The set of all cubic soft sets over U with parameters from E is called cubic soft class and is denoted by $\mathcal{F}_\rho(U, E)$.

Definition 3 Let $\langle \alpha, A \rangle$ and $\langle \beta, B \rangle$ be two cubic soft sets over U. Then, $\langle \alpha, A \rangle$ is called cubic soft set of $\langle \beta, B \rangle$ and write $\langle \alpha, A \rangle \subset \langle \beta, B \rangle$ if

(i) $A \subseteq B$.

(ii) For any $e \in A$, $\alpha(e) \subseteq \beta(e)$.

$\langle \alpha, A \rangle$ and $\langle \beta, B \rangle$ are said to be cubic soft equal and write $\langle \alpha, A \rangle = \langle \beta, B \rangle$ if $\langle \alpha, A \rangle \subset \langle \beta, B \rangle$ and $\langle \beta, B \rangle \subset \langle \alpha, A \rangle$.

Definition 4 The extended union of two cubic soft sets $\langle \alpha, A \rangle$ and $\langle \beta, B \rangle$ over U is called cubic soft set and is denoted by $\langle \omega, C \rangle$, where $C = A \cup B$ and

$$\omega(e) = \begin{cases} \alpha(e), & \text{if } e \in A - B, \\ \beta(e), & \text{if } e \in B - A, \\ \alpha(e) \cup \beta(e), & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $\langle \omega, C \rangle = \langle \alpha, A \rangle \widetilde{\cup} \langle \beta, B \rangle$.

Definition 5 The extended intersection of two cubic soft sets $\langle \alpha, A \rangle$ and $\langle \beta, B \rangle$ over U is called cubic soft set and is denoted by $\langle \omega, C \rangle$, where $C = A \cup B$ and

$$\omega(e) = \begin{cases} \alpha(e), & \text{if } e \in A - B, \\ \beta(e), & \text{if } e \in B - A, \\ \alpha(e) \cap \beta(e), & \text{if } e \in A \cap B \end{cases}$$

for all $e \in C$. This is denoted by $\langle \omega, C \rangle = \langle \alpha, A \rangle \widetilde{\cap} \langle \beta, B \rangle$.

Definition 6 Let $\langle \alpha, A \rangle$ and $\langle \beta, B \rangle$ be two cubic soft sets over U such that $A \cap B \neq \varphi$. The restricted union of $\langle \alpha, A \rangle$ and $\langle \beta, B \rangle$ is defined to be cubic soft set $\langle \omega, C \rangle$, where $C = A \cap B$ and $\omega(e) = \alpha(e) \cup \beta(e)$ for all $e \in C$. This is denoted by $\langle \omega, C \rangle = \langle \alpha, A \rangle \cup \langle \beta, B \rangle$.

Definition 7 Let $\langle\alpha, A\rangle$ and $\langle\beta, B\rangle$ be two cubic soft sets over U such that $A\cap B\neq\varphi$. The restricted intersection of $\langle\alpha, A\rangle$ and $\langle\beta, B\rangle$ is defined to be cubic soft set $\langle\omega, C\rangle$, where $C = A\cap B$ and $\omega(e) = \alpha(e)\cap\beta(e)$ for all $e\in C$. This is denoted by $\langle\omega, C\rangle = \langle\alpha, A\rangle\cap\langle\beta, B\rangle$.

Definition 8 Let $V\subseteq U$. A cubic soft set $\langle\alpha, A\rangle$ over U is said to be relative whole(λ,μ)-cubic soft set (with respect to universe set V and parameter set A), denoted by $\Sigma(V, A)$, if

$$\alpha(e) = f^{\mu}_{\lambda}V \text{ for all } e\in A.$$

Definition 9 Let $\langle\alpha, A\rangle$ and $\langle\beta, B\rangle$ be two cubic soft sets over U. We say that $\langle\alpha, A\rangle$ is an (λ,μ)-cubic soft subset of $\langle\beta, B\rangle$ and write $\langle\alpha, A\rangle\subset_{(\lambda,\mu)}\langle\beta, B\rangle$ if

(i) $A\subseteq B$.

(ii) For any $e\in A$, $\alpha(e)\subseteq\vee q_{(\lambda,\mu)}\beta(e)$.

$\langle\alpha, A\rangle$ and $\langle\beta, B\rangle$ are said to be (λ,μ)-cubic soft equal and write $\langle\alpha, A\rangle\asymp_{(\lambda,\mu)}\langle\beta, B\rangle$ if $\langle\alpha, A\rangle\subset_{(\lambda,\mu)}\langle\beta, B\rangle$ and $\langle\beta, B\rangle\asymp_{(\lambda,\mu)}\langle\alpha, A\rangle$.

The product of two cubic soft sets $\langle\alpha, A\rangle$ and $\langle\beta, B\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$, denoted by $\langle\alpha\circ\beta, C\rangle$, where $C = A\cup B$ and

$$(\alpha\circ\beta)(e) = \begin{cases}\alpha(e), & \text{if } e\in A-B,\\ \beta(e), & \text{if } e\in B-A,\\ \alpha(e)\circ\beta(e),, & \text{if } e\in A\cap B,\end{cases}$$

for all $e\in C$. This is denoted by $\langle\alpha\circ\beta, C\rangle = \langle\alpha, A\rangle\odot(\alpha, B)$.

$(\in_\lambda, \in_\lambda\vee q_\mu)$-cubic soft ideals over an ordered AG-groupoid

Here, we introduce the concepts of $(\in_\lambda, \in_\lambda\vee q_\mu)$-cubic soft left (resp., right) ideals and $(\in_\lambda, \in_\lambda\vee q_\mu)$-cubic soft bi-ideals over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ and investigate the fundamental properties and relationship.of $(\in_\lambda, \in_\lambda\vee q_\mu)$-cubic soft sets and $(\in_\lambda, \in_\lambda\vee q_\mu)$-cubic soft ideals.

Definition 10 A cubic soft set $\langle\alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda\vee q_\mu)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$ if it satisfies:

(1) If $y\leq x$, then $x_{(\widetilde{t}_1,t_2)}\in_\lambda\alpha(e)\Rightarrow y_{(\widetilde{t}_1,t_2)}\in_\lambda\vee q_\mu\alpha(e)$ for all $x, y\in\mathcal{G}$, $e\in A$, $\widetilde{t}_1,\widetilde{\delta}_1,\widetilde{\gamma}_1\in D(0,1]$ such that $\widetilde{\gamma}_1\prec\widetilde{\delta}_1$ and $t_2,\delta_2,\gamma_2\in[0,1)$ such that $\delta_2<\gamma_2$.

(2) $\langle\alpha, A\rangle\odot\langle\alpha, A\rangle\subset_{(\lambda,\mu)}\langle\alpha, A\rangle$.

Definition 11 A cubic soft set $\langle\alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda\vee q_\mu)$-cubic soft left (resp., right) ideal over $\mathcal{G}$ if it satisfies:

(1) If $y\leq x$, then $x_{(\widetilde{t}_1,t_2)}\in_\lambda\alpha(e)\Rightarrow y_{(\widetilde{t}_1,t_2)}\in_\lambda\vee q_\mu\alpha(e)$ for all $x, y\in\mathcal{G}$, $e\in A$, $\widetilde{t}_1,\widetilde{\delta}_1,\widetilde{\gamma}_1\in D(0,1]$ such that $\widetilde{\gamma}_1\prec\widetilde{\delta}_1$ and $t_2,\delta_2,\gamma_2\in[0,1)$ such that $\delta_2<\gamma_2$.

(2) $\sum(\mathcal{G}, A)\odot\langle\alpha, A\rangle\subset_{(\lambda,\mu)}\langle\alpha, A\rangle\ \left(\langle\alpha, A\rangle\odot\sum(\mathcal{G}, A)\subset_{(\lambda,\mu)}\langle\alpha, A\rangle\right)$.

A cubic soft set $\langle\alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft ideal over $\mathcal{G}$ if it is both an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal and an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right ideal over $\mathcal{G}$.

Definition 12 A cubic soft set $\langle\alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal over $\mathcal{G}$ if it satisfies:

(1) If $y \leq x$, then $x_{(\widetilde{t}_1,t_2)} \in_\lambda \alpha(e) \Rightarrow y_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$ for all $x, y \in \mathcal{G}$, $e \in A$, $\widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $t_2, \delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.

(2) $\langle\alpha, A\rangle \odot \langle\alpha, A\rangle \subset_{(\lambda,\mu)} \langle\alpha, A\rangle$.

(3) $\langle\alpha, A\rangle \odot \sum(\mathcal{G}, A) \odot \langle\alpha, A\rangle \subset_{(\lambda,\mu)} \langle\alpha, A\rangle$.

Definition 13 A cubic soft set $\langle\alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$ if for all $a, b \in \mathcal{G}$, $e \in A$, $\widetilde{t}_1, \widetilde{t}_2, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $s_1, s_2, \delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:

(i) If $a \leq b$ and $a_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e) \Rightarrow a_{(\widetilde{t}_1,s_1)} \in_\lambda \vee q_\mu \alpha(e)$

(ii) $a_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e)$ and $b_{(\widetilde{t}_2,s_2)} \in_\lambda \alpha(e) \Rightarrow (ab)_{(r\min\{\widetilde{t}_1,\widetilde{t}_2\},\max\{s_1,s_2\})} \in_\lambda \vee q_\mu \alpha(e)$.

Theorem 1 *Let $\langle\alpha, A\rangle$ be a cubic soft set over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ with left identity. Then, $\langle\alpha, A\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$ if and only if for all $a, b \in \mathcal{G}$, $e \in A, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:*

(1) $r\max\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(a), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$ *with* $a \leq b$.

(2) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(b), \delta_2\}$.

Proof. (i) $\Rightarrow$ (1) Assume that $a, b \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1], t_2, \delta_2, \gamma_2 \in [0,1)$ such that $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ and $\min\{\phi_{\alpha(e)}(a), \gamma_2\}$ $> t_2 \geq \max\{\phi_{\alpha(e)}(b), \delta_2\}$, then $\widetilde{\Psi}_{\alpha(e)}(a) \prec \widetilde{t}_1 \preceq \widetilde{\gamma}_1 \Rightarrow \widetilde{\Psi}_{\alpha(e)}(a) + \widetilde{t}_1 \preceq 2\widetilde{\delta}_1$ and $\phi_{\alpha(e)}(a) > t_2 \geq \gamma_2 \Longrightarrow \phi_{\alpha(e)}(a) + t_2 \geq 2\delta_2$, that is $(a)_{(\widetilde{t}_1,t_2)} \overline{\in_\lambda \vee q_\mu} \alpha(e)$. On the other hand, if $\widetilde{\Psi}_{\alpha(e)}(b) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(b) \leq t_2 < \gamma_2$, then $(b)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$ but $(a)_{(\widetilde{t}_1,t_2)} \overline{\in_\lambda \vee q_\mu} \alpha(e)$. This is a contradiction to the hypothesis. Hence, $r\max\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\} \succeq r\min\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\}$ and $\min\{\phi_{\alpha(e)}(a), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$.

(1) $\Rightarrow$ (i) Assume that $a, b \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1], t_2, \delta_2, \gamma_2 \in [0,1)$ and $(b)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$, then by definition $\widetilde{\Psi}_{\alpha(e)}(b) \succeq \widetilde{t}_1 > \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(b) \leq t_2 < \gamma_2$.

Now, by (1), $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\} \succeq \min\{\widetilde{t}_1,\widetilde{\delta}_1\}$ and $\min\{\phi_{\alpha(e)}(a),\gamma_2\} \leq \max\{\phi_{\alpha(e)}(b),\delta_2\} \leq \min\{t_2,\delta_2\}$.

We have to consider two cases here.

Case (a). If $\widetilde{t}_1 \preceq \widetilde{\delta}_1$ and $t_2 \geq \delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(a) \succeq \widetilde{t}_1 > \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(a) \leq t_2 < \gamma_2$. This implies that $(a)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$.

Case (b). $\widetilde{t}_1 \succ \widetilde{\delta}_1$ and $t_2 < \delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(a) + \widetilde{t}_1 \succ 2\widetilde{\delta}_1$ and $\phi_{\alpha(e)}(a) + t_2 < 2\delta_2$. This implies that $(a)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$.

Hence, from both cases we can write $(a)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$.

(ii) $\Rightarrow$ (2) Let $\langle \alpha, A\rangle$ be a cubic soft set over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ which is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$. Let $a, b \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$, $t_2, \delta_2, \gamma_2 \in [0,1)$ such that

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab),\widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}$$

and

$$\min\{\phi_{\alpha(e)}(ab),\gamma_2\} > t_2 \geq \max\{\phi_{\alpha(e)}(a),\phi_{\alpha(e)}(b),\delta_2\}.$$

Then

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab),\widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \Rightarrow \widetilde{\Psi}_{\alpha(e)}(ab) \prec \widetilde{t}_1 \prec \widetilde{\gamma}_1$$

and

$$\min\{\phi_{\alpha(e)}(ab),\gamma_2\} > t_2 \Rightarrow \phi_{\alpha(e)}(ab) > t_2 > \gamma_2.$$

Thus $(ab)_{(\widetilde{t}_1,t_2)} \overline{\in_\lambda \vee q_\mu} \alpha(e)$. On the other hand, if

$$\widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}$$

and

$$t_2 \geq \max\{\phi_{\alpha(e)}(a),\phi_{\alpha(e)}(b),\delta_2\}$$

we have

$$\widetilde{\Psi}_{\alpha(e)}(a) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1, \widetilde{\Psi}_{\alpha(e)}(b) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1 \text{ and } \phi_{\alpha(e)}(a) \leq t_2 < \gamma_2, \phi_{\alpha(e)}(b) \leq t_2 < \gamma_2.$$

Then $(a)_{(\widetilde{t}_1,t_2)} \in_\lambda \alpha(e)$ and $(b)_{(\widetilde{t}_1,t_2)} \in_\lambda \alpha(e)$ but $(ab)_{(\widetilde{t}_1,t_2)} \overline{\in_\lambda \vee q_\mu} \alpha(e)$.

This is a contradiction to the hypothesis. Hence,

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab),\widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}$$

and

$$\min\{\phi_{\alpha(e)}(ab),\gamma_2\} \leq \max\{\phi_{\alpha(e)}(a),\phi_{\alpha(\epsilon)}(b),\delta_2\}.$$

(2) $\Rightarrow$ (ii) Let there exist $a \in \mathcal{G}$, $e \in A, \widetilde{t}, \widetilde{t}_1 \in D(0,1], s, s_1 \in [0,1)$ such that $(a)_{(\widetilde{t},s)} \in_\lambda \alpha(e), b \in \mathcal{G}$ such that $(b)_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e)$. This implies that $\widetilde{\Psi}_{\alpha(e)}(a) \succeq \widetilde{t} \succ \widetilde{\gamma}_1$, $\phi_{\alpha(e)}(a) \leq s < \gamma_2$ and $\widetilde{\Psi}_{\alpha(e)}(b) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1, \phi_{\alpha(e)}(b) \leq s_1 < \gamma_2$. So

$$r \max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r \min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\} \succeq r \min\{\{\widetilde{t}, \widetilde{t}_1\}, \widetilde{\delta}_1\}$$

and

$$\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(b), \delta_2\} \leq \max\{\{s, s_1\}, \delta_2\}.$$

Now, we have the following cases:

Case (a). If $r \min\{\widetilde{t}, \widetilde{t}_1\} \preceq \widetilde{\delta}_1$ and $\max\{s, s_1\} \geq \delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(ab) \succeq r \min\{\widetilde{t}, \widetilde{t}_1\} \succ \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(ab) \leq \max\{s, s_1\} < \gamma_2$. This implies that $(ab)_{(r \min\{\widetilde{t}, \widetilde{t}_1\}, \max\{\{s,s_1\})} \in_\lambda \alpha(e)$.

Case (b). If $r \min\{\widetilde{t}, \widetilde{t}_1\} \succ \widetilde{\delta}_1$ and $\max\{s, s_1\} < \delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(ab) + r \min\{\widetilde{t}, \widetilde{t}_1\} \succ 2\widetilde{\delta}_1$ and $\phi_{\alpha(e)}(ab) + \max\{\{s, s_1\} < 2\delta_2$. This implies that $(ab)_{(r \min\{\widetilde{t}, \widetilde{t}_1\}, \max\{s,s_1\})} q_\mu \alpha(e)$.

From both cases we get $(ab)_{(r \min\{\widetilde{t}_1, \widetilde{t}_2\}, \max\{s_1,s_2\})} \in_\lambda \vee q_\mu \alpha(e)$. Hence, $\langle \alpha, A\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$. ■

Example 2 Let $\mathcal{G} = \{a, b, c\}$ be an an ordered $\mathcal{AG}$-groupoid with the multiplication table and order given below:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

$$\leq := \{(a,a), (b,b), (c,c), (a,c), (a,b)$$

For $A = \{e_1, e_2, e_3\} \subseteq E$, the cubic soft set $\langle \alpha, A\rangle = \{\alpha(e_1), \alpha(e_2), \alpha(e_3)\}$ over $\mathcal{G}$ is defined as follows.

$$\begin{aligned}
\alpha(e_1) &= \{\langle a, [0.2, 0.3), 0.1\rangle, \langle b, [0.3, 0.4), 0.2\rangle, \langle c, [0.4, 0.5), 0.3\rangle\}, \\
\alpha(e_2) &= \{\langle a, [0.15, 0.16), 0.2\rangle, \langle b, [0.16, 0.18), 0.3\rangle, \langle c, [0.18, 0.2), 0.3\rangle\}, \\
\alpha(e_3) &= \{\langle a, [0.1, 0.12), 0.1\rangle, \langle b, [0.12, 0.14), 0.2\rangle, \langle c, [0.14, 0.16), 0.2\rangle\},
\end{aligned}$$

such that $\widetilde{\gamma}_1 = [0.1, .18) \prec \widetilde{\delta}_1 = [0.3, 0.4)$ and $\delta_2 = 0.3 < \gamma_2 = 0.4$. Then it is easy to verify that $\langle \alpha, A\rangle$ is an $(\in_{([0.1,.18),0.4)}, \in_{([0.1,.18),0.4)} \vee q_{([0.3,0.4),0.3)})$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{G}$.

Definition 14 A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left (resp., right) ideal over $\mathcal{G}$ if for all $a, b \in \mathcal{G}$, $e \in A$, $\widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$and $s_1, \delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:

(i) If $a \leq b$ and $b_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e) \Rightarrow (a)_{(\widetilde{t}_1,s_1)} \in_\lambda \vee q_\mu \alpha(e)$.

(ii) If $b_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e) \Rightarrow (ab)_{(\widetilde{t}_1,s_1)} \in_\lambda \vee q_\mu \alpha(e)$ $(a_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e)$
$\Rightarrow (ab)_{(\widetilde{t}_1,s_1)} \in_\lambda \vee q_\mu \alpha(e))$.

Theorem 2 *A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left(resp., right) ideal if and only if for all $a, b \in \mathcal{G}, e \in A$, $\widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:*

(1) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and*
$\min\{\phi_{\alpha(e)}(a), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$ *with* $a \leq b$.

(2) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and*
$\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$.
$\Big($*resp.,* $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\delta}_1\right\}$ *and*
$\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \delta_2\}\Big)$.

Proof. (i) $\Leftrightarrow$ (1) It is the same as in Theorem 1.

(ii) $\Rightarrow$ (2) Assume that $a, b \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1], t_2, \delta_2, \gamma_2 \in [0,1)$ such that $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ and $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} > t_2 \geq \max\{\phi_{\alpha(e)}(b), \delta_2\}$, then $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \Rightarrow \widetilde{\Psi}_{\alpha(e)}(ab) \prec \widetilde{t}_1 \preceq \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(ab) > t_2 \Rightarrow \phi_{\alpha(e)}(ab) > t_2 \geq \gamma_2$. This implies that $(ab)_{(\widetilde{t}_1,t_2)} \overline{\in_\lambda \vee q_\mu} \alpha(e)$. On the other hand, if $r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\} \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\max\{\phi_{\alpha(e)}(b), \delta_2\} \leq t_2 < \gamma_2$, this implies that $(b)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$, but $(ab)_{(\widetilde{t}_1,t_2)} \overline{\in_\lambda \vee q_\mu} \alpha(e)$ is a contradiction. Hence, $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ and $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$.

(2) $\Rightarrow$ (ii) Assume that $a, b \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1], t_2, \delta_2, \gamma_2 \in [0,1)$ such that $b_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e)$, then by definition we can write $\widetilde{\Psi}_{\alpha(e)}(a) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(a) \leq t_2 < \gamma_2$, therefore $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\} \succeq \min\{\widetilde{t}_1, \widetilde{\delta}_1\}$ and $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\} \leq \{t_2, \delta_2\}$.

We have to consider two cases here.

Case (a). If $\widetilde{t}_1 \preceq \widetilde{\delta}_1$ and $t_2 \geq \delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(ab) \succeq \widetilde{t}_1 > \widetilde{\gamma}_1$ and $\phi_{\alpha(e)}(ab) \leq t_2 < \gamma_2$. This implies that $(ab)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$.

Case (b). $\widetilde{t}_1 \succ \widetilde{\delta}_1$ and $t_2 < \delta_2$, then $\widetilde{\Psi}_{\alpha(e)}(ab) + \widetilde{t}_1 \succ 2\widetilde{\delta}_1$ and $\phi_{\alpha(e)}(ab) + t_2 < 2\delta_2$. This implies that $(ab)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$.

Hence, from both cases we can write $(ab)_{(\widetilde{t}_1,t_2)} \in_\lambda \vee q_\mu \alpha(e)$.
The other case can be seen in a similar way. ∎

A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft ideal over $\mathcal{G}$, if it is both $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left and an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right over $\mathcal{G}$.

Example 3 Let $\mathcal{G} = \{a, b, c\}$ be an ordered $\mathcal{AG}$-groupoid with the multiplication table and order given below:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	b	a

$$\leq := \{(a, a), (b, b), (c, c), (a, c), (a, b)$$

For $A = \{e_1, e_2, e_3\} \subseteq E$, the cubic soft set $\langle \alpha, A\rangle = \{\alpha(e_1), \alpha(e_2), \alpha(e_3)\}$ over $\mathcal{G}$ is defined as follows.

$$\begin{aligned}
\alpha(e_1) &= \{\langle a, [0.2, 0.3), 0.1\rangle, \langle b, [0.3, 0.4), 0.2\rangle, \langle c, [0.4, 0.5), 0.3\rangle\},\\
\alpha(e_2) &= \{\langle a, [0.2, 0.15), 0.1\rangle, \langle b, [0.15, 0.18), 0.3\rangle, \langle c, [0.18, 0.2), 0.4\rangle\},\\
\alpha(e_3) &= \{\langle a, [0.1, 0.15), 0.2\rangle, \langle b, [0.15, 0.20), 0.3\rangle, \langle c, [0.20, 0.25), 0.3\rangle\},
\end{aligned}$$

such that $\widetilde{\gamma}_1 = [0.1, .18) \prec \widetilde{\delta}_1 = [0.19, 0.2)$ and $\delta_2 = 0.53 < \gamma_2 = 0.54$. Then it is easy to varify that $\langle \alpha, A\rangle$ is an $(\in_{([0.1,.18),0.54)}, \in_{([0.1,.18),0.54)} \vee q_{([0.19,0.2),0.53)})$-cubic soft ideal over $\mathcal{G}$.

Lemma 4 *Let $\langle \alpha, A\rangle$ be a cubic soft set over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$, let $e \in A$, $\widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$, then $\langle \alpha, A\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$ cubic soft left (resp., right) over $\mathcal{G}$ if and only if $\langle \alpha, A\rangle$ satisfies the following conditions.*

(1) $a \leq b \Rightarrow r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(a), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}, \forall a, b \in \mathcal{G}$.

(2) $\Sigma\langle \mathcal{G}, A\rangle \odot \langle \alpha, A\rangle \subseteq \vee q_{(\lambda,\mu)}\langle \alpha, A\rangle$ $\left(\textit{resp., } \langle \alpha, A\rangle \odot \Sigma\langle \mathcal{G}, A\rangle \subseteq \vee q_{(\lambda,\mu)}\langle \alpha, A\rangle\right)$.

Proof. The proof is straightforward. ■

Definition 15 A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft ideal over $\mathcal{G}$ if for $a, b, c \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{t}_2, \widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $s_1, s_2, \delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:

(i) If $a \leq b$ and $b_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e) \Rightarrow (a)_{(\widetilde{t}_1,s_1)} \in_\lambda \vee q_\mu \alpha(e)$.

(ii) If $a_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e)$ and $b_{(\widetilde{t}_2,s_2)} \in_\lambda \alpha(e) \Rightarrow (ab)_{(r\min\{\widetilde{t}_1,\widetilde{t}_2\},\max\{s_1,s_2\})} \in_\lambda \vee q_\mu \alpha(e)$.

(iii) If $a_{(\widetilde{t}_1,s_1)} \in_\lambda \alpha(e)$ and $c_{(\widetilde{t}_2,s_2)} \in_\lambda \alpha(e) \Rightarrow ((ab)c)_{(r\min\{\widetilde{t}_1,\widetilde{t}_2\},\max\{s_1,s_2\})} \in_\lambda \vee q_\mu \alpha(e)$.

Theorem 3 *A cubic soft set $\langle \alpha, A\rangle$ over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal over $\mathcal{G}$ if for $a, b, c \in \mathcal{G}$, $e \in A, \widetilde{t}_1, \widetilde{t}_2, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $s_1, s_2, \delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$, the following conditions holds:*

(1) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(a), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(b), \delta_2\}$ *with* $a \leq b$.

(2) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(ab), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(b), \delta_2\}$.

(3) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ab)c), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(c), \widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}((ab)c), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(c), \delta_2\}$.

Proof. (i) $\Leftrightarrow$ (1) It is the same as in Theorem 1.
(ii) $\Leftrightarrow$ (2) It is the same as in Theorem 1.
(iii) $\Leftrightarrow$ (3) Assume that $\langle \alpha, A\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal over an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$. Let $a, b, c \in \mathcal{G}$, $e \in A$, $\widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1], t_2, \delta_2, \gamma_2 \in [0,1)$ such that $r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ab)c), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(c), \widetilde{\delta}_1\right\}$ and $\min\{\phi_{\alpha(e)}((ab)c), \gamma_2\} > t_2 \geq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(c), \delta_2\}$. Then,

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ab)c), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \Rightarrow \widetilde{\Psi}_{\alpha(e)}((ab)c) \prec \widetilde{t}_1 \prec \widetilde{\gamma}_1 \quad \text{and}$$
$$\min\{\phi_{\alpha(e)}((ab)c), \gamma_2\} > t_2 \Rightarrow \phi_{\alpha(e)}((ab)c) > t_2 > \gamma_2.$$

Thus $((ab)c)_{(\widetilde{t}_1, t_2)} \overline{\in_\lambda \vee q_\mu} \alpha(e)$.

On the other hand, if

$$\widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(c), \widetilde{\delta}_1\right\} \text{ and } t_2 \geq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(c), \delta_2\},$$

then we have

$$\widetilde{\Psi}_{\alpha(e)}(a) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1, \widetilde{\Psi}_{\alpha(e)}(c) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1 \text{ and } \phi_{\alpha(e)}(a) \leq t_2 < \gamma_2, \phi_{\alpha(e)}(c) \leq t_2 < \gamma_2.$$

This implies that $(a)_{(\widetilde{t}_1, t_2)} \in_\lambda \alpha(e)$ and $(c)_{(\widetilde{t}_1, t_2)} \in_\lambda \alpha(e)$, but $((ab)c)_{(\widetilde{t}_1, t_2)} \overline{\in_\lambda \vee q_\mu} \alpha(e)$.
This is a contradiction to the hypothesis. Hence,

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ab)c), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(c), \widetilde{\delta}_1\right\} \text{ and}$$
$$\min\{\phi_{\alpha(e)}((ab)c), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(\epsilon)}(c), \delta_2\}.$$

(2) $\Rightarrow$ (ii) Let there exist $a, b, c \in \mathcal{G}$, $e \in A, \widetilde{t}, \widetilde{t}_1 \in D(0,1], s, s_1 \in [0,1)$ such that $(a)_{(\widetilde{t}, s)} \in_\lambda \alpha(e), b \in \mathcal{G}$ such that $(c)_{(\widetilde{t}_1, s_1)} \in_\lambda \alpha(e)$. This implies that $\widetilde{\Psi}_{\alpha(e)}(a) \succeq \widetilde{t} \succ \widetilde{\gamma}_1$, $\phi_{\alpha(e)}(a) \leq s < \gamma_2$ and $\widetilde{\Psi}_{\alpha(e)}(c) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1, \phi_{\alpha(e)}(c) \leq s_1 < \gamma_2$. So

$$r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ab)c), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\Psi}_{\alpha(e)}(b), \widetilde{\delta}_1\right\} \succeq r\min\{\{\widetilde{t}, \widetilde{t}_1\}, \widetilde{\delta}_1\} \text{ and}$$
$$\min\{\phi_{\alpha(e)}((ab)c), \gamma_2\} \leq \max\{\phi_{\alpha(e)}(a), \phi_{\alpha(e)}(b), \delta_2\} \leq \max\{\{s, s_1\}, \delta_2\}.$$

Now, we have the following cases:

Case (a). If $r\min\{\widetilde{t},\widetilde{t}_1\}\preceq\widetilde{\delta}_1$ and $\max\{s,s_1\}\geq\delta_2$, then $\widetilde{\Psi}_{\alpha(e)}((ab)c)\succeq r\min\{\widetilde{t},\widetilde{t}_1\}\succ\widetilde{\gamma}_1$ and $\phi_{\alpha(e)}((ab)c)\leq\max\{s,s_1\}<\gamma_2$. This implies that $((ab)c)_{(r\min\{\widetilde{t},\widetilde{t}_1\},\max\{\{s,s_1\})}\in_\lambda \alpha(e)$.

Case (b). If $r\min\{\widetilde{t},\widetilde{t}_1\}\succ\widetilde{\delta}_1$ and $\max\{s,s_1\}<\delta_2$, then $\widetilde{\Psi}_{\alpha(e)}((ab)c)+r\min\{\widetilde{t},\widetilde{t}_1\}\succ 2\widetilde{\delta}_1$ and $\phi_{\alpha(e)}((ab)c)+\max\{\{s,s_1\}<2\delta_2$. This implies that $(ab)_{(r\min\{\widetilde{t},\widetilde{t}_1\},\max\{s,s_1\})}q_\mu\alpha(e)$.

From both cases we get $((ab)c)_{(r\min\{\widetilde{t}_1,\widetilde{t}_2\},\max\{s_1,s_2\})}\in_\lambda\vee q_\mu\alpha(e)$ for all $a,b,c\in\mathcal{G}$. ■

Example 4 Let $\mathcal{G}=\{0,1,2,3\}$ be an ordered $\mathcal{AG}$-groupoid with multiplication table and order given below.

$\cdot$	0	1	2	3
0	0	0	0	0
1	0	0	0	0
2	0	0	0	1
3	0	0	1	2

$$\leq:=\{(0,0),(1,1),(2,2),(3,3),(0,1)\}$$

For $A=\{e_1,e_2,e_3,e_4\}\subseteq E$, the cubic soft set $\langle\alpha,A\rangle=\{\alpha(e_1),\alpha(e_2),\alpha(e_3),\alpha(e_4)\}$ over $\mathcal{G}$ is defined as follows.

$$\begin{aligned}
\alpha(e_1)&=\{\langle 0,[0.2,0.3),0.1\rangle,\langle 1,[0.3,0.4),0.2\rangle,\langle 2,[0.4,0.5),0.3\rangle,\langle 3,[0.6.0.8),0.5\rangle\},\\
\alpha(e_2)&=\{\langle 0,[0.2,0.15),0.1\rangle,\langle 1,[0.15,0.18),0.3\rangle,\langle 2,[0.18,0.2),0.4\rangle,\langle 3,[0.2.0.28),0.4\rangle\},\\
\alpha(e_3)&=\{\langle 0,[0.1,0.15),0.2\rangle,\langle 1,[0.15,0.20),0.3\rangle,\langle 2,[0.20,0.25),0.3\rangle,\langle 3,[0.25.0.28),0.5\rangle\},\\
\alpha(e_4)&=\{\langle 0,[0.1,0.2),0.5\rangle,\langle 1,[0.2,0.25),0.7\rangle,\langle 2,[0.25,0.3),0.9\rangle,\langle 3,[0.3,0.35),0.9\rangle\}
\end{aligned}$$

such that $\widetilde{\gamma}_1=[0.1,.18)\prec\widetilde{\delta}_1=[0.19,0.2)$ and $\delta_2=0.53<\gamma_2=0.54$. Then it is easy to verify that $\langle\alpha,A\rangle$ is an $(\in_{([0.1,.18),0.54)},\in_{([0.1,.18),0.54)}\vee q_{([0.19,0.2),0.53)})$-cubic soft bi-ideal over $\mathcal{G}$.

Theorem 4 *Let A be a non-empty subset of an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$, then for $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft left (resp., right) ideal the following conditions are equivalent.*

(1) (a) $\sum\langle\mathcal{G},A\rangle\odot\langle\alpha,A\rangle\subset_{(\lambda,\mu)}\langle\alpha,A\rangle$ $\left(\textit{resp.},\ \langle\alpha,A\rangle\odot\sum\langle\mathcal{G},A\rangle\subset_{(\lambda,\mu)}\langle\alpha,A\rangle\right)$.

(b) *If $y\leq x$, then $x_{(\widetilde{t}_1,t_2)}\in_\lambda\alpha(e)\Rightarrow y_{(\widetilde{t}_1,t_2)}\in_\lambda\vee q_\mu\alpha(e)$ for all $x,y\in\mathcal{G}$, $e\in A$, $\widetilde{t}_1,\widetilde{\delta}_1,\widetilde{\gamma}_1\in D(0,1]$ such that $\widetilde{\gamma}_1\prec\widetilde{\delta}_1$ and $t_2,\delta_2,\gamma_2\in[0,1)$ such that $\delta_2<\gamma_2$.*

(2) $x_{(\widetilde{\alpha},\beta)}\in_\lambda\alpha(e)\Rightarrow(yx)_{(\widetilde{\alpha},\beta)}\in_\lambda\alpha(e)$.

(3) (a) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(a),\widetilde{\gamma}_1\right\}\succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(a),\gamma_2\}\leq\max\{\phi_{\alpha(e)}(b),\delta_2\}$ *with $a\leq b$.*

(b) $r\max\left\{\widetilde{\Psi}_{\alpha(e)}(ab),\widetilde{\gamma}_1\right\}\succeq r\min\left\{\widetilde{\Psi}_{\alpha(e)}(b),\widetilde{\delta}_1\right\}$ *and* $\min\{\phi_{\alpha(e)}(ab),\gamma_2\}\leq\max\{\phi_{\alpha(e)}(b),\delta_2\}$.

Proof. It is straightforward. ■

Lemma 5 *Let $\phi \neq A \subseteq \mathcal{G}$, then A is a left (resp., right, bi-) ideal of $\mathcal{G}$ if and only if the cubic characteristic function $\mathcal{X}^{\mu}_{\lambda A} = \left\langle \widetilde{\Psi}_{\mathcal{X}^{\mu}_{\lambda A}}, \phi_{\mathcal{X}^{\mu}_{\lambda A}} \right\rangle$ of $A = \left\langle \widetilde{\Psi}_A, \phi_A \right\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic left (resp., right, bi-) ideal over $\mathcal{G}$, where $\widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.*

Proof. It is the same as in [14]. ■

Let us define the $\in_\lambda \vee q_\mu$-cubic level set for the cubic set $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle$ as

$$[\Sigma]_{(\widetilde{t},\delta)} = \{a \in \mathcal{G} : a_{(\widetilde{t},\delta)} \in_\lambda \vee q_\mu \Sigma\}.$$

Theorem 5 *A cubic set $\Sigma = \left\langle \widetilde{\Psi}_\Sigma, \phi_\Sigma \right\rangle$ is said to be an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic left (resp., right, bi-) ideal of an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ if and only if $\Phi \neq [\Sigma]_{(\widetilde{t},\delta)}$ is left (resp., right, bi-) ideal of $\mathcal{G}$.*

Proof. It is the same as in [14]. ■

Theorem 6 *Let $\mathcal{G}$ be an ordered $\mathcal{AG}$-groupoid and $P \subseteq \mathcal{G}$. Then P is left (resp., right, bi-) ideal of $\mathcal{G}$ if and only if $\Sigma(P, A)$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left (resp., right, bi-) ideal over $\mathcal{G}$ for any $A \subseteq E$.*

Proof. It follows from Theorem 5. ■

Lemma 6 [12] *In an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$, the following are true.*

(i) $A \subseteq (A], \forall\, A \subseteq \mathcal{G}$.

(ii) $A \subseteq B \subseteq \mathcal{G} \Longrightarrow (A] \subseteq (B], \forall\, A, B \subseteq \mathcal{G}$.

(iii) $(A](B] \subseteq (AB], \forall\, A, B \subseteq \mathcal{G}$.

(iv) $(A] = ((A]], \forall\, A \subseteq \mathcal{G}$.

(vi) $((A](B]] = (AB], \forall\, A, B \subseteq \mathcal{G}$.

Main results

This section contains the main results of the paper.

Definition 16 An element a of an ordered $\mathcal{AG}$-groupoid $\mathcal{G}$ is called **intra-regular** if there exist $x, y \in \mathcal{G}$ such that $a \leq (xa^2)y$ and $\mathcal{G}$ is called **intra-regular**, if every element of $\mathcal{G}$ is intra-regular or equivalently, $A \subseteq ((\mathcal{G}A^2)]$, for all $A \subseteq \mathcal{G}$. In this section, we discuss the characterizations of intra-regular ordered $\mathcal{AG}$-groupoids.

From now on, $\mathcal{G}$ will denote an ordered $\mathcal{AG}$-groupoid unless otherwise specified.

Theorem 7 *The following conditions are equivalent for $\mathcal{G}$ with left identity.*

(i) *$\mathcal{G}$ is intra-regular.*

(ii) *For every left ideal L and bi-ideal B of $\mathcal{G}$, $L \cap B \subseteq ((LB]B]$.*

(iii) *For an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal $\langle \alpha, L\rangle$ and for an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal $\langle \beta, B\rangle$ over $\mathcal{G}$, we have $\langle \alpha, L\rangle \widetilde{\cap} \langle \beta, B\rangle \subseteq \vee\, q_{(\lambda,\mu)}\, [\langle \alpha, L\rangle \odot \langle \beta, B\rangle] \odot \langle \beta, B\rangle$.*

Proof. (i) $\Rightarrow$ (iii) Let $\mathcal{G}$ be intra-regular, $\langle \alpha, L\rangle$ be an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal and $\langle \beta, B\rangle$ be an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal over $\mathcal{G}$, respectively. Now let $e \in L \cup B$ and $\langle \alpha, L\rangle \widetilde{\cap} \langle \beta, B\rangle = \langle \omega, L \cup B\rangle$. We consider the following cases.

Case 1. $e \in L - B$. Then $\omega(e) = \alpha(e) = (\alpha \circ \beta)(e)$.

Case 2. $e \in B - L$. Then $\omega(e) = \beta(e) = (\alpha \circ \beta)(e)$.

Case 3. $e \in L \cap B$. Then $\omega(e) = \alpha(e) \cap \beta(e)$ and $(\alpha \circ \beta)(e) = (\alpha(e) \circ \beta(e)) \circ \beta(e)$. Now we show that $\alpha(e) \cap \beta(e) \subseteq \vee q_{(\lambda,\mu)}(\alpha(e) \circ \beta(e)) \circ \beta(e)$. Since $\mathcal{G}$ is intra-regular, then for any $a \in \mathcal{G}$ there exist $x, y \in \mathcal{G}$ such that

$$\begin{aligned} a &\leq (xa^2)y = (a(xa))y = (y(xa))a \leq (y(x((xa^2)y)))a = (x(y((a(xa)y)))a \\ &= (x(a(xa))y^2)a = ((a(xa))(xy^2))a = (((xy^2)(xa))a)a = (((ay^2)x^2)a)a \\ &= (((x^2y^2)a)a)a = (ba)a, \text{ where } b = (x^2y^2)a. \end{aligned}$$

Thus $(ba, a) \in A_a$. Therefore,

$$\begin{aligned} & r\max\left\{((\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)}) \circ \widetilde{\Psi}_{\beta(e)})(a), \widetilde{\gamma}_1\right\} \\ &= r\max\left[r\sup_{(ba,a)\in A_a}\left\{r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)})(ba), \widetilde{\Psi}_{\beta(e)}(a)\right\}\right\}, \widetilde{\gamma}_1\right] \\ &\succeq r\max\left[r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)})(ba), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\ &= r\max\left[r\min\left\{r\sup_{(ba,a)\in A_a}\left\{r\min\left\{\widetilde{\Psi}_{\alpha(e)}(ba), \widetilde{\Psi}_{\beta(e)}(a)\right\}\right\}, \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\ &\succeq r\max\left[r\min\left\{r\min\left\{\widetilde{\Psi}_{\alpha(e)}((x^2y^2)a), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\ &= r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}((x^2y^2)a), \widetilde{\Psi}_{\beta(e)}(a), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\ &= r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}((x^2y^2)a), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\gamma}_1\right\}\right] \\ &\succeq r\min\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\delta}_1\right\}\right] \\ &= r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \cap \widetilde{\Psi}_{\beta(e)})(a), \widetilde{\delta}_1\right\} \end{aligned}$$

and

$$\begin{aligned}
&\min\left\{((\phi_{\alpha(e)}\circ\phi_{\beta(e)})\circ\phi_{\beta(e)})(a),\gamma_2\right\}\\
&=\min\left[r\min_{(ba,a)\in A_a}\left\{\max\left\{(\phi_{\alpha(e)}\circ\phi_{\beta(e)})(ba),\phi_{\beta(e)}(a)\right\}\right\},\gamma_2\right]\\
&\leq\min\left[\max\left\{(\phi_{\alpha(e)}\circ\phi_{\beta(e)})(ba),\phi_{\beta(e)}(a)\right\},\gamma_2\right]\\
&=\min\left[\max\left\{r\min_{(ba,a)\in A_a}\left\{\max\left\{\phi_{\alpha(e)}(ba),\phi_{\beta(e)}(a)\right\}\right\},\phi_{\beta(e)}(a)\right\},\gamma_2\right]\\
&\leq\min\left[\max\left\{\max\left\{\phi_{\alpha(e)}((x^2y^2)a),\phi_{\beta(e)}(a)\right\},\phi_{\beta(e)}(a)\right\},\gamma_2\right]\\
&=\max\left[\min\left\{\phi_{\alpha(e)}((x^2y^2)a),\phi_{\beta(e)}(a),\phi_{\beta(e)}(a)\right\},\gamma_2\right]\\
&=\max\left[\min\left\{\phi_{\alpha(e)}((x^2y^2)a),\gamma_2\right\},\min\left\{\phi_{\beta(e)}(a),\gamma_2\right\},\min\left\{\phi_{\beta(e)}(a),\gamma_2\right\}\right]\\
&\leq\max\left[\max\left\{\phi_{\alpha(e)}(a),\delta_2\right\},\max\left\{\phi_{\beta(e)}(a),\delta_2\right\},\max\left\{\phi_{\beta(e)}(a),\delta_2\right\}\right]\\
&=\max\left\{(\phi_{\alpha(e)}\cap\phi_{\beta(e)})(a),\delta_2\right\}
\end{aligned}$$

Thus, by Lemma 1, $\langle\alpha,L\rangle\widetilde{\cap}\langle\beta,B\rangle\subseteq\vee q_{(\lambda,\mu)}[\langle\alpha,L\rangle\odot\langle\beta,B\rangle]\odot\langle\beta,B\rangle$.

(iii) $\Rightarrow$ (ii) Let L be left ideal and B be bi-ideal of $\mathcal{G}$ with left identity, then by Theorem 6, $\Sigma(L,E)$ and $\Sigma(B,E)$ is an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft left ideal and $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft bi-ideal over $\mathcal{G}$, respectively. Now by using assumption we have $\Sigma(L,E)\cap\Sigma(B,E)\subset_{(\lambda,\mu)}[\Sigma(L,E)\odot\Sigma(B,E)]\odot\Sigma(B,E)$. By using Lemma 3, we have $\mathcal{X}^\mu_{\lambda(L\cap B)}=_{(\lambda,\mu)}\mathcal{X}^\mu_{\lambda L}\cap\mathcal{X}^\mu_{\lambda B}\subseteq\vee q_\mu(\mathcal{X}^\mu_{\lambda L}\odot\mathcal{X}^\mu_{\lambda B})\odot\mathcal{X}^\mu_{\lambda B}=\mathcal{X}^\mu_{\lambda(LB]}\odot\mathcal{X}^\mu_{\lambda B}=_{(\lambda,\mu)}\mathcal{X}^\mu_{\lambda((LB]B)}$. By Lemma 3, $(L\cap B)\subseteq((LB]B]$.

(ii) $\Rightarrow$ (i) Sine $(\mathcal{G}a]$ is a both left and bi-ideal of $\mathcal{G}$ with left identity. Therefore, $a\in(\mathcal{G}a]\cap(\mathcal{G}a]=(((\mathcal{G}a](\mathcal{G}a]](\mathcal{G}a]]=(((\mathcal{G}a](\mathcal{G}a])((\mathcal{G}a]=(((\mathcal{G}a)(\mathcal{G}a))(\mathcal{G}a)]=(((\mathcal{G}\mathcal{G})(aa))(\mathcal{G}a]\subseteq(\mathcal{G}a^2)\mathcal{G}$. Hence, $\mathcal{G}$ is intra-regular. ■

Theorem 8 *The following conditions are equivalent for $\mathcal{G}$ with left identity:*

(i) *$\mathcal{G}$ is intra-regular.*

(ii) $\langle\alpha,L\rangle\widetilde{\cap}(\beta,B)\widetilde{\cap}\langle\omega,R\rangle\subseteq\vee q_{(\lambda,\mu)}[\langle\alpha,L\rangle\odot\langle\beta,B\rangle]\odot[\langle\alpha,L\rangle\odot\langle\omega,R\rangle]$, *where $\langle\alpha,L\rangle$ an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft left ideal, $\langle\beta,R\rangle$ ia an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft right ideal and (ω,B) an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft bi-ideal over $\mathcal{G}$.*

Proof. (i) $\Rightarrow$ (ii) Let $\langle\alpha,L\rangle$ an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft left ideal, $\langle\beta,R\rangle$ ia an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft right ideal and (ω,B) an $(\in_\lambda,\in_\lambda\vee q_\mu)$-cubic soft bi-ideal over an intra-regular $\mathcal{G}$ with left identity. Let a be any element of $\mathcal{G}$, $\langle\alpha,L\rangle\widetilde{\cap}\langle\omega,B\rangle\widetilde{\cap}(\beta,R)=\langle\theta,B\cup L\cup R\rangle$. For any $e\in B\cup L\cup R$. We consider the following cases.

Case 1. $e\in L\backslash(B\cap R)$, then $\alpha(e)=[(\alpha\circ\beta)\circ\omega](e)$.

Case 2. $e\in B\backslash(L\cap R)$, then $\beta(e)=[(\alpha\circ\beta)\circ\omega](e)$.

Case 3. $e\in R\backslash(L\cap B)$, then $\omega(e)=[(\alpha\circ\beta)\circ\omega](e)$.

Case 4. $e\in(L\cap B)\cap R$, then $[(\alpha\circ\beta)\circ\omega](e)=[\alpha(e)\circ\beta(e)]\circ\omega(e)=[\alpha(e)\circ\beta(e)]\circ[\alpha(e)\circ\omega(e)]$.We have to show that $\alpha(e)\cap\beta(e)\cap\omega(e)\subseteq\vee q_{(\lambda,\mu)}(\alpha(e)\circ\beta(e))\circ(\alpha(e)\circ\omega(e))$. Since $\mathcal{G}$ is intra-regular, then for any $a\in\mathcal{G}$ there exist $x,y\in\mathcal{G}$ such that $a\leq(xa^2)y=(a(xa))y=y(xa)\leq y(x((xa^2)y)=y(x(a(xa))y)=$

$y((a(xa))(xy)) = (a(xa))(y(xy)) = ((y(xy))(xa))a = ba$, where $b = (y(xy))(xa)$ and $a \leq (xa^2)y = (x(aa))y = (a(xa))y = (y(xa))a = ca$. Thus, $(c, a) \in A_a$ and $(y(xa), a) \in A_a$. Since $A_a \neq \Phi$, therefore

$$\begin{aligned} & r\max\left\{(\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)})(a), \widetilde{r}_1\right\} \\ &= r\max\left[r\sup_{(ca,a)\in A_a}\left\{r\min\left\{\widetilde{\Psi}_{\alpha(e)}(c), \widetilde{\Psi}_{\beta(e)}(a)\right\}\right\}, \widetilde{\gamma}_1\right] \\ &\succeq r\max\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}(y(xa)), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\ &= r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}(y(xa)), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\gamma}_1\right\}\right] \\ &\succeq r\min\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\delta}_1\right\}\right] \\ &= r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \cap \widetilde{\Psi}_{\beta(e)})(a), \widetilde{\delta}_1\right\} \end{aligned}$$

and

$$\begin{aligned} \min\left\{(\phi_{\alpha(e)} \circ \phi_{\beta(e)})(a), \gamma_2\right\} &= \min\left[r\min_{(ca,a)\in A_a}\left\{\max\left\{\left(\phi_{\alpha(e)}(c), \phi_{\beta(e)}(a)\right)\right\}\right\}, \gamma_2\right] \\ &\leq \min\left[\max\left\{\phi_{\alpha(e)}(y(xa)), \phi_{\beta(e)}(a)\right\}, \gamma_2\right] \\ &= \max\left[\min\left\{\phi_{\alpha(e)}(y(xa)), \gamma_2\right\}, \min\left\{\phi_{\beta(e)}(a), \gamma_2\right\}\right] \\ &\leq \max\left[\max\left\{\phi_{\alpha(e)}(a), \delta_2\right\}, \max\left\{\phi_{\beta(e)}(a), \delta_2\right\}\right] \\ &= \max\left\{\left(\phi_{\alpha(e)} \cap \phi_{\beta(e)}\right)(a), \delta_2\right\} \end{aligned}$$

Thus, by Lemma 1 $\langle\alpha, L\rangle \widetilde{\cap} \langle\beta, B\rangle \subseteq \vee q_{(\lambda,\mu)} \langle\alpha, L\rangle \odot \langle\beta, B\rangle$.

Similarly, we can show that $\langle\alpha, L\rangle \widetilde{\cap} \langle\omega, R\rangle \subseteq \vee q_{(\lambda,\mu)} \langle\alpha, L\rangle \odot \langle\omega, R\rangle$. Hence,

$$\langle\alpha, L\rangle \widetilde{\cap} (\beta, B) \widetilde{\cap} \langle\omega, R\rangle \subseteq \vee q_{(\lambda,\mu)} [\langle\alpha, L\rangle \odot \langle\beta, B\rangle] \odot [\langle\alpha, L\rangle \odot \langle\omega, R\rangle].$$

(ii) $\Rightarrow$ (i) Since $\mathcal{G} \subseteq \mathcal{G}$ with left identity therefore by using Theorem 6 $\Sigma\langle\mathcal{G}, E\rangle$ is cubic soft set over $\mathcal{G}$ and $\mathcal{X}^{\mu}_{\lambda\mathcal{G}}$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right ideal over $\mathcal{G}$. Therefore,

$$\begin{aligned} \langle\alpha, L\rangle \widetilde{\cap} (\beta, B) &= \langle\alpha, L\rangle \widetilde{\cap} (\beta, B) \widetilde{\cap} \mathcal{X}^{\mu}_{\lambda\mathcal{G}} \subseteq \vee q_{(\lambda,\mu)} [\langle\alpha, L\rangle \odot (\beta, B)] \odot [(\beta, B) \odot \mathcal{X}^{\mu}_{\lambda\mathcal{G}}] \\ &\subseteq \vee q_{(\lambda,\mu)} [\langle\alpha, L\rangle \odot (\beta, B)] \odot (\beta, B). \end{aligned}$$

This implies that $\langle\alpha, L\rangle \widetilde{\cap} (\beta, B) \subseteq \vee q_{(\lambda,\mu)} [\langle\alpha, L\rangle \odot (\beta, B)] \odot (\beta, B)$. Hence, by Theorem 7, $\mathcal{G}$ is intra-regular. ∎

Lemma 7 *The following conditions are equivalent for $\mathcal{G}$ with left identity.*

(i) *$\mathcal{G}$ is intra-regular.*

(ii) *$\langle\alpha, R\rangle \widetilde{\cap} \langle\beta, L\rangle = \langle\alpha, R\rangle \odot \langle\beta, L\rangle$,.for an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right ideal $\langle\alpha, R\rangle$ and for an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal $\langle\beta, L\rangle$ over $\mathcal{G}$, such that $\langle\alpha, R\rangle$ is an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft semiprime.*

Proof. The proof is straightforward. ∎

Note that every intra-regular $\mathcal{AG}$-groupoid $\mathcal{G}$, with left identity is regular but the converse is not true in general [18].

Theorem 9 *The following conditions are equivalent for $\mathcal{G}$ with left identity:*

(i) *$\mathcal{G}$ is intra-regular.*

(ii) $\langle \alpha, R\rangle \widetilde{\cap} \langle \omega, B\rangle \widetilde{\cap} \langle \beta, L\rangle \subseteq \vee q_{(\lambda,\mu)} [\langle \alpha, R\rangle \odot \langle \omega, B\rangle] \odot \langle \beta, L\rangle$, *for any* $(\in_\lambda, \in_\lambda \vee q_\mu)$ *-cubic soft right ideal* $\langle \alpha, R\rangle$, $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft bi-ideal* $\langle \omega, B\rangle$ *and* $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft left ideal* $\langle \beta, L\rangle$ *over* $\mathcal{G}$.

(iii) $\langle \alpha, R\rangle \widetilde{\cap} \langle \omega, B\rangle \widetilde{\cap} \langle \beta, L\rangle \subseteq \vee q_{(\lambda,\mu)} [\langle \omega, B\rangle \odot \langle \alpha, R\rangle] \odot \langle \beta, L\rangle$, *for any* $(\in_\lambda, \in_\lambda \vee q_\mu)$ *-cubic soft right ideal* $\langle \alpha, R\rangle$, $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft generalized bi-ideal* $\langle \omega, B\rangle$ *and* $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft left ideal* $\langle \beta, L\rangle$ *over* $\mathcal{G}$.

Proof. (i) $\Rightarrow$ (iii) Assume that $\mathcal{G}$ is intra-regular with left identity. Let $\langle \alpha, R\rangle$, $\langle \omega, B\rangle$ and $\langle \beta, L\rangle$ be any $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft right ideal, $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft bi-ideal and $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft left ideal over $\mathcal{G}$, respectively. Let $\langle \alpha, R\rangle \widetilde{\cap} \langle \omega, B\rangle \widetilde{\cap} (\beta, L) = \langle K, R \cup B \cup L\rangle$. For any $e \in R \cup B \cup L$. We consider the following cases.

Case 1. $e \in R \backslash (B \cap L)$, then $\alpha(e) = [(\alpha \circ \omega) \circ \beta](e)$.

Case 2. $e \in B \backslash (R \cap L)$, then $\beta(e) = [(\alpha \circ \omega) \circ \beta](e)$.

Case 3. $e \in L \backslash (R \cap B)$, then $\omega(e) = [(\alpha \circ \omega) \circ \beta](e)$.

Case 4. $e \in (L \cap B) \cap R$, then $[(\alpha \circ \omega) \circ \beta](e) = [\alpha(e) \circ \omega(e)] \circ \beta(e)$. Since $\mathcal{G}$ is intra-regular, then for any $a \in \mathcal{G}$ there exist $x, y, u \in \mathcal{G}$ such that $a \leq (ua)a$ and $a \leq (xa^2)y$. Now, $au \leq ((xa^2)y)u = (uy)(xa^2) = (ux)(ya^2) = (a^2x)(yu) = ((yu)x)(aa) = ((yu)a)(xa) = (ax)(a(yu)) = (aa)(x(yu)) = ((x(yu))a)a = ((x(yu))(ea))a = ((xe)((yu)a))a = ((ae)((yu)x))a$. Thus, $(((ae)((yu)x)), a) = (d, a) \in A_{au}$, where $d = ((ae)((yu)x))$. Therefore

$$
\begin{aligned}
& r\max\left\{((\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)}) \circ \widetilde{\Psi}_{\omega(e)})(a), \widetilde{\gamma}_1\right\} \\
&= r\max\left[r \sup_{(au,a)\in A_a} \left\{r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)})(au), \widetilde{\Psi}_{\omega(e)}(a)\right\}\right\}, \widetilde{\gamma}_1\right] \\
&\succeq r\max\left[r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)})(au), \widetilde{\Psi}_{\omega(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\
&= r\max\left[r\min\left\{r \sup_{(d,a)\in A_{au}} \left\{r\min\left\{\widetilde{\Psi}_{\alpha(e)}(d), \widetilde{\Psi}_{\beta(e)}(a)\right\}\right\}, \widetilde{\Psi}_{\omega(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\
&\succeq r\max\left[r\min\left\{r\min\left\{\widetilde{\Psi}_{\alpha(e)}(d), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\Psi}_{\omega(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\
&= r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ae)((yu)x)), \widetilde{\Psi}_{\beta(e)}(a), \widetilde{\Psi}_{\omega(e)}(a)\right\}, \widetilde{\gamma}_1\right]
\end{aligned}
$$

$= r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}((ae)((yu)x)), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{\omega(e)}(a), \widetilde{\gamma}_1\right\}\right]$
$\succeq r\min\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{\omega(e)}(a), \widetilde{\delta}_1\right\}\right]$
$= r\min\left\{(\widetilde{\Psi}_{\alpha(e)} \cap \widetilde{\Psi}_{\beta(e)} \cap \widetilde{\Psi}_{\omega(e)})(a), \widetilde{\delta}_1\right\}$
and
$\min\left\{((\phi_{\alpha(e)} \circ \phi_{\beta(e)}) \circ \phi_{\omega(e)})(a), \gamma_2\right\}$
$= \min\left[r\min_{(au,a)\in A_a}\left\{\max\left\{(\phi_{\alpha(e)} \circ \phi_{\beta(e)})(au), \phi_{\omega(e)}(a)\right\}\right\}, \gamma_2\right]$
$\leq \min\left[\max\left\{(\phi_{\alpha(e)} \circ \phi_{\beta(e)})(au), \phi_{\omega(e)}(a)\right\}, \gamma_2\right]$
$= \min\left[\max\left\{\min_{(d,a)\in A_{au}}\left\{\max\left\{\phi_{\alpha(e)}(d), \phi_{\beta(e)}(a)\right\}\right\}, \phi_{\omega(e)}(a)\right\}, \gamma_2\right]$
$\leq \min\left[\max\left\{\max\left\{\phi_{\alpha(e)}(d), \phi_{\beta(e)}(a)\right\}, \phi_{\omega(e)}(a)\right\}, \gamma_2\right]$
$= \max\left[\min\left\{\phi_{\alpha(e)}((ae)((yu)x)), \phi_{\beta(e)}(a), \phi_{\omega(e)}(a)\right\}, \gamma_2\right]$
$= \max\left[\min\left\{\phi_{\alpha(e)}((ae)((yu)x)), \gamma_2\right\}, \min\left\{\phi_{\beta(e)}(a), \gamma_2\right\}, \min\left\{\phi_{\omega(e)}(a), \gamma_2\right\}\right]$
$\leq \max\left[\max\left\{\phi_{\alpha(e)}(a), \delta_2\right\}, \max\left\{\phi_{\beta(e)}(a), \delta_2\right\}, \max\left\{\phi_{\omega(e)}(a), \delta_2\right\}\right]$
$= \max\left\{(\phi_{\alpha(e)} \cap \phi_{\beta(e)} \cap \phi_{\omega(e)})(a), \delta_2\right\}.$

Thus, by Lemma 1, $\langle \alpha, R\rangle \widetilde{\cap} \langle \omega, B\rangle \widetilde{\cap} \langle \beta, L\rangle \subseteq \vee q_{(\lambda,\mu)}[\langle \alpha, R\rangle \odot \langle \omega, B\rangle] \odot \langle \beta, L\rangle$.

(iii) $\Rightarrow$ (ii) It is straightforward.
(ii) $\Rightarrow$ (i) By using the given assumption, it is easy to show that $\langle \alpha, R\rangle \odot \langle \omega, B\rangle \subseteq \vee q_{(\lambda,\mu)}\langle \alpha, R\rangle \widetilde{\cap} \langle \omega, B\rangle$ and therefore by using Lemma 7, $\mathcal{G}$ is intra-regular. ∎

Theorem 10 *The following conditions are equivalent for $\mathcal{G}$ with left identity:*

(i) *$\mathcal{G}$ is intra-regular.*

(ii) $[\langle \alpha, A\rangle \odot \langle \beta, B\rangle] \widetilde{\cap} [\langle \beta, B\rangle \odot \langle \alpha, A\rangle] \supseteq \vee q_{(\lambda,\mu)}\langle \alpha, A\rangle \widetilde{\cap} \langle \beta, B\rangle$, *for any* $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft right ideal* $\langle \alpha, A\rangle$ *and* $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft left ideal* $\langle \beta, B\rangle$ *over* $\mathcal{G}$.

(iii) $[\langle \alpha, A\rangle \odot \langle \beta, B\rangle] \widetilde{\cap} [\langle \beta, B\rangle \odot \langle \alpha, A\rangle] \supseteq \vee q_{(\lambda,\mu)}\langle \alpha, A\rangle \widetilde{\cap} \langle \beta, B\rangle$, *for any* $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft right ideal* $\langle \alpha, A\rangle$ *and* $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft bi-ideal* $\langle \beta, B\rangle$ *over* $\mathcal{G}$.

(iv) $[\langle \alpha, A\rangle \odot \langle \beta, B\rangle] \widetilde{\cap} [\langle \beta, B\rangle \odot \langle \alpha, A\rangle] \supseteq \vee q_{(\lambda,\mu)}\langle \alpha, A\rangle \widetilde{\cap} \langle \beta, B\rangle$, *for any* $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft bi-ideal* $\langle \alpha, A\rangle$ *and for* $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft generalized bi-ideal* $\langle \beta, B\rangle$ *over* $\mathcal{G}$.

(v) $[\langle \alpha, A\rangle \odot \langle \beta, B\rangle] \widetilde{\cap} [\langle \beta, B\rangle \odot \langle \alpha, A\rangle] \supseteq \vee q_{(\lambda,\mu)}\langle \alpha, A\rangle \widetilde{\cap} \langle \beta, B\rangle$, *for* $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft bi-ideals* $\langle \alpha, A\rangle$ *and* $\langle \beta, B\rangle$ *over* $\mathcal{G}$.

(vi) $[\langle \alpha, A\rangle \odot \langle \beta, B\rangle] \widetilde{\cap} [\langle \beta, B\rangle \odot \langle \alpha, A\rangle] \supseteq \vee q_{(\lambda,\mu)}\langle \alpha, A\rangle \widetilde{\cap} \langle \beta, B\rangle$, *for* $(\in_\lambda, \in_\lambda \vee q_\mu)$*-cubic soft generalized bi-ideals* $\langle \alpha, A\rangle$ *and* $\langle \beta, B\rangle$ *over* $\mathcal{G}$.

Proof. (i) $\Rightarrow$ (vi) Let (α, A) and (β, B) be an $(\in_\lambda, \in_\lambda \vee q_\mu)$-cubic soft generalized bi-ideals over an intra-regular $\mathcal{G}$ with left identity. Let $e \in A \cup B$ and $(\alpha, A)\widetilde{\cap}(\beta, B) = (\omega, A \cup B)$.

Now, we consider the following cases.

Case 1. $e \in A - B$. Then $\omega(e) = \alpha(e) = (\alpha \circ \beta)(e)$.

Case 2. $e \in B - A$. Then $\omega(e) = \beta(e) = (\alpha \circ \beta)(e)$.

Case 3. $e \in A \cap B$. Then $\omega(e) = \alpha(e) \cap \beta(e)$ and $(\alpha \circ \beta)(e) = \alpha(e) \circ \beta(e)$.

We have to show that $\alpha(e) \circ \beta(e) \supseteq \vee q_{(\lambda,\mu)} \alpha(\epsilon) \cap \beta(e)$.

Since $\mathcal{G}$ is intra-regular, then for any $a \in \mathcal{G}$ there exist $x, y, u \in \mathcal{G}$ such that $a \leq (xa^2)y$ and $a \leq (au)a$. Now

$$a \leq (xa^2)y = (a(xa))y = (y(xa))a,$$

and

$$\begin{aligned} y(xa) &\leq y(x((xa^2)y)) = y((xa^2)(xy)) = (xa^2)(y(xy)) = (xa^2)(xy^2) \\ &= (xx)(a^2y^2) = a^2(x^2y^2) = (aa)(x^2y^2) \\ &= ((x^2y^2)a)a = ((x^2y^2)((xa^2)y))a \\ &= ((xa^2)((x^2y^2)y))a = ((x(x^2y^2))(a^2y))a \\ &= ((y(x^2y^2))(a^2x))a = (a^2((y(x^2y^2))x))a \\ &= ((x(y(x^2y^2)))a^2)a = (a(x(y(x^2y^2)))a)a \\ &= (av)a, \text{ where } v = a(x(y(x^2y^2)))a. \end{aligned}$$

Hence,

$$a \leq ((av)a)a.$$

Hence, $((av)a, a) \in A_a$. Therefore,

$$\begin{aligned} &r\max\left\{(\widetilde{\Psi}_{\alpha(e)} \circ \widetilde{\Psi}_{\beta(e)})(a), \widetilde{r}_1\right\} \\ &\quad = r\max\left[r \sup_{((av)a,a)\in A_a} \left\{r\min\left\{\left(\widetilde{\Psi}_{\alpha(e)}((av)a), \widetilde{\Psi}_{\beta(e)}(a)\right)\right\}\right\}, \widetilde{\gamma}_1\right] \\ &\quad \succeq r\max\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}((av)a), \widetilde{\Psi}_{\beta(e)}(a)\right\}, \widetilde{\gamma}_1\right] \\ &\quad = r\min\left[r\max\left\{\widetilde{\Psi}_{\alpha(e)}((av)a), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\gamma}_1\right\}\right] \\ &\quad \succeq r\min\left[r\min\left\{\widetilde{\Psi}_{\alpha(e)}(a), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{\beta(e)}(a), \widetilde{\delta}_1\right\}\right] \\ &\quad = r\min\left\{\left(\widetilde{\Psi}_{\alpha(e)} \cap \widetilde{\Psi}_{\beta(e)}\right)(a), \widetilde{\delta}_1\right\} \end{aligned}$$

and

$$\begin{aligned}
&\min\left\{(\phi_{\alpha(e)}\circ\phi_{\beta(e)})(a),\gamma_2\right\}\\
&\quad=\min\left[r\min_{((av)a,a)\in A_a}\left\{\max\left\{\left(\phi_{\alpha(e)}((av)a),\phi_{\beta(e)}(a)\right)\right\}\right\},\gamma_2\right]\\
&\quad\leq\min\left[\max\left\{\phi_{\alpha(e)}((av)a),\phi_{\beta(e)}(a)\right\},\gamma_2\right]\\
&\quad=\max\left[\min\left\{\phi_{\alpha(e)}((av)a),\gamma_2\right\},\min\left\{\phi_{\beta(e)}(a),\gamma_2\right\}\right]\\
&\quad\leq\max\left[\max\left\{\phi_{\alpha(e)}(a),\delta_2\right\},\max\left\{\phi_{\beta(e)}(a),\delta_2\right\}\right]\\
&\quad=\max\left\{\left(\phi_{\alpha(e)}\cap\phi_{\beta(e)}\right)(a),\delta_2\right\}
\end{aligned}$$

This shows that

$$\langle\alpha,A\rangle\odot\langle\beta,B\rangle\supseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle$$

and, similarly, we can show that

$$\langle\beta,B\rangle\odot\langle\alpha,A\rangle\supseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle.$$

Therefore,

$$[\langle\alpha,A\rangle\odot\langle\beta,B\rangle]\widetilde{\cap}[\langle\beta,B\rangle\odot\langle\alpha,A\rangle]\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle.$$

(vi) $\Rightarrow$ (v) $\Rightarrow$ (iv) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are obvious cases.

(ii) $\Rightarrow$ (i) By using (ii), it is easy to show that

$$\langle\alpha,A\rangle\odot\langle\beta,B\rangle\subseteq\vee q_{(\lambda,\mu)}\langle\alpha,A\rangle\widetilde{\cap}\langle\beta,B\rangle$$

and, therefore, by using Lemma 7, $\mathcal{G}$ is intra-regular. ∎

Conclusion

In this paper, we discuss a new approach to soft set through applications of cubic set. By combination of cubic set and soft set, we introduce a new mathematical model which is called cubic soft set. We introduce a new concept of generalized cubic soft sets and apply it to the ideal theory of ordered $\mathcal{AG}$-groupoids. We introduce $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft left (resp., right, bi-) ideals over an ordered $\mathcal{AG}$-groupoids and characterize intra-regular ordered $\mathcal{AG}$-groupoids in terms of $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft sets. In our future research work we will focus on considering other types of $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)}\vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft ideals over an intra-regular ordered $\mathcal{AG}$-groupoids.

Acknowledgements. The authors wish to thank the anonymous reviewer(s) for valuable comments and suggestions.

References

[1] Aktas, H., Cagman, N., *Soft sets and soft groups*, Inform. Sci., 177 (13) (2007), 2726-2735.

[2] Atagun, A.O., Sezgin, A., *Soft substructures of rings, fields and modules*, Comput. Math. Appl., 61 (3) (2011), 592-601.

[3] Cagman, N., Enginoglu, S., *Soft set theory and uni−int decision making*, European Journal of Operational Research, 207, 2 (2010), 848-855.

[4] Dudek, W.A., Khan, M., Khan, N., *Characterizations of intra-regular Abel-Grassmann's groupoids*, Journal of Intelligent and Fuzzy Systems, in press, DOI: 10.3233/IFS-141251.

[5] Feng, F., Li, C.X., Davvaz, B., Ali, M.I., *Soft sets combined with fuzzy sets and rough sets: a tenta approach*, Soft Computing, 14 (2010), 899-911.

[6] Feng, F., Li, Y.M., *Soft subsets and soft product operations*, Inform. Sci., 232 (2013), 44-57.

[7] Irfan Ali, M., Feng, F., Liu, X., Min, W.K., Shabir, M., *On some new operations in soft set theory*, Comput. Math. Appl., 57 (2009), 1547-1553.

[8] Jun, Y.B., Lee, K.J., Kang, M.S., *Cubic structure applied to ideals of BCI-algebra*, Comput. Math. Appl., 62 (9) (2011), 3334-3342.

[9] Jun, Y.B., Kim, C.S., Yang, K.O., *Cubic sets*, Ann. Fuzzy Math. Inform., 4 (1) (2012), 83-98.

[10] Jun, Y.B., Kim, C.S., Kang, M.S., *Cubic subalgebras and ideals of BCK/BCI-algebras*, For East J. Math. Sci., 44 (2010), 239-250.

[11] Kazim, M.A., Naseeruddin, M., *On almost semigroups*, The Alig. Bull. Math., 2 (1972), 1-7.

[12] Khan, M., Yousafzai, F., *On fuzzy ordered Abel-Grassmann's groupoids*, Journal of Mathematics Research, 3 (2011), 27-40.

[13] Muhiuddin, G., Al-roqi, A.M., *Cubic soft sets with applications in BCK/BCL-algebras*, Annals of Fuzzy Mathematics and Informatics, In press.

[14] Khan, M., Jun, Y.B., Gulistan, M., Yaqoob, N., *The generalized version of Jun's cubic sets in semigroups*, Journal of Intelligent and Fuzzy Systems, DOI: 10-3233/IFS-141377.

[15] Molodstov, *Soft set theory first-results*, Comput. Math. Appl., 37 (4-5) (1999), 19-31.

[16] Maji, P.K., Biswas, R., Roy, A.R., *Fuzzy soft sets*, J. Fuzzy Math., 9 (3) (2001), 589-502.

[17] Yin, Y., Zhan, J., *Characterizations of ordered semigroups in terms of fuzzy soft ideals*, Bull. Malays. Math. Sci., 35 (4) (2012), 997-1015.

[18] Zadeh, L.A., *Fuzzy sets*, Inform. Control, 8 (1965), 338-353.

[19] Zhan, J., Jun, Y.B., *Soft BL-algebra based on fuzzy sets*, Comput. Math. Appl., 59 (6) (2010), 2037-2040.

Accepted: 15.06.2015