A GENERALIZED COMMON FIXED POINT THEOREM
FOR SIX SELF-MAPS

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Abstract. A recent common fixed point theorem of Kikina and Kikina (2011) has been extended to two triads of self-maps through the notions of weak compatibility and the property (EA), under an implicit-type relation and restricted completeness, namely orbital completeness of the space.

Keywords: property (EA), implicit relation, orbital completeness, weak compatibility, common fixed point.

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1. Introduction

Throughout this paper, \((X, d)\) denotes a metric space. Given \(x_0 \in X\) and \(f, g\) and \(h\) self-maps on \(X\), the associated sequence \(\langle x_n \rangle_{n=1}^{\infty} \subset X\) with the choice

\[
(1.1) \quad x_{3n-2} = fx_{3n-3}, \quad x_{3n-1} = gx_{3n-2}, \quad x_{3n} = hx_{3n-1} \quad \text{for} \quad n = 1, 2, 3, ...
\]

is an \((f, g, h)\)-orbit at \(x_0\). The metric space \(X\) is \((f, g, h)\)-orbitally complete \([6]\) if every Cauchy sequence in the \((f, g, h)\)-orbit at each \(x_0 \in X\) converges in \(X\).

Kikina and Kikina [6] proved the following

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Theorem 1.1 Let $f$, $g$ and $h$ be self-maps on $X$ satisfying the three conditions:
\begin{align}
\text{(1.2)} & \quad [1 + pd(x, y)]d(fx, gy) \leq p[d(x, fx)d(y, gy) + d(x, gy)d(y, fx)] \\
& \quad + q \max \left\{ d(x, y), d(x, fx), d(y, gy), \frac{1}{2}[d(x, gy) + d(y, fx)] \right\}, \\
\text{(1.3)} & \quad [1 + pd(x, y)]d(gx, hy) \leq p[d(x, gx)d(y, hy) + d(x, hy)d(y, gx)] \\
& \quad + q \max \left\{ d(x, y), d(x, gx), d(y, hy), \frac{1}{2}[d(x, hy) + d(y, gx)] \right\}, \\
\text{(1.4)} & \quad [1 + pd(x, y)]d(hx, fy) \leq p[d(x, hx)d(y, fy) + d(x, fy)d(y, hx)] \\
& \quad + q \max \left\{ d(x, y), d(x, hx), d(y, fy), \frac{1}{2}[d(x, fy) + d(y, hx)] \right\}
\end{align}
for all $x, y \in X$, where
\[
p > \frac{1}{\max \{d(x, y) : x, y \in X\}} \quad \text{with} \quad \max \{d(x, y) : x, y \in X\} > 0
\]
and $0 \leq q < 1$. If $X$ is $(f, g, h)$-orbitally complete, then $f$, $g$ and $h$ will have a unique common fixed point.

In this paper, we first extend the orbital completeness to two triads of self-maps and give an extended version of Theorem 1.1 using weak compatibility and property (EA) under generalized inequalities of (1.2)-(1.4) involving an implicit relation.

2. Preliminaries

Self-maps $f$ and $r$ on $(X, d)$ are known to be commuting if $frx = rfx$ for all $x \in X$, where $fr$ denotes the composition of $f$ and $r$. As a weaker form of it, Sessa [14] introduced weakly commuting maps $f$ and $r$ on $X$ with the choice:
\[
d(fr x, rf x) \leq d(fx, rx) \quad \text{for all} \quad x \in X.
\]
As a further weaker form of commuting mappings, Gerald Jungck [3] introduced the notion of compatibility as follows:

**Definition 2.1** Self-maps $f$ and $r$ on $X$ are said to be compatible if there is a sequence $\langle x_n \rangle_{n=1}^{\infty} \subset X$ such that
\[
\lim_{n \to \infty} fx_n = \lim_{n \to \infty} rx_n = p \quad \text{for some} \quad p \in X
\]
implies that
\[
\lim_{n \to \infty} d(fr x_n, rf x_n) = 0.
\]
It is obvious that every commuting pair is weakly commuting and every weakly commuting pair is compatible. However neither reverse implication is true. For examples, one can refer to [3] and [14].

We observe that both compatibility and noncompatibility of \((f,r)\) guarantee the existence of a sequence \(\langle x_n \rangle \) with the choice (2.2). Self-maps \(f\) and \(r\) on \(X\) satisfy the property (EA) [1] if (2.2) holds good for some \(\langle x_n \rangle \) \(\subseteq X\). The class of compatible maps is contained in the class of weakly compatible maps [4] which commute at their coincidence points. However, weak compatibility and property (EA) are independent of each other [9]. Various form of compatibility and their interrelation was presented in [2]. Weak compatibility have nice applications in dynamical programming (See [10]).

The idea of contraction type conditions involving an implicit relation was first introduced by Popa [13] which covers several contractive conditions and has the ability to unify several fixed point theorems.

For instance, [5] utilized one such implicit relation \(\psi : \mathbb{R}_+^6 \to \mathbb{R}\) which is lower semicontinuous and satisfies the following conditions:

\((C_1)\) \(\psi\) is nondecreasing in the fifth and sixth coordinate variables,
\((C_2)\) For every \(l \geq 0, m \geq 0\), there is a constant \(0 \leq \omega < 1\) such that
\[(2.4) \quad \min\{\psi(l, m, m, l, l + m, 0), \psi(l, m, l, m, 0, l + m)\} \leq 0 \Rightarrow l \leq \omega m,\]
\((C_3)\) \(\psi(l, l, 0, 0, l, l) > 0\) for all \(l > 0\).

In this paper, we first extend the property (EA) and orbital completeness to two triads of self-maps and then obtain an extended generalization of Theorem 1.1 involving weak compatibility and an implicit-type relation which does not require the choice \((C_1)\), unlike in [5].

### 3. Main result

We begin with orbital completeness involving two triads of self-maps as follows:

Given \(x_0 \in X\) and two triads \((f, g, h)\) and \((r, s, t)\) of self-maps on \(X\), if there exist points \(x_1, x_2, x_3, \ldots\) in \(X\) such that

\[
y_{3n-2} = fx_{3n-3} = rx_{3n-2}, \quad y_{3n-1} = gx_{3n-2} = sx_{3n-1},
\]
\[
y_{3n} = hx_{3n-1} = tx_{3n} \quad \text{for } n = 1, 2, 3, \ldots,
\]

then the associated sequence \(\langle y_n \rangle\) \(\subseteq X\) is an \((f, g, h)\)-orbit relative \((r, s, t)\) at \(x_0\).

The space \(X\) is \((f, g, h)\)-orbitally complete at \(x_0\) relative \((r, s, t)\) if every Cauchy sequence in an \((f, g, h)\)-orbit converges in \(X\), and \(X\) is \((f, g, h)\)-orbitally complete relative \((r, s, t)\) if it is \((f, g, h)\)-orbitally complete relative to \((r, s, t)\) at each \(x_0 \in X\).

It may be noted that if \(r = s = t = i\), the identity map, then we get the orbital completeness as discussed in [6].
Similarly, if \( g = h = f \) and \( t = s = r \) in this notion, we get the \( f \)-orbital completeness relative to \( r \) as given in [11].

The property (EA) was extended to two pairs of self-maps in [7]. In fact, self-maps \((f, r)\) and \((g, s)\) share the common property (EA) on \( X \) if there exist sequences \( \langle x_n \rangle \) and \( \langle y_n \rangle \) in \( X \) such that

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} r x_n = \lim_{n \to \infty} g y_n = \lim_{n \to \infty} s y_n = u \quad \text{for some } u \in X. \tag{3.2}
\]

We improve the notion of [7] to a pair of triads of maps as follows:

Let \( f, g, h, r, s \) and \( t \) be self-maps on \( X \). We say that the triads \((f, g, h)\) and \((r, s, t)\) share the common property (EA) if we can find sequences \( \langle x_n \rangle \) and \( \langle z_n \rangle \) in \( X \) such that

\[
\lim_{n \to \infty} f x_n = \lim_{n \to \infty} g y_n = \lim_{n \to \infty} h z_n = \lim_{n \to \infty} r x_n = \lim_{n \to \infty} s y_n = \lim_{n \to \infty} t z_n = u \quad \text{for some } u \in X. \tag{3.3}
\]

If \( r = s = t = i \) in this notion, we get the property (EA) on the collection \( \{f, g, h\} \) as given in [8].

Our main result is

**Theorem 3.1** Let \( f, g, h, r, s \) and \( t \) be self-maps on \( X \) sharing the common property (EA) and satisfying the following inequalities:

\[
\psi(d(fx, gy), d(rx, sy), d(rx, fx), d(sy, gy), d(rx, gy), d(sy, fx)) < 0, \tag{3.4}
\]

\[
\psi(d(gx, hy), d(sx, ty), d(sx, gx), d(ty, hy), d(sx, hy), d(ty, gx)) < 0, \tag{3.5}
\]

\[
\psi(d(hx, fy), d(tx, ry), d(tx, hx), d(ry, fy), d(tx, fy), d(ry, hx)) < 0, \tag{3.6}
\]

for all \( x, y \in X \). Suppose that \( r, s \) and \( t \) are onto. If \((f, r)\), \((g, s)\) and \((h, t)\) are weakly compatible, then all the six maps \( f, g, h, r, s \) and \( t \) will have a unique common fixed point.

**Proof.** Suppose \( f, g, h, r, s \) and \( t \) share the common property. Then we can find sequences \( \langle x_n \rangle \), \( \langle y_n \rangle \) and \( \langle z_n \rangle \) in \( X \) with the choice (3.3). Since \( r \) is onto, we have

\[
u = rp \quad \text{for some } p \in X. \tag{3.7}
\]

Writing \( x = p \) and \( y = x_n \) in (3.4), we get

\[
\psi(d(fp, gx_n), d(rp, sx_n), d(rp, fp), d(sx_n, gx_n), d(rp, gx_n), d(sx_n, fp)) < 0.
\]

Applying the limit as \( n \to \infty \) and then using (3.3) and (3.7), this yields

\[
\psi(d(fp, rp), 0, d(rp, fp), 0, 0, d(rp, fp)) < 0.
\]
which is (2.4) with \( l = d(fp, rp) \) and \( m = 0 \). So, by \((C_2)\) we get \( d(fp, rp) \leq 0 \), that is \( fp = rp = u \). Since \((f, r)\) is weakly compatible, \( f \) and \( r \) commute at the coincidence point \( p \), that is \( frp = rfp \), we get

\[(3.8) \quad fu = ru.\]

Now \( s \) is onto implies that

\[(3.9) \quad u = sq \quad \text{for some} \quad q \in X.\]

Then writing \( x = q \) and \( y = y_n \) in (3.5), we get

\[\psi(d(gq, hy_n), d(sq, ty_n), d(sq, gq), d(ty_n, hy_n), d(sq, hy_n)) < 0.\]

Applying the limit as \( n \to \infty \) and then using (3.3) and (3.9), this yields

\[\psi(d(gq, sq), 0, d(sq, gq), 0, d(sq, gq)) < 0,\]

which is (2.4) with \( l = d(gq, sq) \) and \( m = 0 \). So, by \((C_2)\) we get \( d(gq, sq) \leq 0 \), that is \( gq = sq = u \). By the weak compatibility of, \( g \) and \( s \), we get \( gsq = sgu \) or

\[(3.10) \quad gu = su.\]

Finally, \( t \) is onto implies that

\[(3.11) \quad u = tw \quad \text{for some} \quad w \in X.\]

Then (3.6) with \( x = w \) and and \( y = u_n \) gives

\[\psi(d(hw, fz_n), d(tw, rz_n), d(tw, hw), d(rz_n, fz_n), d(tw, fz_n)) < 0.\]

Applying the limit as \( n \to \infty \) and then using (3.3) and (3.11), this yields

\[\psi(d(hw, tw), 0, d(tw, hw), 0, d(tw, hw)) < 0.\]

This in view of \((C_2)\) with \( l = d(tw, hw) \) and \( m = 0 \) finally gives \( d(tw, hw) \leq 0 \), that is \( tw = hw = u \). By the weak compatibility of, \( h \) and \( t \), we get \( thw = htw \) or

\[(3.12) \quad tu = hu.\]

But from (3.4) with \( x = y = u \), it follows that

\[\psi(d(fu, gu), d(fu, gu), 0, 0, d(fu, gu), d(gu, fu)) < 0,\]

which is nothing but (2.4) with \( l = d(fu, gu) \) and \( m = 0 \). Hence by \((C_2)\), we get

\[d(fu, gu) \leq 0 \quad \text{so that} \quad fu = gu.\]

On one hand, from (3.5) with \( x = u = y \) it follows that

\[\psi(d(gu, hu), d(ru, ru), d(ru, gu), d(ru, hu), d(ru, hu), d(ru, gu)) < 0,\]
or that \( \psi(d(gu, hu), 0, 0, d(fu, hu), d(ru, hu), 0) < 0 \).

If \( d(gu, hu) > 0 \), from \((C_3)\), we see a contradiction that
\[
\psi(d(gu, hu), 0, 0, d(fu, hu), d(ru, hu), 0) > 0.
\]

Hence we must have \( fu = gu \).

Now, writing \( x = y = u \) in (3.5) and then using (3.8), (3.10), (3.12) and \( fu = gu \), we get
\[
\psi(d(gu, hu), d(gu, hu), 0, 0, d(gu, hu), d(hu, gu)) < 0
\]
which, due to \((C_3)\), implies that \( gu = hu \).

In other words, \( u \) is a common coincidence point of all the six maps, that is
\[
(fu = gu = hu = ru = su = tu).
\]

Further, we see from (3.4) with \( x = u \) and \( y = x_n \) becomes
\[
\psi(d(fu, gx_n), d(ru, rx_n), d(ru, fu), d(rx_n, gx_n), d(ru, gu), d(rx_n, fu)) < 0.
\]

Proceeding the limit as \( n \to \infty \), in this and using (3.13), we obtain
\[
\psi(d(fu, u), d(fu, u), 0, 0, d(fu, u), d(fu, u)) < 0,
\]
which would be a contradiction to \((C_3)\) if \( d(fu, u) > 0 \), proving that \( u \) is a fixed point of \( f \) and hence a common fixed point of \( f, g, h \) and \( r \).

**Remark 3.1** Writing \( s = t = r \) in this main result, recently it has been shown in [15] that any two of the three inequalities (3.4)-(3.6) are sufficient to obtain a common fixed point for the four self-maps \( f, g, h \) and \( r \), using the weak compatibility of one of the three pairs \( (f, r) \), \((g, r)\) and \((h, r)\).

Now we write
\[
\psi(l_1, l_2, l_3, l_4, l_5, l_6) = (1 + pl_2)l_1 - p[l_3l_4 + l_5l_6] - q \max \left\{ l_2, l_3, l_4, \frac{1}{2}[l_5 + l_6] \right\},
\]
where \( p \) and \( q \) have the same choice as given in Theorem 1.1.

**Corollary 3.1** Let \( f, g, h, r, s \) and \( t \) be self-maps on \( X \) satisfying following conditions:

\[
[1 + pd(rx, sy)]d(fx, gy) \leq p[d(rx, fx)d(sy, gy) + d(rx, gy)d(sy, fx)]
\]
\[
+ q \max \{d(rx, sy), d(rx, fx), d(sy, gy), \frac{1}{2}[d(rx, gy) + d(sy, fx)]\},
\]

\[
[1 + pd(sx, ty)]d(gx, hy) \leq p[d(sx, gx)d(ty, hy) + d(sx, hy)d(ty, gx)]
\]
\[
+ q \max \{d(sx, ty), d(sx, gx), d(ty, hy), \frac{1}{2}[d(sx, hy) + d(ty, gx)]\},
\]

where \( s, t, r \) are self-maps on \( X \) satisfying following conditions:

\[
[1 + pd(rx, sy)]d(fx, gy) \leq p[d(rx, fx)d(sy, gy) + d(rx, gy)d(sy, fx)]
\]
\[
+ q \max \{d(rx, sy), d(rx, fx), d(sy, gy), \frac{1}{2}[d(rx, gy) + d(sy, fx)]\},
\]

\[
[1 + pd(sx, ty)]d(gx, hy) \leq p[d(sx, gx)d(ty, hy) + d(sx, hy)d(ty, gx)]
\]
\[
+ q \max \{d(sx, ty), d(sx, gx), d(ty, hy), \frac{1}{2}[d(sx, hy) + d(ty, gx)]\},
\]

where \( p \) and \( q \) have the same choice as given in Theorem 1.1.
\[ [1 + pd(tx, ry)]d(hx, fy) \leq p[d(tx, hx)d(ry, fy) + d(tx, fy)d(ry, hx)] \]
\[ + q \max\{d(tx, ry), d(tx, hx), d(ry, fy), \frac{1}{2}d(tx, fy) + d(ry, hx)\} \]

for all \( x, y \in X \), where

\[ p > \frac{1}{\max\{d(x, y) : x, y \in X\}} \]

with \( \max\{d(x, y) : x, y \in X\} > 0 \) and \( 0 < q < 1 \). Suppose that

\[ f(X) \subset r(X), g(X) \subset s(X), h(X) \subset t(X). \]

Suppose that \( X \) is \((f, g, h)\)-orbitally complete relative to \((r, s, t)\), and \( r, s \) and \( t \) are onto. If \((f, r), (g, s)\) and \((h, t)\) are weakly compatible, then all the six maps \( f, g, h, r, s \) and \( t \) will have a unique common fixed point.

**Proof.** Let \( p_0 \in X \). In view of (3.18), there exists an \((f, g, h)\)-orbit relative \((r, s, t)\) at \( p_0 \) with the choice (3.1). By a routine iteration procedure just similar to that of Theorem 1.1, one can prove that \((f, g, h)\)-orbit relative \((r, s, t)\) at \( p_0 \) is Cauchy, and hence converges to some \( u \in X \), since \( X \) is \((f, g, h)\)-orbitally complete at \( p_0 \).

That is

\[ \lim_{n \to \infty} fp_{3n-3} = \lim_{n \to \infty} gp_{3n-2} = \lim_{n \to \infty} hp_{3n-1} = \lim_{n \to \infty} rp_{3n-2} \]
\[ = \lim_{n \to \infty} sp_{3n-1} = \lim_{n \to \infty} tp_{3n} = u \text{ for some } u \in X. \]

Then, using the inequalities (3.15)-(3.17), it is not difficult to see that

\[ \lim_{n \to \infty} fp_{3n-3} = \lim_{n \to \infty} rp_{3n-3} = \lim_{n \to \infty} gp_{3n-2} = \lim_{n \to \infty} sp_{3n-2} \]
\[ = \lim_{n \to \infty} hp_{3n-1} = \lim_{n \to \infty} tp_{3n-1} = u \text{ for some } u \in X. \]

Here we note that (3.19) is a special case of (3.3) with \( x_n = p_{3n-3}, y_n = p_{3n-2} \) and \( z_n = x_{3n-1} \). Therefore, the common fixed point follows from Theorem 3.1. \( \blacksquare \)

When \( r = s = t = I \), the identity map on \( X \) in Corollary 3.1, we find that (1.2), (1.3) and (1.4) are particular cases of the implicit relations of Theorem 3.1. Also it is well-known that \( I \) is onto and commutes with every map and hence is weakly compatible with the maps \( f, g \) and \( h \), and hence a common fixed point follows directly from Theorem 3.1. Thus Theorem 1.1 is a stronger version of Corollary 3.1.

Finally, writing \( h = g = f \) and \( r = s = t \) in Theorem 3.1, we get a particular case of each of (3.4)-(3.6) as

\[ \psi(d(fx, fy), d(rx, ry), d(rx, fx), d(ry, fy), d(rx, fy), d(ry, fx)) < 0 \]

for all \( x, y \in X \). Also the space \( X \) reduces to \( f \)-orbitally complete relative to \( r \) in the sense that every Cauchy sequence in the \((f, r)\)-orbit \( O_{f,r}(x_0) \) at each \( x_0 \) converges in \( X \), where \( O_{f,r}(x_0) \) has the choice

\[ fx_{n-1} = rx_n \text{ for } n = 1, 2, 3, .... \]

Then we have
Corollary 3.2 Let $f$ and $r$ be self-maps on $X$ satisfying the property E. A. and the inequality (3.20). If $r(X)$ is $f$-orbitally complete relative to $r$, then $f$ and $r$ will have a coincidence point. Further if $(f, r)$ is weakly compatible, then $f$ and $r$ will have a unique common fixed point.

The following example shows that $f$-orbital completeness of $r(X)$ is necessary in Corollary 3.2:

**Example 3.1** Let

$$\psi(l_1, l_2, l_3, l_4, l_5, l_6) = l_1^2 - al_2^2 - \frac{bl_5l_6}{l_3^2 + l_4^2 + 1},$$

where $a = 1/2$ and $b = 1/4$. Set $X = \{0, \frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots \}$ with the usual metric $d$. Define $f, r : X \to X$ by

$$f0 = \frac{1}{2}, f\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^{n+1}}$$
and

$$r0 = \frac{1}{2}, r\left(\frac{1}{2^{n-1}}\right) = \frac{1}{2^n},$$

for $n = 1, 2, 3, \ldots$. Then $(f, r)$ satisfies the property E.A. and $r(X) = \left\{\frac{1}{2}, \frac{1}{2^2}, \frac{1}{2^3}, \ldots \right\}$.

For $x_0 = 0$, choose $x_1 = \frac{1}{2}, x_2 = \frac{1}{2^2}, x_3 = \frac{1}{2^3}, \ldots$ so that

$$O_{f,r}(x_0) = \left\{\frac{1}{2^2}, \frac{1}{2^3}, \frac{1}{2^4}, \ldots \right\}$$

while for $x_0 = \frac{1}{2^{n-1}}$, we have

$$O_{f,r}(x_0) = \left\{\frac{1}{2^{n+1}}, \frac{1}{2^{n+2}}, \frac{1}{2^{n+3}}, \ldots \right\}$$

for each $n = 1, 2, 3, \ldots$. In either case, $O_{f,r}(x_0)$ converges to $0 \notin r(X)$.

Thus $r(X)$ is not orbitally complete at each $x_0$. As such the maps $f$ and $r$ do not have a coincidence point, though $X$ is complete.

Since every complete metric space is orbitally complete at each of its points, we have

Corollary 3.3 (Theorem 3.1, [5]) Let $f$ and $r$ be self-maps on $X$ satisfying the property E. A. and the inequality (3.20). If $r(X)$ is complete, then $f$ and $r$ will have a coincidence point. Further, $f$ and $r$ will have a unique common fixed point, provided $(f, r)$ is weakly compatible.
References


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