GENERALIZED HESITANT FUZZY SOFT SETS

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Abstract. In this paper, generalized hesitant fuzzy soft sets and some operations on generalized hesitant fuzzy soft sets are defined and some of their properties are studied. Applications of generalized hesitant fuzzy soft sets in decision-making are investigated.

Keywords: soft sets, fuzzy soft sets, generalized hesitant fuzzy soft sets, similarity, decision-making.

1. Introduction

Decision-making problems referring to evaluating, prioritizing or selecting over some available alternatives are very common in practice [1]. Since it was introduced by Zadeh [2], theories of fuzzy sets serve as an excellent resolution of decision-making under uncertainties. But the modeling tools of Zadeh's fuzzy sets (Z-FSs) are limited whereby two or more sources of vagueness appear simultaneously. Thus several generalizations and extensions of Z-FSs are developed, such as type-2 fuzzy sets [3], [4], type-n fuzzy sets [4], intuitionistic fuzzy sets (IFSs) [5], fuzzy multisets [6] and hesitant fuzzy sets (T-HFSs) [7], [8].

T-HFSs are quite suit for the situation where we have a set of possible values, rather than a margin of error (as in IFSs) or some possibility distribution on the possible values (as in type-2 fuzzy sets) [7], [8]. The motivation to propose the T-HFSs is that when people make a decision, they are usually hesitant and irresolute for one thing or another which makes it difficult to reach a final agreement. "For example, three decision makers give the membership of x into A, and they want to assign 0.4, 0.5 and 0.7, which can be considered as a hesitant fuzzy element $\{0.4, 0.5, 0.7\}$ rather than the convex of 0.4 and 0.7, or the interval between 0.4 and 0.7" [13].

There are some developments on T-HFSs. Torra and Narukawa [7] introduced the extension principle to apply it in decision-making. Xu and Xia [9] developed a series of aggregation operators for hesitant fuzzy information and applied to multi-criteria decision-making. Later, some induced aggregation operators in hesitant fuzzy setting are introduced by Xia et al. [10]. Based on Quasi arithmetic means, Xia et al. [11] discussed some ordered aggregation operators and induced ordered aggregation operators, as well as their application in group decision-making. Some similarity measure and correlation measures are detailed studied in Xu and Xia [12], [13], respectively. Later, Farhadinia study some information measures for hesitant fuzzy sets and interval-valued hesitant fuzzy sets in [14]. Chen et al. [15] study the correlation coefficients of hesitant fuzzy sets and apply to clustering analysis. Bedregal et al. [16] induced some aggregation functions for typical hesitant fuzzy elements.

Qian et al. further generalized the concept of T-HFSs in practice needs and gave the definition of generalized hesitant fuzzy sets [17]. There are mainly three advantages of the extension. First, as the case in T-HFSs, it is very useful to consider all possible memberships with hesitancy rather than considering just an aggregation operator. Second, it can eliminate times of using aggregation operators during the group decision-making process, which can alleviate suffering from less robust decision led by times of aggregations. At last, individual expert can express his/her evaluations by either Z-FSs, IFSs, T-HFSs or the proposed fuzzy sets.

Soft set theory [18] is also a new and important method for dealing with uncertain data. In recent years, research on soft set theory and its generalization has been done by many researchers in mathematics, computer and information science, including the works of fundamental soft set theory [19], soft set theory in abstract algebra and topological space [20]-[23], theory for data analysis, particularly in decision-making [24]-[27]. P.K. Maji et al. [28] defined the fuzzy soft set and studied some properties of this set. And Roy et al. [29] applied fuzzy soft set theory to decision-making problems. Majumdar and Samanta [30] introduced a concept of generalized fuzzy soft sets and their operations and application of generalized fuzzy soft sets in decision-making problem and medical diagnosis problem. Recently, some other generalizations of fuzzy soft sets such as trapezoidal fuzzy soft sets, multi-fuzzy soft sets, Type-2 Fuzzy Soft Sets, generalized interval-valued fuzzy soft sets and fuzzy soft multiset theory are studied in [31]-[35].

From the above analysis, we can see that generalized hesitant fuzzy sets and soft sets are useful tools for dealing with uncertainty and vagueness. Interestingly, it is possible to combine these two sets together and study new operations and properties of these sets, which must make them more important and applicable. In this paper, we will propose new sets based on generalized hesitant fuzzy sets and soft sets and study their properties.

2. Preliminaries

Let U be an initial universe set and E be a collection of all possible parameters with respect to U, where parameters are the characteristics or properties of objects in U. We will call E the universe set of parameters with respect to U.

Definition 2.1. ([18]) A pair (F, A) is called a soft set over U if $A \subset E$ and $F : A \to P(U)$, where $P(U)$ is the set of all subsets of U.

Definition 2.2. ([28]) A pair (F, A) is called a fuzzy soft set over U if $A \subset E$ and $F : A \to I^U$, where I^U denotes the collection of all fuzzy subsets of U.

Definition 2.3. ([5]) Let X be a fixed set, an intuitionistic fuzzy set (IFS) A on X is represented in terms of two functions $\mu : X \to [0, 1]$ and $\nu : X \to [0, 1]$, with the condition $0 \leq \mu(x) + \nu(x) \leq 1$, for all $x \in X$.

Furthermore, $\pi(x) = 1 - \mu(x) - \nu(x)$ is called a hesitancy degree or an intuitionistic index of x in A. In the special case $\pi(x) = 0$, that is, $\mu(x)+\nu(x) = 1$, the IFS A reduces to a FS.

Atanassov [5] gave some basic operations on IFSs, which ensure that the operational results are also IFSs.

Definition 2.4. ([5]) Let a set X be fixed, and let A (represented by the functions μ_A and ν_A), A_1 (represented by the functions μ_{A_1} and ν_{A_1}), A_2 (represented by the functions μ_{A_2} and ν_{A_2}), be three IFSs. Then the following operations are valid:

(1) complement: $A^c = \{\langle x, \nu_A(x), \mu_A(x)\rangle | x \in X\}$.

(2) intersection: $A_1 \cap A_2 = \{\langle x, min\{\mu_{A_1}(x), \mu_{A_2}(x)\}, max\{\nu_{A_1}(x), \nu_{A_2}(x)\}\rangle | x \in X\}$;

(3) union: $A_1 \cup A_2 = \{\langle x, max\{\mu_{A_1}(x), \mu_{A_2}(x)\}, min\{\nu_{A_1}(x), \nu_{A_2}(x)\}\rangle | x \in X\}$.

Sometimes, it is difficult to determine the membership of an element into a fixed set and which may be caused by a doubt among a set of different values. For the sake of a better description of this situation, Torra introduced the concept of T-HFS as a generalization of fuzzy sets. The membership degree of a T-HFS is presented by several possible values in [0, 1]. The definition is cited as follow.

Definition 2.5 ([7],[8]). Let X be a fixed set, then a hesitant fuzzy set (T-HFS) A on X in terms of a function h is that when applied to X returns a subset of $[0, 1]$.

To be easily understood, Xia and Xu [9] expressed the hesitant fuzzy set by the following mathematical symbol: $A = \{\langle x, h_A(x)\rangle \mid x \in X\}$, where $h_A(x)$ is a set of some different values in [0, 1], representing the possible membership degrees of the element $x \in X$ to A. Xia and Xu [9] called $h_A(x)$ a hesitant fuzzy element (HFE), a basic unit of T-HFS.

It is noted that the number of values in different HFE may be different, let $l(h_A(x))$ be the number of values in $h_A(x)$. We arrange the elements in $h_A(x)$ in increasing order, and let $h_A^{\sigma(j)}(x)$ be the jth value in $h_A(x)$.

Example 2.6. Let $X = \{x_1, x_2, x_3\}$ be a reference set. $h_A(x_1) = \{0.1, 0.3, 0.5\}$, $h_A(x_2) = \{0.4, 0.5\}$, $h_A(x_3) = \{0.3, 0.4, 0.5, 0.7\}$. A is a T-HFS, namely

$A = \{\langle x_1, 0.1, 0.3, 0.5\rangle, \langle x_2, 0.4, 0.5\rangle, \langle x_3, 0.3, 0.4, 0.5, 0.7\rangle\}$.

Apparently, this definition encompasses IFSs as a particular case in the form of a nonempty closed interval.

The followings are some basic operations which introduced in [9]-[13] on T-HFS:

Given a hesitant fuzzy set h, we define below its lower and upper bound,

- lower bound: $h^-(x) = minh(x)$ and
- upper bound: $h^+(x) = maxh(x)$.

Qian et.al. [17] defined the generalized hesitant fuzzy set in terms of two functions that return two sets of membership values and nonmembership values, respectively, for each element in the domain as follows.

Definition 2.7. Let X be a fixed set, then a generalized hesitant fuzzy set (GHF set) G on X is described as:

$$G = \left\{ \frac{x}{h(x), g(x)} | x \in X \right\},$$

in which $h(x)$ and $g(x)$ are two sets of some values in [0, 1], denoting the possible membership degrees and nonmembership degrees of the element $x \in X$ to the set G, respectively, with the conditions:

$$0 \leq \mu_i(x), \nu_i(x) \leq 1, 0 \leq \mu_i(x) + \nu_i(x) \leq 1, \; 1 \leq i \leq N_x = |h(x)| = |g(x)|,$$

where $\mu_i(x) \in h(x)$, $\nu_i(x) \in g(x)$, for all $x \in X$. And $|h(x)|$ denote the cardinality of the set $h(x)$. For convenience, the pair $G(x) = (h(x), g(x))$ is called a generalized hesitant fuzzy element (GHFE) denoted by $G = (h, g)$.

Definition 2.8. Given a GHFE represented by G, we define its complement as follows:

$$G^c(x) = \bigcup_{\mu_i(x) \in h(x), \nu_i(x) \in g(x)} \{\{\nu_i(x)\}, \{\mu_i(x)\}\}.$$

Definition 2.9. Given two generalized hesitant fuzzy sets $G_1 = (h_1(x), g_1(x))$ and $G_2 = (h_2(x), g_2(x))$, we define their union represented by $G_1 \cup G_2$ as $G_1 \cup G_2(x) = \{(\mu(x), \nu(x)), \mu(x) \in h_1(x) \cup h_2(x)) | \mu(x) \geq max(h_1^-(x), h_2^-(x)),$ $\nu(x) \in (g_1(x) \cup g_2(x)) | \nu(x) \leq min(g_1^+(x), g_2^+(x))\}$.

Definition 2.10. Given two generalized hesitant fuzzy sets $G_1 = (h_1(x), g_1(x))$ and $G_2 = (h_2(x), g_2(x))$, we define their union represented by $G_1 \cap G_2$ as $G_1 \cap G_2(x) = \{(\mu(x), \nu(x)), \mu(x) \in (h_1(x) \cup h_2(x)) | \mu(x) \leq min(h_1^+(x), h_2^+(x)),$ $\nu(x) \in (g_1(x) \cup g_2(x)) | \nu(x) \geq max(g_1^-(x), g_2^-(x))\}$.

Example 2.11. Let $G_1 = \{\{0.1, 0.3, 0.5\}, \{0.8, 0.6, 0.4\}\}$ and $G_2 = \{\{0.2, 0.4\},$ $\{0.7, 0.5\}\}$ be two GHFEs. Then we have

(1) complement: $G_1^c(x) = \{\{0.8, 0.6, 0.4\}, \{0.1, 0.3, 0.5\}\}$,

(2) union: $G_1 \cup G_2(x) = \{\{0.3, 0.5, 0.2, 0.4\}, \{0.6, 0.4, 0.7, 0.5\}\}$,

(3) intersection: $G_1 \cap G_2(x) = \{\{0.1, 0.3, 0.2, 0.4\}, \{0.8, 0.6, 0.7, 0.5\}\}$.

Assumption 2.12. Notice that the number of values in different GHFEs may be different. Suppose that $|h_M(x)|$ stands for the number of values in $h_M(x)$. Hereafter, the following assumptions are made: (see [12,36-38])

(A1) All the elements in each $h_M(x)$ are arranged in increasing order.

(A2) If, for some $x \in X$, $|h_M(x)| \neq |h_N(x)|$, then $l_x = max\{|h_M(x)|, |h_N(x)|\}$. To have a correct comparison, the two GHFEs $h_M(x)$ and $h_N(x)$ should have the same length l_x. If there are fewer elements in $h_M(x)$ than $h_N(x)$, an extension of $h_M(x)$ should be considered optimistically by repeating its maximum element until it has the same length with $h_N(x)$.

Remark 2.13. If we arrange the membership sequences in increasing order, then the corresponding non-membership sequence may not be in decreasing or increasing order.

Definition 2.14. Let M, N be two GHF set on X. Then, M is a generalized hesitant fuzzy subset of N, if for each $x \in X$, $1 \le i \le l_x$, we have $\mu_i^M(x) \le \mu_i^N(x)$ and $\nu_i^M(x) \ge \nu_i^N(x)$. And denote by $M \sqsubseteq N$.

Example 2.15. Let $X = \{x_1, x_2, x_3\}$ be the discourse set, and

$M = \{\frac{x_1}{(0.4),(0.5)}, \frac{x_2}{(0.4,0.5),(0.5,0.3)}, \frac{x_3}{(0.2,0.3,0.5,0.6),(0.7,0.6,0.4,0.3)}\}$,

$N = \{\frac{x_1}{(0.5,0.7),(0.4,0.2)}, \frac{x_2}{(0.6),(0.3)}, \frac{x_3}{(0.5,0.6),(0.3,0.2)}\}$,

be two GHFS sets on X. Then, in view of Assumption 2.12, the GHF sets M, N can be respectively represented as

$M = \{\frac{x_1}{(0.4,0.4),(0.5,0.5)}, \frac{x_2}{(0.4,0.5),(0.5,0.3)}, \frac{x_3}{(0.2,0.3,0.5,0.6),(0.7,0.6,0.4,0.3)}\}$,

$N = \{\frac{x_1}{(0.5,0.7),(0.4,0.2)}, \frac{x_2}{(0.6,0.6),(0.3,0.3)}, \frac{x_3}{(0.5,0.6,0.6,0.6),(0.3,0.2,0.2,0.2)}\}$.

We can find that $\mu_j^M(x_i) \le \mu_j^N(x_i)$ and $\nu_j^M(x_i) \ge \nu_j^N(x_i)$, for each $x_i \in X$ and each $1 \le j \le l_{x_i}$. Then, M is a generalized hesitant fuzzy subset of N and denote by $M \sqsubseteq N$.

Definition 2.16. For a GHF set G, $s(h) = \frac{1}{|h|} \cdot \sum_{\gamma \in h} \gamma$ and $s(h) = \frac{1}{|g|} \cdot \sum_{\eta \in g} \eta$ is called the score function of h and g, where $|h|$ and $|g|$ represent the number of the elements in h and g.

3. Generalized hesitant fuzzy soft sets

Obviously, by combining the generalized hesitant fuzzy set and soft set, it is natural to define the generalized hesitant fuzzy soft sets model. We first define the generalized hesitant fuzzy soft sets as follows.

Definition 3.1 Let U be an initial universe and E a set of parameters, $A \subseteq E$. Define a function $\tilde{G} : A \to GHF^U$, where GHF^U denote all of the generalized hesitant fuzzy sets over U. Then, a pair $(\tilde{G}, A)$ is called a generalized hesitant fuzzy soft set (GHFS set).

Example 3.2. Let $U = \{x_1, x_2, x_3, x_4\}$. Let $A = \{e_1, e_2, e_3\} \subseteq E$ a set of parameters. Define $\tilde{G} : A \to HFS^U$, as follows:

$\tilde{G}(e_1) = \{\frac{x_1}{(0.4),(0.5)}, \frac{x_2}{(0.4,0.5),(0.5,0.3)}, \frac{x_3}{(0.2,0.3,0.5,0.6),(0.7,0.6,0.4,0.3)}, \frac{x_4}{(0.5,0.6),(0.3,0.2)}\}$,

$\tilde{G}(e_2) = \{\frac{x_1}{(0.1,0.3),(0.6,0.5)}, \frac{x_2}{(0.7,0.8,0.9),(0.2,0.1,0.1)}, \frac{x_3}{(0.2,0.3),(0.6,0.7)}, \frac{x_4}{(0.2,0.4),(0.6,0.5)}\}$,

$\tilde{G}(e_3) = \{\frac{x_1}{(0.2,0.4,0.6),(0.3,0.5,0.3)}, \frac{x_2}{(0.5,0.7,0.9),(0.4,0.3,0.1)}, \frac{x_3}{(0.1,0.2),(0.8,0.6)}, \frac{x_4}{(0.4,0.7),(0.5,0.2)}\}$.

Then $(\tilde{G}, A) = \{\tilde{G}(e_1), \tilde{G}(e_2), \tilde{G}(e_3)\}$ is a GHFS set.

Definition 3.3. Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two GHFS sets over (U, E). Then $(\tilde{F}, A)$ is called a GHFS subset of $(\tilde{G}, B)$ if

(i) $A \subseteq B$,

(ii) $\tilde{F}(e)$ is a generalized hesitant fuzzy subset of $\tilde{G}(e)$, for each $e \in A$.

In this case, the above relationship is denoted by $(\tilde{F}, A)\tilde{\sqsubseteq}(\tilde{G}, B)$. And $(\tilde{G}, B)$ is said to be a GHFS superset of $(\tilde{F}, A)$.

Definition 3.4. Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be two GHFS sets over (U, E). Then $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are said to be GHFS equal if and only if $(\tilde{F}, A)\tilde{\sqsubseteq}(\tilde{G}, B)$ and $(\tilde{G}, B)\tilde{\sqsubseteq}(\tilde{F}, A)$.

Example 3.5. We consider the GHFS set $(\tilde{G}, A)$ given in Example 3.2 and define a GHFS set $(\tilde{M}, B)$ as follows:

$\tilde{M}(e_1) = \{\frac{x_1}{(0.2),(0.7)}, \frac{x_2}{(0.3,0.4),(0.7,0.5)}, \frac{x_3}{(0.1,0.2,0.4,0.5),(0.8,0.7,0.5,0.4)}, \frac{x_4}{(0.2,0.4),(0.6,0.6)}\}$,

$\tilde{M}(e_2) = \{\frac{x_1}{(0.1,0.2),(0.9,0.6)}, \frac{x_2}{(0.3,0.4,0.5),(0.6,0.5,0.4)}, \frac{x_3}{(0.1,0.2),(0.9,0.8)}, \frac{x_4}{(0.2,0.3),(0.7,0.6)}\}$.

Then $(\tilde{M}, B)$ is a GHFS subset of $(\tilde{G}, A)$.

Definition 3.6. The complement of a GHFS set $(\tilde{G}, A)$ is denoted by $(\tilde{G}, A)^c$ and is defined by $(\tilde{G}, A)^c = (\tilde{G}^c, A)$, where $\tilde{G}^c : A \to GHF^U$ is a mapping given by $\tilde{G}^c(e) = \{\frac{x}{g(x),h(x)}, x \in U\}$.

Example 3.7. We consider the GHFS set $(\tilde{G}, A)$ given in Example 3.2. Then,

$\tilde{G}^c(e_1) = \{\frac{x_1}{(0.5),(0.4)}, \frac{x_2}{(0.5,0.3),(0.4,0.5)}, \frac{x_3}{(0.7,0.6,0.4,0.3),(0.2,0.3,0.5,0.6)}, \frac{x_4}{(0.3,0.2),(0.5,0.6)}\}$,

$\tilde{G}^c(e_2) = \{\frac{x_1}{(0.6,0.5),(0.1,0.3)}, \frac{x_2}{(0.2,0.1,0.1),(0.7,0.8,0.9)}, \frac{x_3}{(0.6,0.7),(0.2,0.3)}, \frac{x_4}{(0.6,0.5),(0.2,0.4)}\}$,

$\tilde{G}^c(e_3) = \{\frac{x_1}{(0.3,0.5,0.3),(0.2,0.4,0.6)}, \frac{x_2}{(0.4,0.3,0.1),(0.5,0.7,0.9)}, \frac{x_3}{(0.8,0.6),(0.1,0.2)}, \frac{x_4}{(0.5,0.2),(0.4,0.7)}\}$.

And, in view of Assumption 2.1, the GHFS set $(\tilde{G}, A)^c$ can be respectively represented as

$\tilde{G}^c(e_1) = \{\frac{x_1}{(0.5),(0.4)}, \frac{x_2}{(0.3,0.5),(0.5,0.4)}, \frac{x_3}{(0.3,0.4,0.6,0.7),(0.6,0.5,0.3,0.2)}, \frac{x_4}{(0.2,0.3),(0.6,0.5)}\}$,

$\tilde{G}^c(e_2) = \{\frac{x_1}{(0.5,0.6),(0.3,0.1)}, \frac{x_2}{(0.1,0.1,0.2),(0.8,0.9,0.7)}, \frac{x_3}{(0.6,0.7),(0.2,0.3)}, \frac{x_4}{(0.5,0.6),(0.4,0.2)}\}$,

$\tilde{G}^c(e_3) = \{\frac{x_1}{(0.3,0.3,0.5),(0.2,0.6,0.4)}, \frac{x_2}{(0.1,0.3,0.4),(0.9,0.7,0.5)}, \frac{x_3}{(0.6,0.8),(0.2,0.1)}, \frac{x_4}{(0.2,0.5),(0.7,0.4)}\}$.

Then, $(\tilde{G}, A)^c$ is the complement of the GHFS set $(\tilde{G}, A)$.

Definition 3.8. A GHFS set $(\tilde{G}, A)$ over (U, E) is said to be relative absolute GHFS set denoted by Ω_A, if $h_{\tilde{G}(e)}(x) = 1$ and $g_{\tilde{G}(e)}(x) = 0$ for all $x \in U$ and $e \in A$.

Definition 3.9. A GHFS set $(\tilde{G}, A)$ over (U, E) is said to be relative null GHFS set denoted by Φ_A, if $h_{\tilde{G}(e)}(x) = 0$ and $g_{\tilde{G}(e)}(x) = 1$ for all $x \in U$ and $e \in A$.

Followings are some operations on GHFS set:

Definition 3.10. Union of two GHFS sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ of over U is the GHFS set $(\tilde{H}, B)$, where $C = A \cup B$, and $\forall e \in C$ and $x \in U$,

$$\tilde{H}(e) = \begin{cases} \tilde{F}(e), & \text{if } e \in A - B; \\ \tilde{G}(e), & \text{if } e \in B - A; \\ \tilde{F}(e) \cup \tilde{G}(e), & \text{if } e \in A \cap B. \end{cases}$$

We write $(\tilde{F}, A)\tilde{\cup}(\tilde{G}, B) = (\tilde{H}, C)$.

Example 3.11. Consider Example 3.2 and define a GHFS set $(\tilde{M}, B)$ as follows:

$\tilde{M}(e_1) = \{\frac{x_1}{(0.2),(0.7)}, \frac{x_2}{(0.3,0.4),(0.7,0.6)}, \frac{x_3}{(0.1,0.2,0.4,0.5),(0.8,0.6,0.5,0.4)}, \frac{x_4}{(0.2,0.4),(0.6,0.6)}\}$,

$\tilde{M}(e_2) = \{\frac{x_1}{(0.1,0.2),(0.9,0.6)}, \frac{x_2}{(0.3,0.4,0.5),(0.6,0.5,0.4)}, \frac{x_3}{(0.1,0.2),(0.9,0.8)}, \frac{x_4}{(0.2,0.3),(0.7,0.6)}\}$.

We have $(\tilde{F}, A)\tilde{\cup}(\tilde{M}, B) = (\tilde{H}, C)$, where

$\tilde{H}(e_1) = \{\frac{x_1}{(0.4),(0.5)}, \frac{x_2}{(0.4,0.4,0.5),(0.6,0.5,0.3)}, \frac{x_3}{(0.2,0.3,0.5,0.6,0.2,0.4,0.5),(0.7,0.6,0.4,0.3,0.6,0.5,0.4)}, \frac{x_4}{(0.4,0.5,0.6),(0.6,0.3,0.2)}\}$,

$\tilde{H}(e_2) = \{\frac{x_1}{(0.1,0.2,0.3),(0.6,0.6,0.5)}, \frac{x_2}{(0.4,0.5,0.7,0.8,0.9),(0.5,0.4,0.2,0.1,0.1)}, \frac{x_3}{(0.2,0.2,0.3),(0.8,0.6,0.7)}, \frac{x_4}{(0.2,0.3,0.4),(0.6,0.6,0.5)}\}$,

$\tilde{H}(e_3) = \{\frac{x_1}{(0.2,0.4,0.6),(0.3,0.5,0.3)}, \frac{x_2}{(0.5,0.7,0.9),(0.4,0.3,0.1)}, \frac{x_3}{(0.1,0.2),(0.8,0.6)}, \frac{x_4}{(0.4,0.7),(0.5,0.2)}\}$.

Definition 3.12. Intersection of two GHFS sets $(\tilde{F}, A)$ and $(\tilde{G}, B)$ of over U is the GHFS sets set $(\tilde{H}, B)$, where $C = A \cap B$, and $\forall e \in C$ and $x \in U$, $\tilde{H}(e) = \tilde{F}(e) \cap \tilde{G}(e)$.

We write $(\tilde{F}, A)\tilde{\cap}(\tilde{G}, B) = (\tilde{H}, C)$.

Example 3.13. Consider Example 3.11. We have $(\tilde{F}, A)\tilde{\cap}(\tilde{G}, B) = (\tilde{H}, C)$, where

$\tilde{H}(e_1) = \{\frac{x_1}{(0.2),(0.7)}, \frac{x_2}{(0.3,0.4,0.4),(0.7,0.6,0.5)}, \frac{x_3}{(0.2,0.3,0.5,0.1,0.2,0.4,0.5),(0.7,0.6,0.4,0.8,0.6,0.5,0.4)}, \frac{x_4}{(0.2,0.4,0.5),(0.6,0.6,0.3)}\}$,

$\tilde{H}(e_2) = \{\frac{x_1}{(0.1,0.2,0.1),(0.9,0.6,0.6)}, \frac{x_2}{(0.3,0.4,0.5,0.7),(0.6,0.5,0.4,0.2)}, \frac{x_3}{(0.1,0.2),(0.9,0.8)}, \frac{x_4}{(0.2,0.3,0.2),(0.7,0.6,0.6)}\}$.

By the suggestions given by Molodtsov in [18], we present the notion of AND and OR operations on two GHFS sets as follows.

Definition 3.14. If $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are two GHFS sets of over U, the "$(\tilde{F}, A)$ AND $(\tilde{G}, B)$", denoted by $(\tilde{F}, A)\wedge(\tilde{G}, B)$ is defined by $(\tilde{F}, A)\wedge(\tilde{G}, B)=(\tilde{H}, A\times B)$, where $\tilde{H}(\alpha, \beta)=\tilde{F}(\alpha) \cap \tilde{G}(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Definition 3.15. If $(\tilde{F}, A)$ and $(\tilde{G}, B)$ are two GHFS sets of over U, the "$(\tilde{F}, A)$ OR $(\tilde{G}, B)$", denoted by $(\tilde{F}, A)\vee(\tilde{G}, B)$ is defined by $(\tilde{F}, A)\vee(\tilde{G}, B) = (\tilde{O}, A\times B)$, where $\tilde{O}(\alpha, \beta) = \tilde{F}(\alpha) \cup \tilde{G}(\beta)$, for all $(\alpha, \beta) \in A \times B$.

Example 3.16. Let $U=\{h_1, h_2, h_3\}$. Let $A=\{e_1, e_2\}\subseteq E$ and $B=\{e_2, e_4, e_5\} \subseteq E$ be sets of parameters. Define $(\tilde{F}, A)$ and $(\tilde{G}, B)$ as follows:

$\tilde{F}(e_1) = \{\frac{x_1}{(0.5,0.7),(0.3,0.2)}, \frac{x_2}{(0.8,0.9,1),(0.2,0.1,0)}, \frac{x_3}{(0.2,0.3),(0.7,0.6)}\}$,

$\tilde{F}(e_2) = \{\frac{x_1}{(0.7,1),(0.3,0)}, \frac{x_2}{(0.4,0.5),(0.5,0.4)}, \frac{x_3}{(0.8,0.9),(0.1,0.1)}\}$,

$\tilde{G}(e_2) = \{\frac{x_1}{(0.6,0.8),(0.3,0.2)}, \frac{x_2}{(0.5,0.6),(0.3,0.4)}, \frac{x_3}{(0.5,0.7),(0.4,0.3)}\}$,

$\tilde{G}(e_4) = \{\frac{x_1}{(0.4,0.6),(0.5,0.3)}, \frac{x_2}{(0.6,0.8,0.9),(0.4,0.1,0.05)}, \frac{x_3}{(0.5),(0.3)}\}$,

$\tilde{G}(e_5) = \{\frac{x_1}{(0.8,0.9),(0.2,0.05)}, \frac{x_2}{(0.1,0.3,0.5),(0.8,0.6,0.3)}, \frac{x_3}{(0.4,0.6,0.8),(0.5,0.3,0.2)}\}$.

We have $(\tilde{F}, A) \wedge (\tilde{G}, B) = (\tilde{H}, A \times B)$ and $(\tilde{F}, A) \vee (\tilde{G}, B) = (\tilde{O}, A \times B)$ as follows:

$\tilde{H}(e_1, e_2) = \{\frac{x_1}{(0.5,0.6,0.7),(0.3,0.3,0.2)}, \frac{x_2}{(0.5,0.6,0.8,0.9),(0.3,0.4,0.2,0.1)}, \frac{x_3}{(0.2,0.3,0.5),(0.7,0.6,0.4)}$,

$\tilde{H}(e_1, e_4) = \{\frac{x_1}{(0.4,0.5,0.6),(0.5,0.3,0.3)}, \frac{x_2}{(0.6,0.8,0.8,0.9,0.9),(0.4,0.2,0.1,0.1,0.05)}, \frac{x_3}{(0.2,0.3),(0.7,0.6)}\}$,

$\tilde{H}(e_1, e_5) = \{\frac{x_1}{(0.5,0.7,0.8),(0.3,0.2,0.2)}, \frac{x_2}{(0.1,0.3,0.5,0.8,0.9),(0.8,0.6,0.4,0.2,0.1)}, \frac{x_3}{(0.2,0.3,0.4,0.6),(0.7,0.6,0.5,0.3)}\}$,

$\tilde{H}(e_2, e_2) = \{\frac{x_1}{(0.6,0.7,0.8),(0.3,0.3,0.2)}, \frac{x_2}{(0.4,0.5),(0.5,0.4)}, \frac{x_3}{(0.5,0.7),(0.4,0.3)}\}$,

$\tilde{H}(e_2, e_4) = \{\frac{x_1}{(0.4,0.6,0.7),(0.5,0.3,0.3)}, \frac{x_2}{(0.4,0.5,0.6,0.8),(0.5,0.4,0.4,0.1)}, \frac{x_3}{(0.5),(0.3)}\}$,

$\tilde{H}(e_2, e_5) = \{\frac{x_1}{(0.7,0.8,0.9),(0.3,0.2,0.05)}, \frac{x_2}{(0.1,0.3,0.4,0.5),(0.8,0.6,0.5,0.4)}, \frac{x_3}{(0.4,0.6,0.8),(0.5,0.3,0.2)}\}$.

and

$\tilde{O}(e_1, e_2) = \{\frac{x_1}{(0.6,0.7,0.8),(0.3,0.2,0.2)}, \frac{x_2}{(0.8,0.9,0.1),(0.2,0.1,0)}, \frac{x_3}{(0.3,0.5,0.7),(0.6,0.4,0.3)}$,

$\tilde{O}(e_1,e_4)=\{\frac{x_1}{(0.5,0.6,0.7),(0.3,0.3,0.2)},\frac{x_2}{(0.8,0.8,0.9,0.9,1),(0.2,0.1,0.1,0.05,0)},\frac{x_3}{(0.3,0.5),(0.6,0.3)}\}$,
$\tilde{O}(e_1,e_5)=\{\frac{x_1}{(0.7,0.8,0.9),(0.2,0.2,0.05)},\frac{x_2}{(0.3,0.5,0.8,0.9,1),(0.6,0.3,0.2,0.1,0)},$
$\frac{x_3}{(0.3,0.4,0.6,0.8),(0.6,0.5,0.3,0.2)}\}$,
$\tilde{O}(e_2,e_2)=\{\frac{x_1}{(0.7,0.8,1),(0.3,0.2,0)},\frac{x_2}{(0.5,0.5,0.6),(0.4,0.3,0.4)},\frac{x_3}{(0.7,0.8,0.9),(0.3,0.1,0.1)}\}$,
$\tilde{O}(e_2,e_4)=\{\frac{x_1}{(0.6,0.7,1),(0.3,0.3,0)},\frac{x_2}{(0.5,0.6,0.8,0.9),(0.4,0.4,0.1,0.05)},\frac{x_3}{(0.8,0.9),(0.1,0.1)}\}$,
$\tilde{O}(e_2,e_5)=\{\frac{x_1}{(0.8,0.9,1),(0.2,0.05,0)},\frac{x_2}{(0.3,0.4,0.5,0.5),(0.6,0.5,0.4,0.3)},\frac{x_3}{(0.6,0.8,0.8,0.9),(0.3,0.2,0.1,0.1)}\}$.

4. An application of GHFS set in decision-making

There are several applications of GHFS set theory in several directions. Here we present an application of GHFS sets in the decision-making problem.

First, we give some definitions which are useful in this section:

Definition 4.1. Let $(\tilde{G},A)$ be a GHFS set over (U,E). If for each $e\in A$ and $x\in U$, we use $\mu(x)=h^{-}_{\tilde{G}(e)}(x)$ and the corresponding $\nu(x)$ of $\tilde{G}(e)$ to construct an intuitionistic fuzzy soft set (G,A). Then (G,A) is called the pessimistic reduct intuitionistic fuzzy soft set of $(\tilde{G},A)$.

Example 4.2. Consider Example 3.2, the pessimistic reduct intuitionistic fuzzy soft set (F,A) of $(\tilde{F},A)$ is defined as follow:
$G(e_1)=\{\frac{x_1}{(0.4,0.5)},\frac{x_2}{(0.4,0.5)},\frac{x_3}{(0.2,0.7)},\frac{x_4}{(0.5,0.3)}\}$,
$G(e_2)=\{\frac{x_1}{(0.1,0.6)},\frac{x_2}{(0.7,0.2)},\frac{x_3}{(0.2,0.6)},\frac{x_4}{(0.2,0.6)}\}$,
$G(e_3)=\{\frac{x_1}{(0.2,0.3)},\frac{x_2}{(0.5,0.4)},\frac{x_3}{(0.1,0.8)},\frac{x_4}{(0.4,0.5)}\}$.
Then $(G,A)=\{G(e_1),G(e_2),G(e_3)\}$ is the pessimistic reduct intuitionistic fuzzy soft set of $(\tilde{G},A)$.

Definition 4.3. Let $(\tilde{G},A)$ be a GHFS set over (U,E). If for each $e\in A$ and $x\in U$, we use $\mu(x)=h^{+}_{\tilde{G}(e)}(x)$ and the corresponding $\nu(x)$ of $\tilde{G}(e)$ to construct an intuitionistic fuzzy soft set (G,A). Then (G,A) is called the optimistic reduct intuitionistic fuzzy soft set of $(\tilde{G},A)$.

Example 4.4. Consider Example 3.2, the optimistic reduct intuitionistic fuzzy soft set (G,A) of $(\tilde{G},A)$ is defined as follow:
$G(e_1)=\{\frac{x_1}{(0.4,0.5)},\frac{x_2}{(0.5,0.3)},\frac{x_3}{(0.6,0.3)},\frac{x_4}{(0.6,0.2)}\}$,
$G(e_2)=\{\frac{x_1}{(0.3,0.5)},\frac{x_2}{(0.9,0.1)},\frac{x_3}{(0.3,0.7)},\frac{x_4}{(0.4,0.5)}\}$,
$G(e_3)=\{\frac{x_1}{(0.6,0.3)},\frac{x_2}{(0.9,0.1)},\frac{x_3}{(0.2,0.6)},\frac{x_4}{(0.7,0.2)}\}$.
Then $(G,A)=\{G(e_1),G(e_2),G(e_3)\}$ is the optimistic reduct intuitionistic fuzzy soft set of $(\tilde{G},A)$.

Definition 4.5. Let $(\tilde{G},A)$ be a GHFS set over (U,E). If for each $e\in A$ and $x\in U$, we use $\mu(x)=s(h_{\tilde{G}(e)}(x))$ and the corresponding $\nu(x)=s(g_{\tilde{G}(e)}(x))$ of $\tilde{G}(e)$ to construct an intuitionistic fuzzy soft set (G,A). Then (G,A) is called the neutral reduct intuitionistic fuzzy soft set of $(\tilde{G},A)$.

Example 4.6. Consider Example 3.2, the neutral reduct intuitionistic fuzzy soft set (F,A) of $(\tilde{F},A)$ is defined as follow:

$G(e_1) = \{\frac{x_1}{(0.4,0.5)}, \frac{x_2}{(0.45,0.4)}, \frac{x_3}{(0.4,0.5)}, \frac{x_4}{(0.55,0.25)}\}$,
$G(e_2) = \{\frac{x_1}{(0.2,0.55)}, \frac{x_2}{(0.8,0.13)}, \frac{x_3}{(0.25,0.65)}, \frac{x_4}{(0.3,0.55)}\}$,
$G(e_3) = \{\frac{x_1}{(0.4,0.36)}, \frac{x_2}{(0.7,0.27)}, \frac{x_3}{(0.15,0.7)}, \frac{x_4}{(0.55,0.35)}\}$.
Then $(G, A) = \{G(e_1), G(e_2), G(e_3)\}$ is the neutral reduct intuitionistic fuzzy soft set of $(\tilde{F}, A)$.

In the following, we will use comparison table introduced by A.R. Roy and P.K. Maji [29] to solve a decision-making problem which is based on the operations of generalized hesitant fuzzy soft sets.

Example 4.7. Let us consider a GHFS set which describes evaluations of the investment value of some stocks. Suppose that there are six stocks in the universe $U = \{x_1, x_2, x_3, x_4, x_5, x_6\}$ under consideration, and $E = \{e_1, e_2, e_3, e_4, e_5, e_6\}$ is the set of decision parameters, where e_i $(i = 1, 2, 3, 4, 5, 6)$ stands for the parameters "earnings per share(eps)", "net assets per share", "net profit growth rate", "market share growth ratio", "asset-liability ratio" and "price to earning ratio (ps)", respectively. Let A, B and C denote three subsets of the parameters E. Also let A represents the profitability of each stock and B represents the growth of each stock. $A = \{$earnings per share(eps), net assets per share$\}$ and $B = \{$net profit growth rate, market share growth ratio$\}$. The subset C represents the risk of each stock, i.e., $C = \{$asset-liability ratio, price to earning ratio (ps)$\}$.

Assuming that the GHFS set $(\tilde{F}, A)$ describes the "stocks having high profi-tabi-lity", the GHFS set $(\tilde{G}, B)$ describes the "stocks having high growth" and the GHFS set $(\tilde{H}, C)$ describes the "stocks having low investing risk". Our purpose is to find out the best stock for the investor based on experts' evaluations.

After a serious and careful analysis, a group of stock experts construct the following generalized hesitant fuzzy soft sets:
$(\tilde{F}, A) = \{\tilde{F}(e_1), \tilde{F}(e_2)\}\}$ and
$\tilde{F}(e_1) = \{\frac{x_1}{(0.6,0.7,0.8),(0.3,0.3,0.1)}, \frac{x_2}{(0.2,0.3),(0.6,0.5)}, \frac{x_3}{(0.1,0.3,0.4),(0.8,0.7,0.6)}, \frac{x_4}{(0.5,0.7,0.8),(0.3,0.2,0.2)}, \frac{x_5}{(0.5,0.6,0.7),(0.3,0.4,0.3)}, \frac{x_6}{(0.4,0.6,0.8),(0.4,0.3,0.2)}\}$,
$\tilde{F}(e_2) = \{\frac{x_1}{(0.3,0.5,0.7),(0.6,0.3,0.3)}, \frac{x_2}{(0.2,0.3,0.5),(0.7,0.5,0.2)}, \frac{x_3}{(0.3,0.4,0.5),(0.6,0.4,0.5)}, \frac{x_4}{(0.3,0.5,0.6),(0.6,0.3,0.2)}, \frac{x_5}{(0.7,0.8,0.9),(0.3,0.1,0.05)}, \frac{x_6}{(0.3,0.5,0.7),(0.6,0.5,0.15)}\}$.
$(\tilde{G}, B) = \{\tilde{G}(e_3), \tilde{G}(e_4)\}$ and
$\tilde{G}(e_3) = \{\frac{x_1}{(0.5,0.6),(0.4,0.2)}, \frac{x_2}{(0.4,0.5,0.6),(0.5,0.3,0.3)}, \frac{x_3}{(0.2,0.4,0.6),(0.7,0.5,0.4)}, \frac{x_4}{(0.4,0.5),(0.5,0.4)}, \frac{x_5}{(0.3,0.4,0.5),(0.6,0.5,0.2)}, \frac{x_6}{(0.2,0.4,0.6),(0.7,0.6,0.2)}\}$,
$\tilde{G}(e_4) = \{\frac{x_1}{(0.4,0.6),(0.5,0.2)}, \frac{x_2}{(0.3,0.4,0.5),(0.65,0.5,0.5)}, \frac{x_3}{(0.2,0.3),(0.65,0.6)}, \frac{x_4}{(0.4),(0.3)}, \frac{x_5}{(0.4,0.6,0.9),(0.6,0.3,0.1)}, \frac{x_6}{(0.7,0.8),(0.2,0.1)}\}$.
$(\tilde{H}, C) = \{\tilde{H}(e_5), \tilde{H}(e_6)\}$ and
$\tilde{H}(e_5) = \{\frac{x_1}{(0.75,0.85),(0.15,0.05)}, \frac{x_2}{(0.55,0.65),(0.1,0.35)}, \frac{x_3}{(0.65,0.75),(0.2,0.1)}, \frac{x_4}{(0.75,0.85),(0.15,0.05)}, \frac{x_5}{(0.65,0.85),(0.25,0.15)}, \frac{x_6}{(0.75,0.95),(0.2,0.01)}\}$,
$\tilde{H}(e_6) = \{\frac{x_1}{(0.65),(0.15)}, \frac{x_2}{(0.72,0.83),(0.18,0.12)}, \frac{x_3}{(0.65),(0.22)}, \frac{x_4}{(0.45),(0.15)}, \frac{x_5}{(0.72,0.88),(0.2,0.12)}, \frac{x_6}{(0.66,0.82),(0.22,0.13)}\}$.

Let $(\tilde{F}, A)$ and $(\tilde{G}, B)$ be any two GHFS sets over the common universe U. After performing some operations (like AND,OR etc.) on the GHFS sets for some particular parameters of A and B, we obtain another GHFS set. The newly obtained GHFS set is termed as resultant GHFS set of $(\tilde{F},A)$ and $(\tilde{G},B)$

Considering the above two GHFS sets $(\tilde{F},A)$ and $(\tilde{G},B)$ if we perform "$(\tilde{F},A)$ OR $(\tilde{G},B)$". Then we can get the following resultant new GHFS set: $(\tilde{K},A\times B)$ defined as following,

$$\tilde{K}(e_1,e_3)=\{\frac{x_1}{(0.6,0.6,0.7,0.8),(0.2,0.3,0.3,0.1)},\frac{x_2}{(0.3,0.4,0.5,0.6),(0.5,0.5,0.3,0.3)},\frac{x_3}{(0.2,0.3,0.4,0.4,0.6),(0.7,0.7,0.6,0.5,0.4)},\frac{x_4}{(0.5,0.5,0.7,0.8),(0.4,0.3,0.2,0.2)},\frac{x_5}{(0.4,0.5,0.5,0.6,0.7),(0.5,0.2,0.3,0.4,0.3)},\frac{x_6}{(0.4,0.4,0.6,0.6,0.8),(0.6,0.4,0.2,0.3,0.2)}\},$$

$$\tilde{K}(e_1,e_4)=\{\frac{x_1}{(0.6,0.6,0.7,0.8),(0.2,0.3,0.3,0.1)},\frac{x_2}{(0.3,0.4,0.5),(0.5,0.5,0.5)},\frac{x_3}{(0.2,0.3,0.3,0.4),(0.65,0.7,0.6,0.6)},\frac{x_4}{(0.5,0.7,0.8),(0.3,0.2,0.2)},\frac{x_5}{(0.5,0.6,0.6,0.7,0.9),(0.3,0.4,0.3,0.3,0.1)},\frac{x_6}{(0.6,0.7,0.8,0.8),(0.3,0.2,0.2,0.1)}\},$$

$$\tilde{K}(e_2,e_3)=\{\frac{x_1}{(0.5,0.5,0.6,0.7),(0.4,0.3,0.2,0.3)},\frac{x_2}{(0.3,0.4,0.5,0.5,0.6),(0.5,0.5,0.2,0.3,0.3)},\frac{x_3}{(0.3,0.4,0.4,0.5,0.6),(0.6,0.4,0.5,0.5,0.4)},\frac{x_4}{(0.4,0.5,0.5,0.6),(0.5,0.4,0.3,0.2)},\frac{x_5}{(0.4,0.5,0.7,0.8,0.9),(0.5,0.2,0.3,0.1,0.05)},\frac{x_6}{(0.3,0.4,0.5,0.6,0.7),(0.6,0.6,0.5,0.2,0.15)}\},$$

$$\tilde{K}(e_2,e_4)=\{\frac{x_1}{(0.4,0.5,0.6,0.7),(0.5,0.3,0.2,0.3)},\frac{x_2}{(0.3,0.3,0.4,0.5,0.5),(0.5,0.65,0.5,0.2,0.5)},\frac{x_3}{(0.3,0.4,0.5),(0.6,0.4,0.5)},\frac{x_4}{(0.4,0.5,0.6),(0.3,0.3,0.2)},\frac{x_5}{(0.6,0.7,0.8,0.9,0.9),(0.3,0.3,0.1,0.1,0.05)},\frac{x_6}{(0.5,0.7,0.7,0.8),(0.5,0.15,0.2,0.1)}\}.$$

Considering the above two GHFS sets $(\tilde{K},A\times B)$ and $(\tilde{H},C)$ if we perform "$(\tilde{K},A\times B)$ AND $(\tilde{H},C)$". Then we can get the following resultant new GHFS set: $(\tilde{R},A\times B\times C)$ defined as following,

$$\tilde{R}(e_1,e_3,e_5)=\{\frac{x_1}{(0.6,0.6,0.7,0.75,0.8),(0.2,0.3,0.3,0.15,0.1)},\frac{x_2}{(0.3,0.4,0.5,0.6),(0.5,0.5,0.3,0.3)},\frac{x_3}{(0.2,0.3,0.4,0.4,0.6,0.65),(0.7,0.7,0.6,0.5,0.4,0.2)},\frac{x_4}{(0.5,0.5,0.7,0.75,0.8),(0.4,0.3,0.2,0.15,0.2)},\frac{x_5}{(0.4,0.5,0.5,0.6,0.65,0.7),(0.5,0.2,0.3,0.4,0.25,0.3)},\frac{x_6}{(0.4,0.4,0.6,0.6,0.75,0.8),(0.6,0.4,0.2,0.3,0.2,0.2)}\},$$

$$\tilde{R}(e_1,e_3,e_6)=\{\frac{x_1}{(0.6,0.6,0.65,0.7),(0.2,0.3,0.15,0.3)},\frac{x_2}{(0.3,0.4,0.5,0.6,0.72),(0.5,0.5,0.3,0.3,0.18)},\frac{x_3}{(0.2,0.3,0.4,0.4,0.6),(0.7,0.7,0.6,0.5,0.4)},\frac{x_4}{(0.5,0.5,0.7),(0.4,0.3,0.2)},\frac{x_5}{(0.4,0.5,0.5,0.6,0.7,0.72),(0.5,0.2,0.3,0.4,0.3,0.2)},\frac{x_6}{(0.4,0.4,0.6,0.6,0.66,0.8),(0.6,0.4,0.2,0.3,0.22,0.2)}\},$$

$$\tilde{R}(e_1,e_4,e_5)\{\frac{x_1}{(0.6,0.6,0.7,0.75,0.8),(0.2,0.3,0.3,0.15,0.1)},\frac{x_2}{(0.3,0.4,0.5),(0.5,0.5,0.5)},\frac{x_3}{(0.2,0.3,0.3,0.4,0.65),(0.65,0.7,0.6,0.6,0.2)},\frac{x_4}{(0.5,0.7,0.75,0.8),(0.3,0.2,0.15,0.2)},\frac{x_5}{(0.5,0.6,0.6,0.65,0.7,0.85),(0.3,0.4,0.3,0.25,0.3,0.15)},\frac{x_6}{(0.6,0.7,0.75,0.8,0.8),(0.3,0.2,0.2,0.2,0.1)}\},$$

$$\tilde{R}(e_1,e_4,e_6)=\{\frac{x_1}{(0.6,0.6,0.65,0.7),(0.2,0.3,0.15,0.3)},\frac{x_2}{(0.3,0.4,0.5,0.72),(0.5,0.5,0.5,0.18)},\frac{x_3}{(0.2,0.3,0.3,0.4),(0.65,0.7,0.6,0.6)},\frac{x_4}{(0.5,0.7),(0.3,0.2)},\frac{x_5}{(0.5,0.6,0.6,0.7,0.72,0.88),(0.3,0.4,0.3,0.3,0.2,0.12)},\frac{x_6}{(0.6,0.66,0.7,0.8),(0.3,0.22,0.2,0.2)}\},$$

$$\tilde{R}(e_2,e_3,e_5)=\{\frac{x_1}{(0.5,0.5,0.6,0.7,0.75),(0.4,0.3,0.2,0.3,0.15)},\frac{x_2}{(0.3,0.4,0.5,0.5,0.6),(0.5,0.5,0.2,0.3,0.3)},\frac{x_3}{(0.3,0.4,0.4,0.5,0.6,0.65),(0.6,0.4,0.5,0.5,0.4,0.2)},\frac{x_4}{(0.4,0.5,0.5,0.6,0.75),(0.5,0.4,0.3,0.2,0.15)},\frac{x_5}{(0.4,0.5,0.65,0.7,0.8,0.85),(0.5,0.2,0.25,0.3,0.1,0.15)},\frac{x_6}{(0.3,0.4,0.5,0.6,0.7,0.75),(0.6,0.6,0.5,0.2,0.15,0.2)}\},$$

$$\tilde{R}(e_2,e_3,e_6)=\{\frac{x_1}{(0.5,0.5,0.6),(0.4,0.3,0.2)},\frac{x_2}{(0.3,0.4,0.5,0.5,0.6,0.72),(0.5,0.5,0.2,0.3,0.3,0.18)},\frac{x_3}{(0.3,0.4,0.4,0.5,0.6),(0.6,0.4,0.5,0.5,0.4)},\frac{x_4}{(0.4,0.5,0.5),(0.5,0.4,0.3)},\frac{x_5}{(0.4,0.5,0.7,0.72,0.8,0.88),(0.5,0.2,0.3,0.2,0.1,0.12)},\frac{x_6}{(0.3,0.4,0.5,0.6,0.66,0.7),(0.6,0.6,0.5,0.2,0.22,0.15)}\},$$

$$\tilde{R}(e_2,e_4,e_5)=\{\frac{x_1}{(0.4,0.5,0.6,0.7,0.75),(0.5,0.3,0.2,0.3,0.15)},\frac{x_2}{(0.3,0.3,0.4,0.5,0.5),(0.5,0.65,0.5,0.2,0.5)},\frac{x_3}{(0.3,0.4,0.5,0.65),(0.6,0.4,0.5,0.2)},\frac{x_4}{(0.4,0.5,0.6,0.75),(0.3,0.3,0.2,0.15)},\frac{x_5}{(0.6,0.65,0.7,0.8,0.85,0.9),(0.3,0.25,0.3,0.1,0.15,0.1)},\frac{x_6}{(0.5,0.7,0.7,0.75,0.8),(0.5,0.15,0.2,0.2,0.1)}\},$$

$$\tilde{R}(e_2,e_4,e_6)=\{\frac{x_1}{(0.4,0.5,0.6),(0.5,0.3,0.2)},\frac{x_2}{(0.3,0.3,0.4,0.5,0.5,0.72),(0.5,0.65,0.5,0.2,0.5,0.18)},\frac{x_3}{(0.3,0.4,0.5),(0.6,0.4,0.5)},\frac{x_4}{(0.4,0.5),(0.3,0.3)},\frac{x_5}{(0.6,0.7,0.72,0.8,0.88,0.9),(0.3,0.3,0.2,0.1,0.12,0.1)},\frac{x_6}{(0.5,0.66,0.7,0.7),(0.5,0.22,0.15,0.2)}\}.$$

And the GHFS set $(\tilde{R}, A\times B\times C)$ describes the evaluations of the stocks having high profitability or high growth and low investing risk by the group of experts.

Definition 4.8. [29] (Comparison table) It is a square table in which number of rows and number of column are equal and both are labeled by the object name of the universe such as $x_1, x_2, ..., x_n$ and the entries are c_{ij}, where c_{ij} = the number of parameters for which the value of x_i exceeds or equal to the value of x_j.

Algorithm:

(i) Input the set $A \subseteq E$ of choice of parameters of the investor.

(ii) Consider the reduced intuitionistic fuzzy soft set obtained by the GHFS set in tabular form.

(iii) Compute the comparison table of membership function and non-membership function of the reduced intuitionistic fuzzy soft set.

(iv) Compute the membership score and non-membership score.

(v) Compute the final score by subtracting non-membership score from membership score.

(vi) Find the maximum score, if it occurs in i-th row then the investor will choose x_i.

Note. In step (ii), the reduced intuitionistic fuzzy soft set we can choose the pessimistic reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$, the optimistic reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$, or the neutral reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$, depending on the decision-maker's actual situations. For example, we use the optimistic reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$ in step (ii). The tabular representation of the optimistic reduct fuzzy soft set of $(\tilde{R}, A\times B\times C)$ will be as.

Table 1. Tabular representation of membership function

$\cdot$	(e_1,e_3,e_5)	(e_1,e_3,e_6)	(e_1,e_4,e_5)	(e_1,e_4,e_6)
x_1	0.8	0.7	0.8	0.7
x_2	0.6	0.72	0.5	0.72
x_3	0.65	0.6	0.65	0.4
x_4	0.8	0.7	0.8	0.7
x_5	0.7	0.72	0.85	0.88
x_6	0.8	0.8	0.8	0.8

$\cdot$	(e_2,e_3,e_5)	(e_2,e_3,e_6)	(e_2,e_4,e_5)	(e_2,e_4,e_6)
x_1	0.75	0.6	0.75	0.6
x_2	0.6	0.72	0.5	0.72
x_3	0.65	0.6	0.65	0.5
x_4	0.75	0.5	0.75	0.5
x_5	0.85	0.88	0.9	0.9
x_6	0.75	0.7	0.8	0.7

Table 2. Comparison table of the above table

$\cdot$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	8	4	8	8	1	3
x_2	4	8	4	4	1	2
x_3	1	4	8	2	0	0
x_4	6	4	7	8	1	3
x_5	7	8	8	7	8	6
x_6	8	6	8	8	2	8

Table 3. Membership score table

$\cdot$	RowSum(a)	ColumnSum(b)	MembershipScore(a-b)
x_1	32	34	-2
x_2	23	34	-11
x_3	15	43	-28
x_4	29	37	-8
x_5	44	13	31
x_6	40	22	18

Table 4. Tabular representation of non-membership function

$\cdot$	(e_1, e_3, e_5)	(e_1, e_3, e_6)	(e_1, e_4, e_5)	(e_1, e_4, e_6)
x_1	0.1	0.3	0.1	0.3
x_2	0.3	0.18	0.5	0.18
x_3	0.2	0.4	0.2	0.6
x_4	0.2	0.2	0.2	0.2
x_5	0.3	0.2	0.15	0.12
x_6	0.2	0.2	0.1	0.2

$\cdot$	(e_2, e_3, e_5)	(e_2, e_3, e_6)	(e_2, e_4, e_5)	(e_2, e_4, e_6)
x_1	0.15	0.2	0.15	0.2
x_2	0.3	0.18	0.5	0.18
x_3	0.2	0.4	0.2	0.5
x_4	0.15	0.3	0.15	0.3
x_5	0.15	0.12	0.1	0.1
x_6	0.2	0.15	0.1	0.2

Table 5. Comparison table of the above table

$\cdot$	x_1	x_2	x_3	x_4	x_5	x_6
x_1	8	4	0	4	6	6
x_2	4	8	4	4	7	5
x_3	8	4	8	8	7	8
x_4	6	4	2	8	7	7
x_5	3	2	1	3	8	4
x_6	4	3	2	4	6	8

Table 6. Non-membership score table

$\cdot$	RowSum(a)	ColumnSum(b)	Non-embershipScore(a-b)
x_1	28	33	-5
x_2	32	25	7
x_3	43	17	26
x_4	34	31	3
x_5	21	41	-20
x_6	27	38	-11

Table 7. Final score table

$\cdot$	MembershipScore(m)	Non-membershipScore(n)	FinalScore(m-n)
x_1	-2	-5	3
x_2	-11	7	-18
x_3	-28	26	-54
x_4	-8	3	-11
x_5	31	-20	51
x_6	18	-11	29

Clearly, the maximum score is 51 scored by the stock x_5. Decision: The investor will choose x_5. In case, if the investor does not want to choose x_5 due to certain reasons, its second choice will be x_6. This is the result obtained by the optimistic reduct fuzzy soft set of $(\tilde{R}, A\times B\times C)$. Other cases, e.g., the pessimistic reduct intuitionistic fuzzy soft set or the neutral reduct intuitionistic fuzzy soft set of $(\tilde{G}, A)$, are similar to the above analysis.

Remark 4.9. After step (ii), we can also adopt Jiang's algorithm by using adjustable approach to intuitionistic fuzzy soft sets based decision-making [39] or Zhang's rough set approach to intuitionistic fuzzy soft set based decision making [25] to get the final optimal decision, especially when there are too many "optimal choices" to be chosen.

5. Similarity between two generalized hesitant fuzzy soft sets

In several problems it is often required to compare two sets. The sets may be fuzzy, may be vague etc. We often interested to know whether two patterns or images are identical or approximately identical or at least to what degree they are identical. Several researchers have studied the problem of similarity measurement between fuzzy sets, fuzzy numbers and vague sets. Recently Majumdar and Samanta [40], [41] have studied the similarity measure of soft sets and fuzzy soft sets. Chang Wang and Anjing Qu studied the entropy, similarity measure and distance on vague soft sets in [42]. Y.C. Jiang et al. studied entropy, similarity measure and distance on intuitionistic fuzzy soft sets interval-valued fuzzy soft sets in [43]. Similarity measures have extensive application in several areas such as pattern recognition [44], image processing [45], cluster analysis [46], approximate reasoning [47], medical diagnosis [48], decision-making [49] etc.

Let $U = \{x_1, x_2, ..., x_n\}$ be the universal set of elements and $E = \{e_1, e_2, ..., e_m\}$ be the universal set of parameters. Let $(\tilde{F}, E)$ and $(\tilde{G}, E)$ be two GHFS sets over the parametrized universe (U, E).

Definition 5.1. We first define the Hamming distance $d((\tilde{F}, E), (\tilde{G}, E))$ by the equation:

$$d((\tilde{F}, E), (\tilde{G}, E)) = \sum_{i=1}^{m}\sum_{j=1}^{n}\frac{1}{2}\cdot\left[\frac{1}{l_{x_j}}\cdot\sum_{k=1}^{l_{x_j}}|\mu_F^{\sigma(k)}(e_i)(x_j) - \mu_G^{\sigma(k)}(e_i)(x_j)| + \frac{1}{l_{x_j}}\cdot\sum_{k=1}^{l_{x_j}}|\nu_F^{\sigma(k)}(e_i)(x_j) - \nu_G^{\sigma(k)}(e_i)(x_j)|\right].$$

Then, define the normalized Hamming distance between $\tilde{F}(e_i)$ and $\tilde{G}(e_i)$ by the following formula:

$D((\tilde{F}, E), (\tilde{G}, E)) = \frac{d((\tilde{F},E),(\tilde{G},E))}{nm}$.

And we find the similarity by using the simple formula:

$S((\tilde{F}, E), (\tilde{G}, E)) = 1 - D((\tilde{F}, E), (\tilde{G}, E))$.

Example 5.2. Consider the following two GHFS sets where $U = \{x_1, x_2, x_3\}$ $E = \{e_1, e_2\}$. Define $(\tilde{F}, E)$ and $(\tilde{G}, E)$ as follows:

$\tilde{F}(e_1) = \{\frac{x_1}{(0.3,0.5),(0.5,0.2)}, \frac{x_2}{(0.6,0.9),(0.2,0.1)}, \frac{x_3}{(0.2,0.4),(0.6,0.5)}\}$,
$\tilde{F}(e_2) = \{\frac{x_1}{(0.7,0.8),(0.1,0.2)}, \frac{x_2}{(0.5,0.7),(0.3,0.2)}, \frac{x_3}{(0.5,0.8),(0.3,0.1)}\}$.
$\tilde{G}(e_1) = \{\frac{x_1}{(0.4,0.7),(0.5,0.3)}, \frac{x_2}{(0.6,0.7),(0.3,0.2)}, \frac{x_3}{(0.2,0.5),(0.3,0.4)}\}$,
$\tilde{G}(e_2) = \{\frac{x_1}{(0.6,0.7),(0.3,0.1)}, \frac{x_2}{(0.6,0.8),(0.3,0.1)}, \frac{x_3}{(0.5,0.7),(0.3,0.2)}\}$.

Here, the Hamming distance

$$d((\tilde{F}, E), (\tilde{G}, E)) = \sum_{i=1}^{2}\sum_{j=1}^{3}\frac{1}{2}\cdot\left[\frac{1}{l_{x_j}}\cdot\sum_{k=1}^{l_{x_j}}|\mu_F^{\sigma(k)}(e_i)(x_j) - \mu_G^{\sigma(k)}(e_i)(x_j)| + \frac{1}{l_{x_j}}\cdot\sum_{k=1}^{l_{x_j}}|\nu_F^{\sigma(k)}(e_i)(x_j) - \nu_G^{\sigma(k)}(e_i)(x_j)|\right]$$

$= \frac{\frac{1}{2}(|0.3-0.4|+|0.5-0.7|+|0.5-0.5|+|0.2-0.3|)+\frac{1}{2}(|0.6-0.6|+|0.9-0.7|+|0.2-0.3|+|0.1-0.2|)+\frac{1}{2}(|0.2-0.2|+|0.4-0.5|+|0.6-0}{2}$
$+ \frac{\frac{1}{2}(|0.7-0.6|+|0.8-0.7|+|0.1-0.3|+|0.2-0.1|)+\frac{1}{2}(|0.5-0.6|+|0.7-0.8|+|0.3-0.3|+|0.2-0.1|)+\frac{1}{2}(|0.5-0.5|+|0.8-0.7|+|0.3-0.}{2}$
$= 0.575$.

The normalized Hamming distance

$$D((\tilde{F}, E), (\tilde{G}, E)) = \frac{d((\tilde{F}, E), (\tilde{G}, E))}{nm} = \frac{0.575}{6} \approx 0.096.$$

Hence the similarity between the two GHFS sets $(\tilde{F}, E)$ and $(\tilde{G}, E)$ will be

$$S((\tilde{F}, E), (\tilde{G}, E)) = 1 - 0.096 = 0.904.$$

Proposition 5.3. *Let $(\tilde{F}, E)$, $(\tilde{G}, E)$ and $(\tilde{H}, E)$ be three GHFS sets over the parametrized universe (U, E). Then the following holds:*

(i) $S((\tilde{F}, E), (\tilde{G}, E)) = S((\tilde{G}, E), (\tilde{F}, E))$,

(ii) $0 \leq S((\tilde{F}, E), (\tilde{G}, E)) \leq 1$,

(iii) $(\tilde{F}, E) = (\tilde{G}, E) \Leftrightarrow S((\tilde{F}, E), (\tilde{G}, E)) = 1$,

(iv) $(\tilde{F}, E)\tilde{\sqsubseteq}(\tilde{G}, E)\tilde{\sqsubseteq}(\tilde{H}, E) \Rightarrow S((\tilde{F}, E), (\tilde{H}, E)) \leq S((\tilde{G}, E), (\tilde{H}, E))$.

Proof. The proofs are straightforward and follow from definition.

6. An application of this similarity measure in decision-making

This technique of similarity measure between two GHFS sets can also be applied to decision-making problems.

In what follows, we give an example adapted from [50] and [12] to illustrate our distance measures for GHFS sets:

Example 6.1. Energy is an indispensable factor for the socio-economic development of societies. Thus the correct energy policy affects economic development and environment, and so, the most appropriate energy policy selection is very important. Suppose that there are four alternatives (energy projects) A_i (i = 1,2,3,4) to be invested, each energy project has two stages x_1, x_2, and four attributes to be considered: e_1: technological; e_2: environmental; e_3: socio-political; e_4: economic (more details about them can be found in [50,12]). Several decision makers are invited to evaluate the performance of the four alternatives. For an alternative under an attribute, although all of the decision makers provide their evaluated values, some of these values may be repeated. However, a value repeated more times does not indicate that it has more importance than other values repeated less times. For example, the value repeated one time may be provided by a decision maker who is an expert at this area, and the value repeated twice may be provided by two decision makers who are not familiar with this area. In such cases, the value repeated one time may be more important than the one repeated twice. To get a more reasonable result, it is better that the decision makers give their evaluations anonymously. We only collect all of the possible values for an alternative under an attribute, and each value provided only means that it is a possible value, but its importance is unknown. Thus the times that the values repeated are unimportant, and it is reasonable to allow these values repeated many times appear only once. The GHFS is just a tool to deal with such cases, and all possible evaluations for an alternative under the attributes can be considered as a GHFS.

Suppose that the ideal alternative is $A = 1$ seen as a special GHFS, we can calculate the distance between each alternative and the ideal alternative using our distance measures.

Our model generalized hesitant fuzzy soft set for ideal alternative A is given in Table 8.

Table 8. Tabular representation of Model GHFS set

$(\tilde{A}, E)$	e_1	e_2	e_3	e_4
x_1	(1, 0)	(1, 0)	(1, 0)	(1, 0)
x_2	(1, 0)	(1, 0)	(1, 0)	(1, 0)

Table 9. Tabular representation of GHFS set for the first energy projects

$(\tilde{A}_1, E)$	e_1	e_2	e_3	e_4
x_1	$(0.6, 0.7, 0.9), (0.2, 0.1, 0.1)$	$(0.7, 0.8), (0.2, 0.1)$	$(0.8, 0.9), (0.2, 0.1)$	$(0.7, 0.9), (0.2, 0.1)$
x_2	$(0.7, 0.8), (0.2, 0.1)$	$(0.6, 0.7), (0.3, 0.2)$	$(0.5, 0.7), (0.3, 0.4)$	$(0.8, 0.9), (0.2, 0.1)$

Table 10. Tabular representation of GHFS set for the second energy projects

$(\tilde{A}_2, E)$	e_1	e_2	e_3	e_4
x_1	(0.3,0.5),(0.5,0.3)	(0.7,0.9),(0.3,0.1)	(0.1,0.5),(0.8,0.5)	(0.3,0.4),(0.6,0.5)
x_2	(0.4,0.6),(0.3,0.3)	(0.6,0.7),(0.2,0.1)	(0.3,0.5),(0.4,0.3)	(0.5,0.6),(0.3,0.3)

Table 11. Tabular representation of GHFS set for the third energy projects

$(\tilde{A}_3, E)$	e_1	e_2	e_3	e_4
x_1	$(0.3, 0.5, 0.7), (0.4, 0.3, 0.2)$	$(0.4, 0.6), (0.3, 0.2)$	$(0.5, 0.7), (0.2, 0.2)$	$(0.3, 0.5), (0.2, 0.4)$
x_2	$(0.1, 0.2), (0.7, 0.8)$	$(0.3, 0.4), (0.1, 0.2)$	$(0.5, 0.6), (0.2, 0.3)$	$(0.7, 0.8), (0.2, 0.1)$

Table 12. Tabular representation of GHFS set for the third energy projects

$(\tilde{A}_4, E)$	e_1	e_2	e_3	e_4
x_1	(0.4,0.7),(0.3,0.2)	(0.8,0.9),(0.1,0.1)	(0.3,0.4),(0.5,0.4)	(0.6,0.7),(0.3,0.2)
x_2	(0.5,0.6),(0.3,0.3)	(0.3,0.5),(0.5,0.3)	(0.6,0.8),(0.3,0.2)	(0.5,0.8),(0.2,0.1)

After simple computing, we have the follow results:

Table 13. Tabular representation of similarity of A_i and A

$S((\tilde{A}, E), (\tilde{A}_1, E))$	$S((\tilde{A}, E), (\tilde{A}_2, E))$	$S((\tilde{A}, E), (\tilde{A}_3, E))$	$S((\tilde{A}, E), (\tilde{A}_4, E))$
0.78125	0.565625	0.590625	0.659375

Thus, the ranking of the four energy projects is $A_1 \succ A_4 \succ A_3 \succ A_2$.

7. Concluding remarks

In this paper, the concept of generalized hesitant fuzzy soft sets and some operations on generalized hesitant fuzzy soft sets are defined and some of their properties are studied. Applications of generalized hesitant fuzzy soft sets in decision-making are investigated. As we can see, the generalized hesitant fuzzy soft sets are effective for the decision makers and easier to be applied in real-life applications. In the following paper, we will study entropy, similarity and distance measure of generalized hesitant fuzzy soft sets and their applications in decision-making as in [12,42,43]. Also, we will investigate the applications of generalized hesitant fuzzy soft sets with Multiple Criterion Decision-Making (MCDM) problems as in [31] in the following paper. The weighted decision making problem discussed in [32] about generalized hesitant fuzzy soft sets should also be investigated in the future. And the theory of generalized hesitant fuzzy soft sets can also be use in the classification for gene expression engineering as in [51]. We are hopeful that this modified concept will be helpful in dealing with several problems related to uncertainty and will yield more natural results.

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