APPLICATIONS OF THE UNITING ELEMENTS METHOD

R. Mahjoob

Department of Mathematics
Faculty of Mathematics, Statistics and Computer Sciences
Semnan University, Semnan
Iran
e-mail: mahjoob@profs.semnan.ac.ir

T. Vougiouklis

Democritus University of Thrace
School of Education
681 00 Alexandroupolis
Greece
e-mail: tvougiou@eled.duth.gr

Abstract. We present the Uniting Elements Method introduced by Corsini-Vougiouklis in 1989 [5]. Some applications of the uniting elements method on the classical algebra are presented and finally we connect them with ∂-structures and the other classes of hyperstructures.

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1. Basic definitions

The largest class of hyperstructures is the one which satisfy the weak properties. These are called H_v-structures introduced in 1990 [10], and they proved to have a lot of applications on several applied science such as linguistic, biology, chemistry, physics, and so on. The H_v-structures satisfy the *weak axioms* where the non-empty intersection replaces the equality. The H_v-structures can be used in models as an organized devise as well.

Recall some basic definitions:

Definitions 1.1. A set H equipped with at least one hyperoperation (we abbreviate by *hope* any *hyperoperation*) $. : H \times H \longrightarrow P(H)$ is called the *hyperstructure*. We abbreviate by *WASS* the *weak associativity*:$(xy)z \cap x(yz) \neq \emptyset,\ \forall x, y, z \in H$ and by *COW* the *weak commutativity*: $xy \cap yx \neq \emptyset,\ \forall x, y \in H$.

The hyperstructure $(H, .)$ is called *H_v-semigroup* if it is WASS and is called H_v-group if it is reproductive H_v-semigroup, i.e., $xH = Hx = H, \forall x \in H$. The hyperstructure $(R, +, .)$ is called H_v-ring if $(+)$ and $(.)$ are WASS, the reproduction axiom is valid for $(+)$ and $(.)$ is *weak distributive* with respect to $(+)$, i.e.,

$$x(y+z) \cap (xy+xz) \neq \emptyset, \ (x+y)z \cap (xz+yz) \neq \emptyset, \ \forall x, y, z \in R.$$

For more definitions and results on H_v-structures one can seen in books and papers as [1], [2], [3], [4], [6], [7], [9], [13]. An extreme class of the H_v-structures is the following: An H_v-structure is called *very thin* iff all hopes are operations except one, which has all hyperproducts singletons except only one, which has cardinality more than one.

The fundamental relations β^*,γ^* and ε^* are defined, in H_v-groups, H_v-rings and H_v-vector spaces, respectively, as the smallest equivalences so that the quotient would be group, ring and vector space, respectively [11],[12],[13]. The way to find the fundamental classes is given by analogous theorems to the following one:

Theorem. *Let $(H, .)$ be an H_v-group and denote by $\boldsymbol{U}$ the set of all finite products of elements of H. We define the relation β in H as follows: $x\beta y$ iff $\{x, y\} \subset \mathbf{u}$ where $\mathbf{u} \in \mathbf{U}$. Then the fundamental relation β^* is the transitive closure of the relation β.*

Remark that the main point of the proof is that the β guaranties the validity of the following: Take two elements x, y such that $\{x, y\} \subset \mathbf{u} \in \mathbf{U}$ and any hyperproduct where one of the elements x, y, is used. Then, if this element is replaced by the other, the new hyperproduct is inside the same fundamental class where the first hyperproduct is. Therefore, if the 'hyperproducts' of the above β-classes are 'products', then, they are fundamental classes. Analogous remarks for the relations γ and ε, are also applied.

An element is called *single* if its fundamental class is a singleton.

Motivation for H_v-structures. *We know that the quotient of a group with respect to an invariant subgroup is a group. Marty states that, the quotient of a group with respect to any subgroup is a hypergroup. Now, the quotient of a group with respect to any partition is an H_v-group.*

Let $(H, .), (H, \otimes)$ H_v-semigroups defined on the same set H. $(.)$ is called *smaller* than $(\otimes)$, and $(\otimes)$ greater than $(.)$, iff there exists automorphism

$$f \in Aut(H, \otimes) \ \text{ such that } \ xy \subset f(x \otimes y), \forall x, y \in H.$$

Then we write $. \leq \otimes$ and we say that $(H, \otimes)$ *contains* the $(H, .)$. If $(H, .)$ is a structure then it is called *basic structure* and $(H, \otimes)$ is called H_b-*structure.*

The Little Theorem. *Greater hopes of hopes which are WASS or COW, are also WASS and COW, respectively.*

The fundamental relations are used for general definitions of hyperstructures. Thus, to define the general H$_v$-field one uses the fundamental relation γ^*: The H$_v$-ring $(R, +, .)$ is called *H$_v$-field* if the quotient R/γ^* is a field [11],[12]. The *H$_v$-module* is an H$_v$-group over an H$_v$-ring if the weak distributivity and a mixed weak associativity on all hopes, is valid. In an analogous way the *H$_v$-vector spaces* and the *H$_v$-algebra* can be defined [11], [16].

The definition of the H$_v$-field introduced a new class of hyperstructures [14]: The H$_v$-semigroup (H,.) is called *h/v-group* if the quotient H/β^* is a group. The h/v-groups are a generalization of the H$_v$-groups because in h/v-groups the reproductivity is not necessarily valid. In a similar way the h/v-rings, h/v-fields, h/v-modulus, h/v-vector spaces etc, are defined.

The general definition of an H$_v$-Lie algebra was given in [13], [16] as follows:

Definitions 1.3. Let $(\mathbf{L}, +)$ be H$_v$-vector space over the field $(\mathbf{F}, +, .)$, $\varphi : \mathbf{F} \longrightarrow \mathbf{F}/\gamma^*$, the canonical map and $\omega_F = \{x \in F : \varphi(x) = 0\}$, where 0 is the zero of the fundamental field $\mathbf{F}/\gamma^*$. Similarly, let ω_L be the core of the canonical map $\varphi' : \mathbf{L} \longrightarrow \mathbf{L}/\varepsilon^*$ and denote by the same symbol 0 the zero of $\mathbf{L}/\varepsilon^*$. Consider the *bracket (commutator) hope:*

$$[\ ,\ ] : L \times L \longrightarrow P(L) : (x, y) \longrightarrow [x, y].$$

Then $\mathbf{L}$ is an *H$_v$*-Lie algebra over $\mathbf{F}$ if the following axioms are satisfied:

(L1) The bracket hope is bilinear, i.e.,
$[\lambda_1 x_1 + \lambda_2 x_2, y] \cap (\lambda_1[x_1, y] + \lambda_2[x_2, y]) \neq \emptyset$
$[x, \lambda_1 y_1 + \lambda_2 y_2] \cap (\lambda_1[x, y_1] + \lambda_2[x, y_2]) \neq \emptyset, \ \forall x, x_1, x_2, y, y_1, y_2 \in L, \ \lambda_1, \lambda_2 \in F,$

(L2) $[x, x] \cap \omega_L \neq \emptyset, \ \forall x \in L,$

(L3) $([x, [y, z] + [y, [z, x]] + [z, [x, y]]) \cap \omega_L \neq \emptyset, \ \forall x, y, z \in L.$

A general way to define hopes, from given operations [9], [11] is the following:

Definitions 1.4. Let $(G, .)$ be a groupoid, then for every set $P \subset G, P \neq \emptyset$, we define the following hopes called *P-hopes*:

$$\underline{P} : x\underline{P}y = (xP)y \cup x(Py),$$

$$\underline{P}_r : x\underline{P}_r y = (xy)P \cup x(yP), \underline{P}_l : x\underline{P}_l y = (Px)y \cup P(xy), \forall x, y \in G.$$

The $(G, P), (G, \underline{P}_r)$ and $(G, \underline{P}_l)$ are called *P-hyperstructures.* If $(G, .)$ is semigroup, then (G, P) is a semihypergroup but we do not know for $(G, \underline{P}_r), (G, \underline{P}_l)$. In some cases, mainly depending on the choice of P, the $(G, \underline{P}_r), (G, \underline{P}_l)$ can be associative or WASS. If in G, more operations are defined then for each operation several P-hopes can be defined.

Constructions 1.5. Let $(G, .)$ be abelian group and $P \subset G$, with more than one elements. We define a hope $(\times_P)$ as follows:

$$x \times_P y = \begin{cases} x.P.y = \{x.h.y \mid h \in P\} & \text{if } x \neq e \text{ and } y \neq e \\ x.y & \text{if } x = e \text{ or } y = e \end{cases}$$

and we call this *P_e-hope.* The hyperstructure $(G, \times_P)$ is an abelian H$_v$-group.

During the last decades, the hyperstructures have a variety of applications not only in other branches of mathematics but also in many other sciences including social ones. These applications are on biomathematics, conchology, inheritance, hadronic physics, to mention but a few. The hyperstructure theory is closely related to fuzzy theory; thus, hyperstructures can now be widely applicable in industry and production, too.

The Lie-Santilli theory on *isotopies* was born in 1970's [8] to solve Hadronic Mechanics problems. Santilli proposed a 'lifting'of the n-dimensional trivial unit matrix of a normal theory into a nowhere singular, symmetric, real-valued, positive defined, n-dimensional new matrix. The original theory is reconstructed such as to admit the new matrix as left and right unit. The *isofields* needed in this theory correspond into the hyperstructures were introduced by Santilli and Vougiouklis in 1999 and they are called *e-hyperfields*. The H_v-fields can give e-hyperfields which can be used in the isotopy theory in applications as in physics or biology, see [8].

A new application, which combines hyperstructure theory and fuzzy, is to replace in questionnaires the scale of Likert by the bar of Vougiouklis & Vougiouklis [17]. The suggestion is the following:

Definition 1.6. In every question substitute the Likert scale with 'the bar' whose poles are defined with $'0'$ on the left end, and $'1'$ on the right end:

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The subjects/participants are asked instead of deciding and checking a specific grade on the scale, to cut the bar at any point s/he feels expresses her/his answer to the specific question. The use of the bar of Vougiouklis & Vougiouklis instead of a scale of Likert has several advantages during both the filling-in and the research processing. The final suggested length of the bar, according to the Golden Ratio, is 6.2cm.

Now, we intend to remove elements from hyperstructures or classical structures.

Definitions 1.7. Let $(H, .)$ be hypergroupoid.

We say that we *remove* $h \in H$, if we simply consider the restriction of the hope $(.)$ on the $H - \{h\}$.

We say that an $\underline{h} \in H$ *absorbs* $h \in H$ if we replace h, whenever it appears, by $\underline{h}$.

We say that $\underline{h} \in H$ *merges* with $h \in H$, if we take as the product of any $x \in H$ by $\underline{h}$, the union of the results of x with both h and $\underline{h}$, and we consider h and $\underline{h}$ as one class, with representative $\underline{h}$.

The above definitions can be applied on more complicated structures analogously. When we remove an element, then the hope may become partial. If the removing element does not appeared in any result, then a hyperstructure more strict may obtained.

The representation problem of H_v-structures by H_v-matrices is the following (see [11]):

The *H_v-matrix* is a matrix with entries of an H_v-ring or H_v-field. The hyperproduct of two H_v-matrices $\mathbf{A} = (a_{ij})$ and $\mathbf{B} = (b_{ij})$, of type $m \times n$ and $n \times r$, respectively, is defined in the usual manner but it is a set of $m \times r$ H_v-matrices:

$$\mathbf{A}.\mathbf{B} = (a_{ij}).(b_{ij}) = \left\{ \mathbf{C} = (c_{ij}) \mid c_{ij} \in \oplus \sum a_{ik} b_{kj} \right\},$$

where $(\oplus)$ denotes the *n-ary circle hope* on the hyperaddition, i.e. the sum of products of elements of the H_v-ring is the union of the sets obtained with all possible parentheses put on them. The hyperproduct is not WASS.

Definitions 1.8. Let $(H, .)$ be H_v-group, consider an H_v-ring or H_v-field, $(R, +, .)$ and a set

$$\mathbf{M}_R = \{(a_{ij}) \mid a_{ij} \in R\},$$

then is called the *H_v-matrix representation*, any map

$$\mathbf{T} : H \longrightarrow M_R, h \longrightarrow \mathbf{T}(h) \text{ such that } \mathbf{T}(h_1 h_2) \cap \mathbf{T}(h_1)\mathbf{T}(h_2) \neq \emptyset, \ \forall h_1, h_2 \in H.$$

If $\mathbf{T}(h_1 h_2) \subset \mathbf{T}(h_1)\mathbf{T}(h_2), \forall h_1, h_2 \in H$, then $\mathbf{T}$ is an *inclusion representation*, if

$$\mathbf{T}(h_1 h_2) = \mathbf{T}(h_1)\mathbf{T}(h_2) = \{\mathbf{T}(h) \mid h \in h_1 h_2\}, \forall h_1, h_2 \in H,$$

then $\mathbf{T}$ is a *good representation*.

The main theorem of the theory of representations is:

Theorem. *A necessary condition in order to have an inclusion representation $\boldsymbol{T}$ of the H_v-group $(H, .)$ by $n \times n$ H_v-matrices over the H_v-ring $(R, +, .)$ is the following:*

For all classes $\beta^*(a), a \in H$ there must exist elements $a_{ij} \in R, i, j \in \{1, ..., n\}$ such that

$$\boldsymbol{T}(\beta^*(a)) \subset \left\{ \boldsymbol{A} = (a'_{ij}) \mid a'_{ij} \in \gamma^*(a_{ij}), i, j \in \{1, ..., n\} \right\}.$$

2. The theta ∂-hopes

We present now a large class of hopes defined in any groupoid with a map f on it, which is denoted by 'theta' ∂, since the motivation is the property which the derivative has on the product of functions, see [15], [16], [17].

Definitions 2.1. Let $(G, .)$ be groupoid (respectively, hypergroupoid) and $f : G \longrightarrow G$ be a map. We define a hope ∂, called *theta-hope*, on G as follows

$$x \partial y = \{f(x).y, x.f(y)\}, \forall x, y \in G. (resp.\ \ x \partial y = (f(x).y) \cup (x.f(y)), \forall x, y \in G)$$

If (.) is commutative, then (∂) is commutative. If (.) is *COW*, then (∂) is *COW*.

Let $(G, .)$ be groupoid (resp. hypergroupoid) and $f : G \longrightarrow \mathbf{P}(G) - \{\emptyset\}$, multivalued map. We define the *theta-hope* (∂), on G as follows

$$x\partial y = (f(x).y) \cup (x.f(y)), \forall x, y \in G$$

Let $(G, .)$ be a groupoid and $f_i : G \longrightarrow G, i \in I$, be set of maps on G. We consider the map $f_\cup : G \longrightarrow \mathbf{P}(G)$ such that $f_\cup(x) = \{f_i(x)|i \in I\}$, called *the union* of the $f_i(x)$. We define the *union theta-hrope* (∂), on G if we consider the $f_\cup(x)$. A special case for given f, is to take the union with the identity: We consider the map $\underline{f} \equiv f \cup (id)$, so $\underline{f}(x) = \{x, f(x)\}, \forall x \in G$, which we call *b-theta-hope*. Then we have

$$x\partial y = \{xy, f(x).y, x.f(y)\}, \forall x, y \in G.$$

Motivation for the definition of the theta-hope is the map *derivative* where only the multiplication of functions can be used. Therefore, in these terms, for any functions $s(x), t(x)$, we have $s\partial t = \{s't, st'\}$ where $(')$ denotes the derivative.

Properties 2.2. [15] If $(G, .)$ is a semigroup, then:

(a) For every f, the (∂) is *WASS*. If f is homomorphism then (∂) remains *WASS*.

(b) If f is homomorphism and projection, i.e., $f^2 = f$, then (∂) is associative.

(c) If $(G, .)$ is a semigroup then, for every f, the b-theta-operation (∂) is *WASS*.

(d) *Reproductivity.* If (.) is reproductive then $(\partial)$is also reproductive, because

$$x\partial G = \bigcup_{g \in G} \{f(x).g, x.f(g)\} = G \ \textit{and} \ G\partial x = \bigcup_{g \in G} \{f(g).x, g.f(x)\} = G$$

(e) *Commutativity.* If (.) is commutative, then (∂) is commutative. If f is into the center of G, then (∂) is a commutative. If (.) is a COW then, (∂) is a COW.

(f) *Unit elements.* u is a right unit element if $x \in x\partial u = \{f(x).u, x.f(u)\}$. So $f(u) = e$, where e be a unit in $(G, .)$. The elements of the kernel of f, are the units of (G, ∂).

(g) *Inverse elements.* Let $(G, .)$ is a monoid with unit e and u be a unit in (G, ∂), then $f(u) = e$. For given x, the element $x' = (f(x))^{-1}u$ and $x'' = u(f(x))^{-1}$, are the right and left inverses, respectively. We have two-sided inverses iff $f(x)u = uf(x)$.

Definitions 2.3. Let $(R,+,.)$ be a ring and $f : R \longrightarrow R, g : R \longrightarrow R$ be two maps. We define two hopes (∂_+) and $(\partial_.)$, called both *theta-hopes*, on R as follows

$$x\partial_+ y = \{f(x)+y, x+f(y)\} \ \text{and} \ x\partial_. y = \{g(x)y, xg(y)\}, \forall x, y \in G$$

The hyperstructure $(R, \partial_+, \partial_.)$, called *theta*, is an H_v-near-ring, i.e., satisfy all H_v-ring axioms, except the weak distributivity.

Some results and examples:

Let $(G,.)$ be group and $f(x) = a$, a constant map on G. Then $(G, \partial)/\beta^*$ is singleton. If $f(x) = e$, then $x\partial y = \{x, y\}$ which is the smallest incidence hope.

Consider all polynomials of first degree $g_i(x) = a_i x + b_i$, and as map the derivative, we have

$$g_1 \partial g_2 = \{a_1 a_2 x + a_1 b_2, a_1 a_2 x + b_1 a_2\},$$

so it is a hope inside the set of first degree polynomials. Moreover all polynomials $x + c$, where c be a constant, are units.

Several results can be obtained using ∂-hopes [15]:

Theorems 2.4.

(a) Consider the group of integers $(\mathbb{Z},+)$ and $n \neq 0$ be a natural number. Take the map f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbb{Z} - \{0\}$. Then

$$(\mathbb{Z}, \partial)/\beta^* \cong (\mathbb{Z}_n, +).$$

(b) Take the ring of integers $(\mathbb{Z},+,.)$ and fix $n \neq 0$ a natural number. Consider f such that $f(0) = n$ and $f(x) = x, \forall x \in \mathbb{Z} - \{0\}$. Then $(\mathbb{Z}, \partial_+, \partial_.)$, where ∂_+ and $\partial_.$ are the ∂-hopes refereed to the addition and the multiplication respectively, is an H_v-near-ring, with

$$(\mathbb{Z}, \partial_+, \partial_.)/\gamma^* \cong (\mathbb{Z}_n).$$

(c) Take the $(\mathbb{Z},+,.)$ and $n \neq 0$ a natural. Take f such that $f(n) = 0$ and $f(x) = x, \forall x \in \mathbb{Z} - \{n\}$. Then $(\mathbb{Z}, \partial_+, \partial_.)$ is an H_v-ring, moreover,

$$(\mathbb{Z}, \partial_+, \partial_.)/\gamma^* \cong (\mathbb{Z}_n).$$

Special case of the above is for $n = p$, prime, then $(\mathbb{Z}, \partial_+, \partial_.)$ is an H_v-field.

3. Uniting elements

The *uniting elements* method was introduced by Corsini-Vougiouklis [5] in 1989. With this method one puts in the same class, two or more elements. This leads, through hyperstructures, to structures satisfying additional properties.

Definition 3.1. The *Uniting Elements Method* is described as follows: Let G be an algebraic structure and let d be a property, which is not valid. Suppose that d is described by a set of equations; then, consider the partition in G for which it is put together, in the same partition class, every pair of elements that causes the non-validity of the property d. The quotient by this partition G/d is an H_v-structure. Then, quotient out the H_v-structure G/d by the fundamental relation β^*, a stricter structure $(G/d)\beta^*$ for which the property d is valid, is obtained.

The merging procedure is in fact the first step of the Uniting Elements Method, i.e., we do not take the quotient of the obtaining hyperstructure by the fundamental relation. From the definition of merging elements we obtain the following. If $\underline{h}$ merges with the element h in a groupoid $(H, .)$, then in merged hyperstructure (H, o) we have $\forall x \in H$,

$$\underline{h}ox = \{\underline{h}.x, h.x\}, \;\; xo\underline{h} = \{x.\underline{h}, x.h\}, \;\; \underline{h}o\underline{h} = \{\underline{h}.\underline{h}, \underline{h}.h, h.\underline{h}, h.h\}$$

and if $(H, .)$ is a hypergroupoid, we have $\forall x \in H$,

$$\underline{h}ox = (\underline{h}.x) \cup (h.x), \;\; xo\underline{h} = (x.\underline{h}) \cup (x.h), \;\; \underline{h}o\underline{h} = (\underline{h}.\underline{h}) \cup (\underline{h}.h) \cup (h.\underline{h}) \cup (h.h)$$

Moreover if (.) is *WASS*, *COW*, reproductive then the merge construction remains *WASS*, *COW*, reproductive, respectively.

An interesting application of the Uniting Elements Method is when more, than one, properties are desired. The reason for this is that some of the properties lead straighter to the classes than others. Therefore, it is better to apply the straightforward classes followed by the more complicated ones. The commutativity is one of the easy applicable properties. Moreover it is clear that the reproductivity property is also easily applicable. One can do this because the following is valid.

Theorem 3.2. [11] *Let $(G, .)$ be a groupoid, and*

$$F = \{f_1, ..., f_m, f_{m+1}, ..., f_{m+n}\}$$

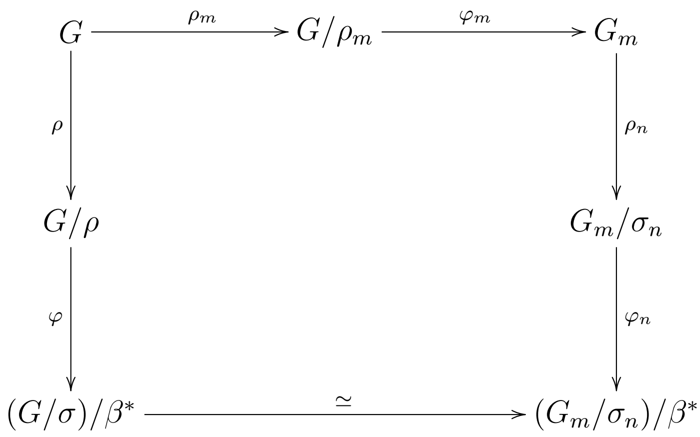
be a system of equations on G consisting of two subsystems

$$F_m = \{f_1, ..., f_m\} \text{ and } F_n = \{f_{m+1}, ..., f_{m+n}\}.$$

Let σ, σ_m be the equivalence relations defined by the Uniting Elements procedure using the systems F and F_m respectively, and let σ_n be the equivalence relation defined using the induced equations of F_n on the groupoid $G_m = (G/\sigma_m)/\beta^*$. Then

$$(G/\sigma)/\beta^* \cong (G_m/\sigma_n)/\beta^*,$$

i.e., the following diagram is commutative



From the above it is clear that the fundamental structure is very important, and even more so if it is known from the beginning. This is the problem to construct hyperstructures with desired fundamental structures [14], [16].

Now, we introduce the following generalization:

Generalization of the Uniting Elements Method. Using this basic Theorem we remark that it is not necessary to apply the Uniting Elements Method to obtain a total property, but to obtain a part of a property and then in next steps to obtain the property. Therefore, one can enlarge some operations or hopes, and even more only some results, by putting together elements which cause the non-validity of the property. Thus, we can enlarge the classes to obtain desired properties.

An opposite problem, to the Uniting Elements Method, is to 'enlarge' given hyperstructures [14]. In representation theory enlargement or reduction, are useful if the H_v-structures have the same fundamental structure.

4. Applications

The applications of the Uniting Elements Method are referred mainly on the classical algebraic structures, however we can apply the method on hyperstructures as well. In the following, we present some special and general applications of the Uniting Elements Method and, moreover, we connect them with ∂-hopes.

Let now present a special example on the generalization of the Uniting Elements Method, where we enlarge only one product:

Example 4.1. Consider the non-commutative quaternion group

$$Q = \{1, -1, i, -i, j, -j, k, -k\}$$

with defining relations:

$$i^2 = j^2 = k^2 = -1, ij = k, jk = i, ki = j.$$

Denote

$$\underline{i} = \{i, -i\}, \underline{j} = \{j, -j\}, \underline{k} = \{k, -k\}, \underline{1} = \{1, -1\}.$$

Suppose that we enlarge only the product $ij = k$ as follows $ij = \{k, -k\} = \underline{k}$. Then we have $i(ij) = i\underline{k} = \underline{j}$, $j(ij) = j\underline{k} = \underline{i}$, $k(ij) = k\underline{k} = \underline{1}$. Therefore, we have four fundamental equivalent classes, the $\underline{1}, \underline{i}, \underline{j}, \underline{k}$. The Cayley table is

*	$\underline{\mathbf{1}}$	$\underline{\mathbf{i}}$	$\underline{\mathbf{j}}$	$\underline{\mathbf{k}}$
$\underline{\mathbf{1}}$	$\underline{\mathbf{1}}$	$\underline{\mathbf{i}}$	$\underline{\mathbf{j}}$	$\underline{\mathbf{k}}$
$\underline{\mathbf{i}}$	$\underline{\mathbf{i}}$	$\underline{\mathbf{1}}$	$\underline{\mathbf{k}}$	$\underline{\mathbf{j}}$
$\underline{\mathbf{j}}$	$\underline{\mathbf{j}}$	$\underline{\mathbf{k}}$	$\underline{\mathbf{1}}$	$\underline{\mathbf{i}}$
$\underline{\mathbf{k}}$	$\underline{\mathbf{k}}$	$\underline{\mathbf{j}}$	$\underline{\mathbf{i}}$	$\underline{\mathbf{1}}$

Therefore, the fundamental group becomes isomorphic to the group $(\mathbb{Z}_2 \times \mathbb{Z}_2, +)$.

Basic general results on the topic are the following (see [5], [11]):

Definition. Denote U the set of all finite products of elements of a hypergroupoid $(H, .)$. Consider the relation L defined as follows:

$$xLy \iff \text{there exists } u \in U \text{ such that } ux \cap uy \neq \emptyset.$$

Then the transitive closure L^* of L is called *left fundamental reproductivity relation*. Similarly, *the right fundamental reproductivity relation* R^* is defined.

Theorem 4.2. *If $(H, .)$ is a commutative semihypergroup, i.e., the strong commutativity and the strong associativity is valid, then the strong expression of the above $L : ux = uy$, has the property: $L^* = L$.*

Theorem 4.3. *Let $(S, .)$ be commutative semigroup with one element $w \in S$ such that the set wS is finite. Consider the transitive closure L^* of the relation L defined as follows:*

$$xLy \iff \textit{there exists } z \in S \textit{ such that } zx = zy.$$

Then $\langle S/L^*, \circ \rangle / \beta^*$ is a finite commutative group, where $(\circ)$ is the induced operation on classes of S/L^*.

Proof. For the proofs one can follow [5] and [11]. ∎

An open problem is to prove that L^*, is the smallest equivalence relation such that H/L^*, is reproductive.

Theorem 4.4. [16] *Let $(H, \otimes)$ be an H_b-semigroup where the basic semigroup $(H, .)$ is commutative and which has at least one element $w \in H$ such that the set $w.H$ is finite. Then $(H/L^*, \otimes)$, where L is the relation: xLy iff there exists $z \in H$ such that $zx = zy$ and $(\otimes)$ is the induced hyperoperation on classes, i.e., L is defined with respect to $(.)$, is finite commutative h/v-group.*

Remarks 4.5. We can see the results of Theorem 2.4, in terms of the generalized Uniting Elements Method, especially in the cases (a) and (c) where strict fundamental structures are obtained. Therefore, in the case (a), since we have, for the given map f,

$$0\partial 0 = n, \;\; 0\partial x = \{n+x, x\}, \forall x \in \mathbb{Z} - \{0\}, \;\; x\partial y = x+y, \forall x, y \in \mathbb{Z} - \{0\}$$

we remark that we only enlarge the sums $0\partial x, \forall x \in \mathbb{Z} - \{0\}$, by adding in the results the element $n+x$.

In the case (c), since we have, for the given map f,

$$n\partial_+ n = n, \;\; n\partial_+ x = \{x, n+x\}, \; \forall x \in \mathbb{Z} - \{n\}, \;\; x\partial_+ y = x+y, \; \forall x \in \mathbb{Z} - \{n\}$$

and

$$n\partial_. n = 0, \;\; n\partial_. x = \{0, nx\}, \; \forall x \in \mathbb{Z} - \{n\}, \;\; x\partial_. y = xy, \; \forall x \in \mathbb{Z} - \{n\}$$

we remark that we only enlarge the sums $n\partial_+ x, \forall x \in \mathbb{Z} - \{n\}$, by adding in the results the element x, and enlarge the products $n\partial_. x, \forall x \in \mathbb{Z} - \{n\}$, by adding the element 0.

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