INTERVAL-VALUED FUZZY HYPERGRAPH AND INTERVAL-VALUED FUZZY HYPEROPERATIONS[1]

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Abstract. We first construct two interval-valued fuzzy hyperoperations by the use of $[\alpha, \beta]$-cut on an interval-valued $[\alpha, \beta]$-fuzzy hypergraph and list some properties of them. Then we construct another two interval-valued fuzzy hyperoperations by the use of $\langle\alpha, \beta\rangle$-cut on an interval-valued $\langle\alpha, \beta\rangle$-fuzzy hypergraph and also list some properties of them. Lastly, we study the interval-valued fuzzy hyperstructure associated to the latter two interval-valued fuzzy hyperoperations and we can see that the hyperstructure is almost an interval-valued superlattice.

Keywords: interval-valued fuzzy hyperoperation; interval-valued fuzzy hypergraph; quasi-hypergroup; superlattice.

1. Introduction and preliminaries

The connections between graphs and hyperoperations had been looked into by several researchers (see for instance [3], [8]). Corsini [4] and Ali [1] studied the connections between hypergraphs and hyperoperations. Feng used fuzzy hypergraphs to construct fuzzy hyperoperations in [6], [7].

In this paper, we first construct two interval-valued fuzzy hyperoperations by the use of $[\alpha, \beta]$-cut on an interval-valued $[\alpha, \beta]$-fuzzy hypergraph and list some properties of them. Then we construct another two interval-valued fuzzy hyperoperations by the use of $\langle\alpha, \beta\rangle$-cut on an interval-valued $\langle\alpha, \beta\rangle$-fuzzy hypergraph

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and also list some properties of them. Lastly, we study the interval-valued fuzzy hyperstructure associated to the latter two interval-valued fuzzy hyperoperations and we can see that the hyperstructure is almost an interval-valued superlattice.

We recall some notations of interval-valued fuzzy hyperstructure theory.

All intervals in $[0,1]$ is denoted by $I([0,1])$.

Definition 1.1. For all $[a,b],[x,y] \in I([0,1])$, we define

$$\begin{aligned} [a,b] \vee [x,y] &\doteq [a \vee x, b \vee y]. \\ [a,b] \wedge [x,y] &\doteq [a \wedge x, b \wedge y]. \\ [a,b]' &= [1-b, 1-a]. \end{aligned}$$

Proposition 1.2. $\forall A,B,C \in I([0,1])$, *we have the following properties*

(1) $A \vee A = A, A \wedge A = A$;

(2) $A \vee B = B \vee A, A \wedge B = B \wedge A$;

(3) $(A \vee B) \vee C = A \vee (B \vee C), (A \wedge B) \wedge C = A \wedge (B \wedge C)$;

(4) $A \wedge (A \vee B) = A, A \vee (A \wedge B) = A$;

(5) $(A \vee B) \wedge C = (A \wedge C) \vee (B \wedge C), (A \wedge B) \vee C = (A \vee C) \wedge (B \vee C)$;

(6) $A \vee [0,0] = A, A \wedge [0,0] = [0,0], A \vee [1,1] = [1,1], A \wedge [1,1] = A$;

(7) $(A')' = A$;

(8) $(A \vee B)' = A' \wedge B', (A \wedge B)' = A' \vee B'$.

Proof. A straightforward verification. ■

Definition 1.3. $\forall [a,b],[x,y] \in I([0,1])$, we define

$$[a,b] \le [x,y] \doteq a \le x, b \le y.$$

It is easy to see that $\langle I([0,1]); \le \rangle$ is a partial ordered set.

An interval-valued fuzzy subset of a nonempty set H is a function $M : H \to I([0,1])$. The collection of all interval-valued fuzzy subsets of H is denoted by $IF(H)$.

Given $A,B \in IF(H)$, we give the following definitions

$$\begin{aligned} A \subseteq B &\doteq A(x) \le B(x), & \forall x \in H. \\ A = B &\doteq A(x) = B(x), & \forall x \in H. \\ (A \cup B)(x) &\doteq A(x) \vee B(x), & \forall x \in H. \\ (A \cap B)(x) &\doteq A(x) \wedge B(x), & \forall x \in H. \\ A'(x) &\doteq (A(x))', & \forall x \in H. \end{aligned}$$

Note that $\emptyset$ and H can be seen as interval-valued fuzzy subsets of H and we set $\emptyset(x) = [0,0]$, $H(x) = [1,1]$, $\forall x \in H$.

Proposition 1.4. $\forall A, B, C \in IF(H)$, *we have the following properties*

(1) $A \cup A = A,\ A \cap A = A$;

(2) $A \cup B = B \cup A,\ A \cap B = B \cap A$;

(3) $(A \cup B) \cup C = A \cup (B \cup C),\ (A \cap B) \cap C = A \cap (B \cap C)$;

(4) $A \cap (A \cup B) = A,\ A \cup (A \cap B) = A$;

(5) $(A \cup B) \cap C = (A \cap C) \cup (B \cap C),\ (A \cap B) \cup C = (A \cup C) \cap (B \cup C)$;

(6) $A \cup \emptyset = A, A \cap \emptyset = \emptyset,\ A \cup H = H, A \cap H = A$.

(7) $(A')' = A$;

(8) $(A \cup B)' = A' \cap B',\ (A \cap B)' = A' \cup B'$.

Proof. A straightforward verification. ∎

An interval-valued fuzzy hypergroupoid $\langle H; * \rangle$ is a nonempty set H endowed with an interval-valued fuzzy hyperoperation (i.e., a function $*$ from $H \times H$ to $IF(H)$).

Given an interval-valued fuzzy hyperoperation $* : H \times H \to IF(H)$, for all $a \in H$, $B \in IF(H)$, the interval-valued fuzzy subset $a * B$ of H is defined by

$$(a * B)(x) \doteq \vee_{B(b) \subseteq (0,1]} (a * b)(x).$$

The readers can consult [2], [5], [9] to learn more about hyperstructures and fuzzy sets.

2. Interval-valued fuzzy hyperoperations obtained from $[\alpha, \beta]$-cut

The $[\alpha, \beta]$-cut of a fuzzy subset M of H is defined by

$$M_{[\alpha,\beta]} \doteq \{x \in H | M(x) \subseteq [\alpha, \beta]\}.$$

Definition 2.1. H is a nonempty set, $\{A_i\}_i$ is a family of interval-valued fuzzy subsets of H, if there exists $[\alpha, \beta] \in I([0,1])$ with $\alpha \neq 0$ such that

$$\cup_i (A_i)_{[\alpha,\beta]} = H,$$

then $\langle H; \{A_i\}_i \rangle$ is called an interval-valued $[\alpha, \beta]$-fuzzy hypergraph.

2.1. The interval-valued fuzzy hyperoperation $\overline{*}_{\cup}$

In this section, we use $[\alpha, \beta]$-cut to construct an interval-valued fuzzy hyperoperation $\overline{*}_{\cup}$.

Definition 2.2. Let $\Gamma = \langle H; \{A_i\}_i \rangle$ be an interval-valued $[\alpha, \beta]$-fuzzy hypergraph, set $E_{[\alpha,\beta]}(x) = \cup_{A_i(x)\subseteq[\alpha,\beta]} A_i$. The interval-valued fuzzy hypergroupoid $H_\Gamma = \langle H; \overline{*}_\cup \rangle$ where the interval-valued fuzzy hyperoperation $\overline{*}_\cup$ is defined by

$$x \overline{*}_\cup y = E_{[\alpha,\beta]}(x) \cup E_{[\alpha,\beta]}(y), \quad \forall x, y \in H$$

is called an interval-valued $[\alpha, \beta]$-fuzzy hypergraph hypergroupoid.

Proposition 2.3. *The interval-valued $[\alpha, \beta]$-fuzzy hypergraph hypergroupoid H_Γ has the following properties for any $x, y \in H$:*

(1) $x \overline{*}_\cup y = x \overline{*}_\cup x \cup y \overline{*}_\cup y$;

(2) $x \in (x \overline{*}_\cup x)_{[\alpha,\beta]}$.

Proof. (1) $x \overline{*}_\cup y = E_{[\alpha,\beta]}(x) \cup E_{[\alpha,\beta]}(y) = (E_{[\alpha,\beta]}(x) \cup E_{[\alpha,\beta]}(x)) \cup (E_{[\alpha,\beta]}(y) \cup E_{[\alpha,\beta]}(y)) = x \overline{*}_\cup x \cup y \overline{*}_\cup y$.
(2) $(x \overline{*}_\cup x)(x) = (E_{[\alpha,\beta]}(x))(x) = (\cup_{A_i(x)\subseteq[\alpha,\beta]} A_i)(x) = \vee_{A_i(x)\subseteq[\alpha,\beta]} A_i(x) \subseteq [\alpha, \beta]$. ∎

Remark 2.4. $\forall x, y \in H, \quad y \in (x \overline{*}_\cup x)_{[\alpha,\beta]} \not\Rightarrow x \in (y \overline{*}_\cup y)_{[\alpha,\beta]}$.

For example, $\Gamma = \langle \{a, b\}; \{A_1, A_2, A_3\} \rangle$, where

$$\begin{aligned} A_1 &= \frac{[0.2, 0.25]}{a} + \frac{[0.1, 0.3]}{b}, \\ A_2 &= \frac{[0.2, 0.3]}{a} + \frac{[0.25, 0.3]}{b}, \\ A_3 &= \frac{[0.4, 0.5]}{a} + \frac{[0.2, 0.3]}{b}. \end{aligned}$$

Then $(A_1)_{[0.2,0.3]} = \{a\}, (A_2)_{[0.2,0.3]} = \{a, b\}, (A_3)_{[0.2,0.3]} = \{b\}$, and so, Γ is an interval-valued $[0.2, 0.3]$-fuzzy hypergraph since $\cup_{i=1}^{i=3} (A_i)_{[0.2,0.3]} = \{a, b\}$.

$b \in (a \overline{*}_\cup a)_{[0.2,0.3]}$ since $(a \overline{*}_\cup a)(b) = A_1(b) \vee A_2(b) = [0.1, 0.3] \vee [0.25, 0.3] = [0.25, 0.3] \subseteq [0.2, 0.3]$.

But $a \overline{\in} (b \overline{*}_\cup b)_{[0.2,0.3]}$ since $(b \overline{*}_\cup b)(a) = A_2(a) \vee A_3(a) = [0.2, 0.3] \vee [0.4, 0.5] = [0.4, 0.5] \not\subseteq [0.2, 0.3]$.

Lemma 2.5. *For all $[u, v] \subseteq [\alpha, \beta] \subseteq [0, 1]$ and $[x, y] \subseteq [0, 1]$ we have*

$$[u, v] \vee [x, y] \subseteq [\alpha, 1].$$

Proof. From $[u, v] \vee [x, y] = [u \vee x, v \vee y]$, $\alpha \leq v \vee x$ and $b \vee y \leq 1$ we can easily see that the conclusion is correct. ∎

Proposition 2.6. *If an interval-valued hyperstructure $\langle H; \overline{*}_\cup \rangle$ satisfying* (1) *and* (2) *of the previous proposition, then it also satisfies*

(3) $\{x, y\} \subseteq (x \overline{*}_\cup y)_{[\alpha,1]}$;

(4) $x \overline{*}_\cup y = y \overline{*}_\cup x$;

(5) $(x\overline{*}_\cup H)_{[\alpha,1]} = H$;

(6) $\langle H; \{x\overline{*}_\cup x\}_{x\in H}\rangle$ *is an interval-valued* $[\alpha,\beta]$*-fuzzy hypergraph;*

(7) $x\overline{*}_\cup x\overline{*}_\cup x = \cup_{(x\overline{*}_\cup x)(z)\subseteq(0,1]} z\overline{*}_\cup z$;

(8) $(x\overline{*}_\cup x)\overline{*}_\cup(x\overline{*}_\cup x) = x\overline{*}_\cup x\overline{*}_\cup x$.

Proof. Let us use (1), (2) of the previous proposition to prove (3)-(8).

(3) $(x\overline{*}_\cup y)(x) = (x\overline{*}_\cup x \cup y\overline{*}_\cup y)(x) = (x\overline{*}_\cup x)(x) \vee (y\overline{*}_\cup y)(x) \subseteq [\alpha, 1]$.
So $x \in (x\overline{*}_\cup y)_{[\alpha,1]}$. Similarly we can prove $y \in (x\overline{*}_\cup y)_{[\alpha,1]}$.

(4) $x\overline{*}_\cup y = x\overline{*}_\cup x \cup y\overline{*}_\cup y = y\overline{*}_\cup y \cup x\overline{*}_\cup x = y\overline{*}_\cup x$.

(5) for any $y \in H$, $(x\overline{*}_\cup H)(y) = (\cup_{t\in H} x\overline{*}_\cup t)(y) = (\cup_{t\in H}(x\overline{*}_\cup x \cup t\overline{*}_\cup t))(y) = ((y\overline{*}_\cup y) \cup (\cup_{t\in H}(x\overline{*}_\cup x \cup t\overline{*}_\cup t)))(y) = (y\overline{*}_\cup y)(y) \vee (\cup_{t\in H} x\overline{*}_\cup t)(y) \subseteq [\alpha, 1]$ and thus $y \in (x\overline{*}_\cup H)_{[\alpha,1]}$.

So, $H \subseteq (x\overline{*}_\cup H)_{[\alpha,1]}$ and, finally, $H = (x\overline{*}_\cup H)_{[\alpha,1]}$.

(6) From $x \in (x\overline{*}_\cup x)_{[\alpha,\beta]}$ we know $\cup_{x\in H}(x\overline{*}_\cup x)_{[\alpha,\beta]} = H$.
And then $< H; \{x\overline{*}_\cup x\}_{x\in H} >$ is an interval-valued $[\alpha, \beta]$-fuzzy hypergraph.

(7) $x\overline{*}_\cup x\overline{*}_\cup x = \cup_{(x\overline{*}_\cup x)(z)\subseteq(0,1]} z\overline{*}_\cup x = \cup_{(x\overline{*}_\cup x)(z)\subseteq(0,1]}(z\overline{*}_\cup z) \cup (x\overline{*}_\cup x)$
$= \cup_{(x\overline{*}_\cup x)(z)\subseteq(0,1]} z\overline{*}_\cup z$.

(8) $(x\overline{*}_\cup x)\overline{*}_\cup(x\overline{*}_\cup x) = \cup_{(x\overline{*}_\cup x)(a)\subseteq(0,1],(x\overline{*}_\cup x)(b)\subseteq(0,1]} a\overline{*}_\cup b$
$= \cup_{(x\overline{*}_\cup x)(a)\subseteq(0,1],(x\overline{*}_\cup x)(b)\subseteq(0,1]}(a\overline{*}_\cup a \cup b\overline{*}_\cup b) = \cup_{(x\overline{*}_\cup x)(a)\subseteq(0,1]} a\overline{*}_\cup a = x\overline{*}_\cup x\overline{*}_\cup x$. ■

Remark 2.7. An interval-valued $[\alpha, \beta]$-fuzzy quasi-hypergroup is an interval-valued fuzzy hypergroupoid such that

$$(x * H)_{[\alpha,\beta]} = H = (H * x)_{[\alpha,\beta]}, \quad \forall x \in H.$$

From (4) and (5) of the previous we know that H_Γ is a commutative interval-valued $[\alpha, \beta]$-fuzzy quasi-hypergroup if $\beta = 1$.

2.2. The interval-valued fuzzy hyperoperation $\overline{*}_\cap$

In this section, we use $[\alpha, \beta]$-cut to construct an interval-valued fuzzy hyperoperation $\overline{*}_\cap$.

Definition 2.8. Let $\Gamma = \langle H; \{A_i\}_i\rangle$ be an interval-valued $[\alpha, \beta]$-fuzzy hypergraph, set $D_{[\alpha,\beta]}(x) = \cap_{A_i(x)\subseteq[\alpha,\beta]} A_i$. The fuzzy hyperoperation $\overline{*}_\cap$ is defined by

$$x\overline{*}_\cap y \doteq D_{[\alpha,\beta]}(x) \cup D_{[\alpha,\beta]}(y), \quad \forall x, y \in H.$$

Proposition 2.9. *For all* $x, y \in H$, *we have:*

(1) $x\overline{*}_\cap y = x\overline{*}_\cap x \cup y\overline{*}_\cap y$;

(2) $x \in (x\overline{*}_\cap x)_{[\alpha,\beta]}$.

Proof. (1) $x\overline{*}_\cap y = D_{[\alpha,\beta]}(x) \cup D_{[\alpha,\beta]}(y) = (D_{[\alpha,\beta]}(x) \cup D_{[\alpha,\beta]}(x)) \cup (D_{[\alpha,\beta]}(y) \cup D_{[\alpha,\beta]}(y)) = x\overline{*}_\cap x \cup y\overline{*}_\cap y$.

(2) $(x\overline{*}_\cap x)(x) = (D_{[\alpha,\beta]}(x))(x) = \wedge_{A_i(x)\subseteq[\alpha,\beta]} A_i(x) \subseteq [\alpha,\beta]$. ■

Proposition 2.10. $\langle H; \overline{*}_\cap\rangle$ *also has the following properties.*

(3) $\{x,y\} \subseteq (x\overline{*}_\cap y)_{[\alpha,1]}$;

(4) $x\overline{*}_\cap y = y\overline{*}_\cap x$;

(5) $(x\overline{*}_\cap H)_{[\alpha,1]} = H$;

(6) $\langle H; \{x\overline{*}_\cap x\}_{x\in H}\rangle$ *is an interval-valued* $[\alpha,\beta]$*-fuzzy hypergraph;*

(7) $x\overline{*}_\cap x\overline{*}_\cap x = \cup_{(x\overline{*}_\cap x)(z)\subseteq(0,1]} z\overline{*}_\cap z$;

(8) $(x\overline{*}_\cap x)\overline{*}_\cap(x\overline{*}_\cap x) = x\overline{*}_\cap x\overline{*}_\cap x$.

Proof. Applied Proposition 2.6. ■

Remark 2.11. From (4), (5) of the previous proposition we know that $\langle H; \overline{*}_\cap\rangle$ is also a commutative interval-valued $[\alpha,\beta]$-fuzzy quasi-hypergroup if $\beta = 1$.

Remark 2.12. $\forall x, y \in H$,

$$y \in (x\overline{*}_\cap x)_{[\alpha,\beta]} \not\Rightarrow x \in (y\overline{*}_\cap y)_{[\alpha,\beta]}.$$

For example,

$$\Gamma = \langle \{a,b\}; \{A_1, A_2, A_3\}\rangle\,,$$

where

$$\begin{aligned} A_1 &= \frac{[0.2,0.25]}{a} + \frac{[0.2,0.4]}{b}, \\ A_2 &= \frac{[0.2,0.3]}{a} + \frac{[0.2,0.25]}{b}, \\ A_3 &= \frac{[0.1,0.2]}{a} + \frac{[0.2,0.3]}{b}. \end{aligned}$$

Then

$$(A_1)_{[0.2,0.3]} = \{a\},\ (A_2)_{[0.2,0.3]} = \{a,b\},\ (A_3)_{[0.2,0.3]} = \{b\},$$

and so Γ is an interval-valued $[0.2,0.3]$-fuzzy hypergraph since

$$\cup_{i=1}^{i=3}(A_i)_{[0.2,0.3]} = \{a,b\}.$$

$b \in (a\overline{*}_\cup a)_{[0.2,0.3]}$ since

$$(a\overline{*}_\cap a)(b) = A_1(b) \wedge A_2(b) = [0.2,0.4] \wedge [0.2,0.25] = [0.2,0.25] \subseteq [0.2,0.3].$$

But $a\overline{\in}(b\overline{*}_\cup b)_{[0.2,0.3]}$ since

$$(b\overline{*}_\cup b)(a) = A_2(a) \wedge A_3(a) = [0.2,0.3] \wedge [0.1,0.2] = [0.1,0.2] \not\subseteq [0.2,0.3].$$

3. Interval-valued fuzzy hyperoperations obtained from $\langle\alpha,\beta\rangle$-cut

Definition 3.1. M is an interval-valued fuzzy subset of H, set

$$M(x)=[M^-(x),M^+(x)],\ \forall x\in H.$$

$M^-:H\to[0,1]$ and $M^+:H\to[0,1]$ are called lower fuzzy set and upper fuzzy set about M.

The $\langle\alpha,\beta\rangle$-cut of an interval-valued fuzzy subset M of H is defined by

$$M_{\langle\alpha,\beta\rangle}\doteq\{x\in H\mid M^-(x)\geq\alpha,M^+(x)\geq\beta\}.$$

Definition 3.2. H is a nonempty set, $\{A_i\}_i$ is a family of interval-valued fuzzy subsets of H, if there exists $\alpha,\beta\in(0,1]$ such that

$$\cup_i(A_i)_{\langle\alpha,\beta\rangle}=H,$$

then $\langle H;\{A_i\}_i\rangle$ is called an interval-valued $\langle\alpha,\beta\rangle$-fuzzy hypergraph.

3.1. The interval-valued fuzzy hyperoperation $\overline{\circ}_\cup$

In this section, we use $<\alpha,\beta>$-cut to construct an interval-valued fuzzy hyperoperation $\overline{\circ}_\cup$.

Definition 3.3. Let $\Gamma=\langle H;\{A_i\}_i\rangle$ be an interval-valued $\langle\alpha,\beta\rangle$-fuzzy hypergraph, set $E_{\langle\alpha,\beta\rangle}(x)=\cup_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i$. The interval-valued fuzzy hypergroupoid $H_\Gamma=\langle H;\overline{\circ}_\cup\rangle$ where the interval-valued fuzzy hyperoperation $\overline{\circ}_\cup$ is defined by

$$x\overline{\circ}_\cup y=E_{\langle\alpha,\beta\rangle}(x)\cup E_{\langle\alpha,\beta\rangle}(y),\quad\forall x,y\in H$$

is called an interval-valued $<\alpha,\beta>$-fuzzy hypergraph hypergroupoid.

Proposition 3.4. *The interval-valued $\langle\alpha,\beta\rangle$-fuzzy hypergraph hypergroupoid H_Γ has the following properties for any $x,y\in H$:*

(1) $x\overline{\circ}_\cup y=x\overline{\circ}_\cup x\cup y\overline{\circ}_\cup y$;

(2) $x\in(x\overline{\circ}_\cup x)_{\langle\alpha,\beta\rangle}$.

Proof. (1) $x\overline{\circ}_\cup y=E_{\langle\alpha,\beta\rangle}(x)\cup E_{\langle\alpha,\beta\rangle}(y)=(E_{\langle\alpha,\beta\rangle}(x)\cup E_{\langle\alpha,\beta\rangle}(x))\cup(E_{[\alpha,\beta]}(y)\cup E_{\langle\alpha,\beta\rangle}(y))=x\overline{\circ}_\cup x\cup y\overline{\circ}_\cup y$.

(2) $(x\overline{\circ}_\cup x)(x)=(E_{\langle\alpha,\beta\rangle}(x))(x)=(\cup_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i)(x)=\vee_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i(x)$.

Since $\cup_i(A_i)_{\langle\alpha,\beta\rangle}=H$, then for all $x\in H$ there exists some A_i such that $x\in(A_j)_{\langle\alpha,\beta\rangle}$ and thus $A_j^-(x)\geq\alpha,A_j^+(x)\geq\beta$.

So $(x\overline{\circ}_\cup x)^-(x)=\vee_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^-(x)\geq\alpha$.

Similarly, $(x\overline{\circ}_\cup x)^+(x)=\vee_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^+(x)\geq\beta$.

So $x\in(x\overline{\circ}_\cup x)_{\langle\alpha,\beta\rangle}$. ∎

Proposition 3.5. *If an interval-valued hyperstructure* $< H; \overline{\circ}_\cup >$ *satisfying* (1) *and* (2) *of the previous proposition, then it also satisfies*

(3) $\{x, y\} \subseteq (x\overline{\circ}_\cup y)_{\langle\alpha,\beta\rangle}$;

(4) $x\overline{\circ}_\cup y = y\overline{\circ}_\cup x$;

(5) $(x\overline{\circ}_\cup H)_{\langle\alpha,\beta\rangle} = H$;

(6) $\langle H; \{x\overline{\circ}_\cup x\}_{x\in H}\rangle$ *is an interval-valued* $\langle\alpha, \beta\rangle$*-fuzzy hypergraph;*

(7) $x\overline{\circ}_\cup x\overline{\circ}_\cup x = \cup_{(x\overline{\circ}_\cup x)(z)\subseteq(0,1]} z\overline{\circ}_\cup z$;

(8) $(x\overline{\circ}_\cup x)\overline{\circ}_\cup(x\overline{\circ}_\cup x) = x\overline{\circ}_\cup x\overline{\circ}_\cup x$.

Proof. Let's use the (1) and (2) of the previous proposition to prove (3)-(8).

(3) Since $(x\overline{\circ}_\cup y)(x) = (x\overline{\circ}_\cup x \cup y\overline{\circ}_\cup y)(x) = (\vee_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta} A_i(x)) \vee (\vee_{A_i^-(y)\geq\alpha, A_i^+(y)\geq\beta} A_i(x))$, then

$$(x\overline{\circ}_\cup y)^+(x) = (\vee_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta} A_i^+(x)) \vee (\vee_{A_i^-(y)\geq\alpha, A_i^+(y)\geq\beta} A_i^+(x)) \geq \beta.$$

Similarly, $(x\overline{\circ}_\cup y)^-(x) \geq \alpha$. And so $x \in (x\overline{\circ}_\cup y)_{\langle\alpha,\beta\rangle}$. The other part is proved similarly.

(4) $x\overline{\circ}_\cup y = x\overline{\circ}_\cup x \cup y\overline{\circ}_\cup y = y\overline{\circ}_\cup y \cup x\overline{\circ}_\cup x = y\overline{\circ}_\cup x$.

(5) $\forall y \in H$, $(x\overline{\circ}_\cup H)(y) = (\cup_{t\in H}(x\overline{\circ}_\cup t))(y) = (\cup_{t\in H}(x\overline{\circ}_\cup x \cup \overline{\circ}_\cup t))(y) = (y\overline{\circ}_\cup y \cup (\cup_{t\in H}(x\overline{\circ}_\cup t)))(y) = (\vee_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta} A_i(y)) \vee (\cup_{t\in H}(x\overline{\circ}_\cup t))(y)$.

Thus $(x\overline{\circ}_\cup H)^-(y) \geq \alpha$ and $(x\overline{\circ}_\cup H)^+(y) \geq \beta$. And so $(x *_\cup H)_{\langle\alpha,\beta\rangle} \supseteq H$. Therefore, $(x *_\cup H)_{\langle\alpha,\beta\rangle} = H$.

(6) From $x \in (x\overline{\circ}_\cup x)_{\langle\alpha,\beta\rangle}$ we know $\cup_{x\in H}(x\overline{\circ}_\cup x)_{\alpha,\beta} = H$.

And then $\langle H; \{x\overline{\circ}_\cup x\}_{x\in H}\rangle$ is an interval-valued $\langle\alpha, \beta\rangle$-fuzzy hypergraph.

(7) $x\overline{\circ}_\cup x\overline{\circ}_\cup x = \cup_{(x\overline{\circ}_\cup x)(z)\subseteq(0,1]} z\overline{\circ}_\cup x = \cup_{(x\overline{\circ}_\cup x)(z)\subseteq(0,1]}((z\overline{\circ}_\cup z) \cup (x\overline{\circ}_\cup x)) = \cup_{(x\overline{\circ}_\cup x)(z)\subseteq(0,1]} z\overline{\circ}_\cup z$.

(8) $(x\overline{\circ}_\cup x)\overline{\circ}_\cup(x\overline{\circ}_\cup x) = \cup_{(x\overline{\circ}_\cup x)(u)\subset(0,1],(x\overline{\circ}_\cup x)(v)\subset(0,1]} u\overline{\circ}_\cup v = \cup_{(x\overline{\circ}_\cup x)(u)\subset(0,1],(x\overline{\circ}_\cup x)(v)\subset(0,1]}(u\overline{\circ}_\cup u \cup v\overline{\circ}_\cup v) = \cup_{(x\overline{\circ}_\cup x)(u)\subset(0,1]} u\overline{\circ}_\cup u = x\overline{\circ}_\cup x\overline{\circ}_\cup x$. ∎

Remark 3.6. An interval-valued $\langle\alpha, \beta\rangle$-fuzzy quasi-hypergroup is an interval-valued fuzzy hypergroupoid such that

$$(x * H)_{\langle\alpha,\beta\rangle} = H = (H * x)_{\langle\alpha,\beta\rangle}, \quad \forall x \in H.$$

From (4) and (5) of the previous proposition we know that H_Γ is a commutative interval-valued $\langle\alpha, \beta\rangle$-fuzzy quasi-hypergroup.

3.2. The interval-valued fuzzy hyperoperation $\overline{\circ}_\cap$

Definition 3.7. *Let* $\Gamma = \langle H; \{A_i\}_i\rangle$ *be an interval-valued* $\langle\alpha, \beta\rangle$*-fuzzy hypergraph, set* $D_{\langle\alpha,\beta\rangle}(x) = \cap_{A_i^-(x)\geq\alpha, A_i^+(x)\geq\beta} A_i$. *The interval-valued fuzzy hyperoperation* $\overline{\circ}_\cap$ *is defined by*

$$x\overline{\circ}_\cap y = D_{\langle\alpha,\beta\rangle}(x) \cup D_{\langle\alpha,\beta\rangle}(y), \quad \forall x, y \in H.$$

Proposition 3.8. *The interval-valued fuzzy hyperoperation* $\overline{\circ}_\cap$ *has the following properties for any* $x, y \in H$*:*

(1) $x\overline{\circ}_\cap y = x\overline{\circ}_\cap x \cup y\overline{\circ}_\cap y$;

(2) $x \in (x\overline{\circ}_\cap x)_{\langle\alpha,\beta\rangle}$.

Proof. (1) $x\overline{\circ}_\cap y = D_{\langle\alpha,\beta\rangle}(x) \cup D_{\langle\alpha,\beta\rangle}(y) = (D_{\langle\alpha,\beta\rangle}(x) \cup D_{\langle\alpha,\beta\rangle}(x)) \cup (D_{[\alpha,\beta]}(y) \cup D_{\langle\alpha,\beta\rangle}(y)) = x\overline{\circ}_\cap x \cup y\overline{\circ}_\cap y$.

(2) $(x\overline{\circ}_\cap x)(x)$=$(D_{\langle\alpha,\beta\rangle}(x))(x)$=$(\cap_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i)(x)$=$\wedge_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i(x)$.

Since $\cup_i(A_i)_{\langle\alpha,\beta\rangle} = H$, then for all $x \in H$ there exists some A_i such that $x \in (A_j)_{\langle\alpha,\beta\rangle}$ and thus $A_j^-(x) \geq \alpha$, $A_j^+(x) \geq \beta$.

So $(x\overline{\circ}_\cup x)^-(x) = \wedge_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^-(x) \geq \alpha$ and

$(x\overline{\circ}_\cup x)^+(x) = \wedge_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^+(x) \geq \beta$.

So $x \in (x\overline{\circ}_\cap x)_{<\alpha,\beta>}$. ∎

Proposition 3.9. *The interval-valued fuzzy hyperoperation* $\overline{\circ}_\cup$ *also satisfies*

(3) $\{x, y\} \subseteq (x\overline{\circ}_\cap y)_{\langle\alpha,\beta\rangle}$;

(4) $x\overline{\circ}_\cap y = y\overline{\circ}_\cap x$;

(5) $(x\overline{\circ}_\cap H)_{\langle\alpha,\beta\rangle} = H$;

(6) $\langle H; \{x\overline{\circ}_\cap x\}_{x\in H}\rangle$ *is an interval-valued* $\langle\alpha,\beta\rangle$*-fuzzy hypergraph;*

(7) $x\overline{\circ}_\cap x\overline{\circ}_\cap x = \cup_{(x\overline{\circ}_\cap x)(z)\subseteq(0,1]} z\overline{\circ}_\cap z$;

(8) $(x\overline{\circ}_\cap x)\overline{\circ}_\cap(x\overline{\circ}_\cap x) = x\overline{\circ}_\cap x\overline{\circ}_\cap x$.

Proof. Applied Proposition 3.5. ∎

Remark 3.10. From (4) and (5) of the previous we know that $\langle H; \overline{\circ}_\cap\rangle$ is also a commutative interval-valued $\langle\alpha,\beta\rangle$-fuzzy quasi-hypergroup .

Proposition 3.11. $\forall x, y \in H,\ y \in (x\overline{*}_\cap x)_{\langle\alpha,\beta\rangle} \Rightarrow (y\overline{*}_\cap y)(x) \leq (x\overline{*}_\cap x)(x)$.

Proof. Since $(x\overline{*}_\cap x)^-(y) = \wedge_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^-(y) \geq \alpha$, then $A_i^-(y) \geq \alpha$ for all $A_i \in IF(H)$, where $A_i^-(x) \geq \alpha$. Similarly, $A_i^+(y) \geq \beta$ for all $A_i \in IF(H)$, where $A_i^+(x) \geq \beta$. Thus

$$\begin{aligned}(y\overline{*}_\cap y)^-(x) &= \wedge_{A_i^-(y)\geq\alpha,A_i^+(y)\geq\beta}A_i^-(x)\\ &\leq (\wedge_{A_i^-(y)\geq\alpha,A_i^+(y)\geq\beta,A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^-(x))\\ &= \wedge_{A_i^-(x)\geq\alpha,A_i^+(x)\geq\beta}A_i^-(x)\\ &= (x\overline{*}_\cap x)^-(x).\end{aligned}$$

Similarly, $(y\overline{*}_\cap y)^+(x) \leq (x\overline{*}_\cap x)^+(x)$.

Therefore, $(y\overline{*}_\cap y)(x) \leq (x\overline{*}_\cap x)(x)$. ∎

3.3. The interval-valued fuzzy hyperstructure $\langle H; \overline{\circ}_\cap, \overline{\circ}_\cup \rangle$

Definition 3.12. Given the interval-valued fuzzy hyperoperations $\widetilde{\triangle}$, $\widetilde{\bigtriangledown}$ on X, we say that $\left\langle X; \widetilde{\triangle}, \widetilde{\bigtriangledown} \right\rangle$ is an interval-valued $\langle \alpha, \beta \rangle$-fuzzy superlattice iff the following properties hold for all $x, y, z \in X$.

(1) $x \in (x\widetilde{\triangle}x)_{\langle \alpha,\beta \rangle}$, $x \in (x\widetilde{\bigtriangledown}x)_{\langle \alpha,\beta \rangle}$.

(2) $x\widetilde{\triangle}y = y\widetilde{\triangle}x$, $x\widetilde{\bigtriangledown}y = y\widetilde{\bigtriangledown}x$.

(3) $((x\widetilde{\triangle}y)\widetilde{\triangle}z)_{\langle \alpha,\beta \rangle} = (x\widetilde{\triangle}(y\widetilde{\triangle}z))_{\langle \alpha,\beta \rangle}$, $((x\widetilde{\bigtriangledown}y)\widetilde{\bigtriangledown}z)_{\langle \alpha,\beta \rangle} = (x\widetilde{\bigtriangledown}(y\widetilde{\bigtriangledown}z))_{\langle \alpha,\beta \rangle}$.

(4) $x \in ((x\widetilde{\triangle}y)\widetilde{\bigtriangledown}x)_{\langle \alpha,\beta \rangle}$, $x \in ((x\widetilde{\bigtriangledown}y)\widetilde{\triangle}x)_{\langle \alpha,\beta \rangle}$.

(5) $y \in (x\widetilde{\bigtriangledown}y)_{\langle \alpha,\beta \rangle} \Leftrightarrow x \in (x\widetilde{\triangle}y)_{\langle \alpha,\beta \rangle}$.

(6) $x, y \in (x\widetilde{\bigtriangledown}y)_{\langle \alpha,\beta \rangle} \Rightarrow x = y$.

(7) $y \in (x\widetilde{\bigtriangledown}y)_{\langle \alpha,\beta \rangle}$ et $z \in (y\widetilde{\bigtriangledown}z)_{\langle \alpha,\beta \rangle} \Rightarrow z \in (x\widetilde{\bigtriangledown}z)_{\langle \alpha,\beta \rangle}$.

Proposition 3.13. *The fuzzy hyperstructure $\langle H; \overline{*}_\cap, \overline{*}_\cup \rangle$ has the following properties for all $x, y, z \in H$.*

(1) $x \in (x\overline{\circ}_\cap x)_{\langle \alpha,\beta \rangle}$, $x \in (x\overline{\circ}_\cup x)_{\langle \alpha,\beta \rangle}$.

(2) $x\overline{\circ}_\cap y = y\overline{\circ}_\cap x$, $x\overline{\circ}_\cup y = y\overline{\circ}_\cup x$.

(3) $((x\overline{\circ}_\cap y)\overline{\circ}_\cap z)_{\langle \alpha,\beta \rangle} \cap (x\overline{\circ}_\cap(y\overline{\circ}_\cap z))_{\langle \alpha,\beta \rangle} \neq \emptyset$,
$((x\overline{\circ}_\cup y)\overline{\circ}_\cup z)_{\langle \alpha,\beta \rangle} \cap (x\overline{\circ}_\cup(y\overline{\circ}_\cup z))_{\langle \alpha,\beta \rangle} \neq \emptyset$.

(4) $x \in ((x\overline{\circ}_\cap y)\overline{\circ}_\cup x)_{\langle \alpha,\beta \rangle}$, $x \in ((x\overline{\circ}_\cup y)\overline{\circ}_\cap x)_{\langle \alpha,\beta \rangle}$.

(5) $y \in (x\overline{\circ}_\cup y)_{\langle \alpha,\beta \rangle} \Leftrightarrow x \in (x\overline{\circ}_\cap y)_{\langle \alpha,\beta \rangle}$.

(6) $x, y \in (x\overline{\circ}_\cup y)_{\langle \alpha,\beta \rangle} \not\Rightarrow x = y$.

(7) $y \in (x\overline{\circ}_\cup y)_{\langle \alpha,\beta \rangle}$ *et* $z \in (y\overline{\circ}_\cup z)_{\langle \alpha,\beta \rangle} \Rightarrow z \in (x\overline{\circ}_\cup z)_{\langle \alpha,\beta \rangle}$.

Proof. It is enough to prove (3), (4) and (6). The others are clear.

(3) Since $((x\overline{\circ}_\cap y)\overline{\circ}_\cap z)^-(y) \geq (y\overline{\circ}_\cap z)^-(y) \geq \alpha$ and $(x\overline{\circ}_\cap(y\overline{\circ}_\cap z))^-(y) \geq (x\overline{\circ}_\cap y)^-(y) \geq \alpha$.

Similarly, $((x\overline{\circ}_\cap y)\overline{\circ}_\cap z)^+(y) \geq \beta$ and $(x\overline{\circ}_\cap(y\overline{\circ}_\cap z))^+(y) \geq \beta$.

Then $y \in ((x\overline{\circ}_\cap y)\overline{\circ}_\cap z)_{\langle \alpha,\beta \rangle} \cap (x\overline{\circ}_\cap(y\overline{\circ}_\cap z))_{\langle \alpha,\beta \rangle}$.

And so $((x\overline{\circ}_\cap y)\overline{\circ}_\cap z)_{\langle \alpha,\beta \rangle} \cap (x\overline{\circ}_\cap(y\overline{\circ}_\cap z))_{\langle \alpha,\beta \rangle} \neq \emptyset$.

The other part is proved similarly.

(4)$((x\overline{\circ}_\cap y)\overline{\circ}_\cup x)^-(x) \geq (x\overline{\circ}_\cup x)^-(x) \geq \alpha$;
$((x\overline{\circ}_\cup y)\overline{\circ}_\cap x)^-(x) \geq (x\overline{\circ}_\cap x)^-(x) \geq \alpha$.

Similarly, $((x\overline{\circ}_\cap y)\overline{\circ}_\cup x)^+(x) \geq \beta$; $((x\overline{\circ}_\cup y)\overline{\circ}_\cap x)^+(x) \geq \beta$.

So $x \in ((x\overline{\circ}_\cap y)\overline{\circ}_\cup x)_{\langle \alpha,\beta \rangle}$.

The other part is proved similarly.

(6)For example, the interval-valued $\langle 0.2, 0.5\rangle$-fuzzy hypergraph is

$$\langle \{x, y\}; \{A_1, A_2\}\rangle ,$$

where

$$A_1 = \frac{[0.2, 0.5]}{x} + \frac{[0.4, 0.8]}{y} \text{ and } A_2 = \frac{[0.2, 0.3]}{x} + \frac{[0.2, 0.5]}{y}.$$

Of course we have $x, y \in x\overline{\circ}_{\cap} y$. But $x \neq y$. ∎

Remark 3.14. From the previous proposition we can see that the hyperstructure $\langle H; \overline{\circ}_{\cap}, \overline{\circ}_{\cup}\rangle$ is almost an interval-valued $\langle \alpha, \beta\rangle$-fuzzy superlattice.

4. Further research

Definition 4.1. for all $[a, b], [x, y] \in I([0, 1])$,$p \in [0, 1]$ we define

$$[a, b] \vee_p [x, y] \doteq [(a \vee x) \wedge p, (b \vee y) \vee p'].$$
$$[a, b] \wedge_p [x, y] \doteq [(a \wedge x) \wedge p, (b \wedge y) \vee p'].$$

Definition 4.2. Given $A, B \in IF(H)$, we give the following definitions

$$(A \cup_p B)(x) \doteq A(x) \vee_p B(x), \forall x \in H.$$
$$(A \cap_p B)(x) \doteq A(x) \wedge_p B(x), \forall x \in H.$$

The present work can be extended in several directions. Let us indicate some possibilities.

1. We can use the above definitions to reconstruct interval-valued fuzzy hyperoperations and study the properties of them and the relations between them.

2. One may explore the interval-valued fuzzy join space by the use of the hyperoperations.

3. One can also investigate the hyperstructure associated to the product of interval-valued hypergraphs (see [1]).

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