GLOBAL ASYMPTOTIC STABILITY FOR A STOCHASTIC COMPETITION AND COOPERATION MODEL OF TWO ENTERPRISES

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Abstract. This paper deals with a stochastic competition and cooperation model of two enterprises. Some very verifiable criteria on the global stability of the positive equilibrium of the deterministic system are established. An example with its computer simulations is given to illustrate our main theoretical findings.

Keywords: competition and cooperation model; global stability; stochastic perturbation; enterprise.

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1. Introduction

Competition and cooperation of two enterprises has been the hot topic of international community, many governments and researchers. During the past several decades, numerous authors have widely investigated enterprises' inborn path, operating mechanism, management process, risk and efficiency, evolvement cycle, economic performance and so on [1]. For example, Tian and Nie [2] investigated the competition and cooperation model of two enterprises based on ecosystem

$$(1.1)\qquad \begin{cases} \dfrac{dx_1(t)}{dt} = r_1(t)x_1(t)\left[1 - \dfrac{x_1(t)}{K} - \dfrac{\alpha(x_2(t)-c_2)^2}{K}\right], \\ \dfrac{dx_2(t)}{dt} = r_2(t)x_2(t)\left[1 - \dfrac{x_2(t)}{K} - \dfrac{\beta(x_1(t)-c_1)^2}{K}\right], \end{cases}$$

where $x_1(t), x_2(t)$ represent the output of interprises A and B, r_1, r_2 are the intrinsic growth rates, K denotes the carrying capacity of mark under nature unlimited conditions, α, β are the competitive coefficient of two enterprises, c_1, c_2 are the initial production of two enterprises. Let $a_1 = \frac{r_1}{K}, a_2 = \frac{r_2}{K}, b_1 = \frac{r_1\alpha}{K}, b_2 = \frac{r_2\beta}{K}$ in system (1.1). Then system (1.1) takes the form

$$(1.2) \qquad \begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1 - a_1x_1(t) - b_1(x_2(t) - c_2)^2], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2 - a_2x_2(t) + b_2(x_1(t) - c_1)^2]. \end{cases}$$

In real world, two enterprises are constantly in the competition and cooperation, and when an enterprise suffers damage from another one by competition, another one could benefit, and two enterprises can simultaneously benefit by cooperation, meantime, the duration time of output for enterprises would also play an important role. Motivated by the viewpoint, Liao et al. [3] modified system (1.2) as follows

$$(1.3) \qquad \begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1 - a_1x_1(t-\tau_1) - b_1(x_2(t-\tau_2) - c_2)^2], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2 - a_2x_2(t-\tau_1) + b_2(x_1(t-\tau_2) - c_1)^2], \end{cases}$$

where τ_1 is nonnegative constant which stands for the gestation periodic of production for two enterprises, τ_2 in the first equation of the system (1.3) represents the block delay of enterprise B to A, and τ_2 in the second equation of the system (1.3) represents the promoting delay of enterprise A to B. By choosing the two delays τ_1 and τ_2 as bifurcation parameters, Liao et al. [3] analyzed the effect of different delays on the dynamics of model (1.3). If $\tau_1 = \tau_2 = \tau$, then model (1.3) takes the form

$$(1.4) \qquad \begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1 - a_1x_1(t-\tau) - b_1(x_2(t-\tau) - c_2)^2], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2 - a_2x_2(t-\tau) + b_2(x_1(t-\tau) - c_1)^2]. \end{cases}$$

By regarding the time delay τ as bifurcation parameter, Liao et al. [4] investigated the stability and Hopf bifurcation nature of model (1.4).

In 2012, Li and Zhang [1] studied the model (1.2) with nonconstant coefficients, which can be derived as a modified model (1.2) with variable coefficients

$$(1.5) \qquad \begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1(t) - a_1(t)x_1(t) - b_1(t)(x_2(t) - c_2(t))^2], \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2(t) - a_2(t)x_1(t) + b_2(t)(x_1(t) - c_1(t))^2]. \end{cases}$$

Using coincidence degree theory and differential inequality theory, Xu [5] established some sufficient conditions to ensure the existence of periodic solutions of model (1.5).

Considering that the realistic models require the inclusion of the effect of changing environment and the output of enterprises A and B are subjected to rapid change at certain instants in time, Xu and Shao [6] focused on the existence and global attractivity of periodic solution for the following enterprise clusters based on ecology theory with impulse and varying coefficients

$$
(1.6)\qquad \begin{cases} \dfrac{dx_1(t)}{dt} = x_1(t)[r_1(t) - a_1(t)x_1(t) - b_1(t)(x_2(t) - c_2(t))^2],\ t \neq t_k, \\ \dfrac{dx_2(t)}{dt} = x_2(t)[r_2(t) - a_2(t)x_1(t) + b_2(t)(x_1(t) - c_1(t))^2],\ t \neq t_k, \\ \Delta_i(t_k) = x_i(t_k^+) - x_i(t_k^-) = -\gamma_{ik}(t_k),\ i = 1, 2,\ k = 1, 2, ..., q, \end{cases}
$$

where $\Delta_i(t_k) = x_i(t_k^+) - x_i(t_k^-)$ denotes the impulses at moments t_k and $t_1 < t_2 < \cdots$ is a strictly increasing sequence such that $\lim_{k\to+\infty} t_k = +\infty$ and q is a positive integer. Applying continuation theorem of coincidence degree theory, Xu and Shao [6] obtained a set of sufficient conditions that ensure the existence of at least a positive periodic solution, and by constructing a suitable Lyapunov functional, the uniqueness and global attractivity of the positive periodic solution are included.

Numerous scholars argue [7]-[18] that in the real word the enterprises usually live in a changing circumstances which have important effect on the intrinsic growth rates. So we can think that the intrinsic growth rate can be expressed as an average rate plus an error term. In view of the well known central limit theorem, we know that the error term follows a normal distribution [19]-[21]. Therefore the error term can be denoted by a white noise $\sigma_i(t)\dot{W}_i(t)(i = 1, 2)$, where $\sigma_i^2(t)$ denotes the intensity of the noise and $\dot{W}_i(t)$ denotes a standard white noise, where $W_i(t)$ represents a Brownian motion defined on a complete probability space (Ω, F, P). Motivated by the discussed above, we can let the growth rates r_1 and r_2 become $r_1 + \sigma_1\dot{W}_1(t)$ and $r_2 + \sigma_2\dot{W}_2(t)$, respectively. In view of model (1.1), we can establish the stochastic competition and cooperation model of two enterprises which takes the form:

$$
(1.7)\qquad \begin{cases} dx_1 = x_1(t)\left[1 - \dfrac{x_1(t)}{K} - \dfrac{\alpha(x_2(t) - c_2)^2}{K}\right][r_1 dt + \sigma_1 dW_1(t)], \\ dx_2 = x_2(t)\left[1 - \dfrac{x_2(t)}{K} - \dfrac{\beta(x_1(t) - c_1)^2}{K}\right][r_2 dt + \sigma_2 dW_2(t)]. \end{cases}
$$

The initial conditions of system (1.7) are given by

$$
(1.8)\qquad x_{10} = \varphi_1(0) > 0, x_{20} = \varphi_2(0) > 0.
$$

The remainder of the paper is organized as follows: in Section 2, some sufficient conditions for the existence of global positive solution of system (1.7) with the initial conditions (1.8) are established. Moreover, the sufficient condition is obtained to ensure that the equilibrium of system (1.7) with the initial conditions (1.8) is globally asymptotically stable. In Section 3, an example with its computer simulations is given to illustrate the feasibility and effectiveness of our main results. A brief conclusion is drawn in Section 4.

2. Main results

In this section, we shall present our main results.

Theorem 2.1 *For any given initial value* $(x_{10}, x_{20}) \in R^2_+$, *system* (1.7) *has a unique global solution* $(x_1(t), x_2(t))$ *almost sure (a.s.), where* $R^2_+ = \{x_1 > 0,\ x_2 > 0\}$.

Proof. Consider the following system

$$(2.1)\qquad \begin{cases} dh_1 = \Big\{ r_1 \Big[1 - \dfrac{e^{h_1}}{K} - \dfrac{\alpha(e^{h_2} - c_2)^2}{K}\Big] \\ \qquad - \dfrac{\sigma_1^2}{2}\Big[1 - \dfrac{e^{h_1}}{K} - \dfrac{\alpha(e^{h_2} - c_2)^2}{K}\Big]^2 \Big\} dt \\ \qquad + \sigma_1 \Big[1 - \dfrac{e^{h_1}}{K} - \dfrac{\alpha(e^{h_2} - c_2)^2}{K}\Big] dW_1(t), \\ dh_2 = \Big\{ r_2 \Big[1 - \dfrac{e^{h_2}}{K} - \dfrac{\beta(e^{h_1} - c_1)^2}{K}\Big] \\ \qquad - \dfrac{\sigma_2^2}{2}\Big[1 - \dfrac{e^{h_2}}{K} - \dfrac{\beta(e^{h_1} - c_1)^2}{K}\Big]^2 \Big\} dt \\ \qquad + \sigma_2 \Big[1 - \dfrac{e^{h_2}}{K} - \dfrac{\beta(e^{h_1} - c_1)^2}{K}\Big] dW_2(t) \end{cases}$$

with initial value $h_{10} = \ln x_{10}, h_{20} = \ln x_{20}$.

Obviously, the coefficients of (2.1) satisfy the local Lipschitz condition, then there is unique local solution $(h_1(t), h_2(t))$ on $[0, \tau_e)$, where τ_e is the explosion time. Thus, in view of Itö's formula, $(x_1(t), x_2(t)) = (e^{h_1(t)}, e^{h_2(t)})$ is the unique positive local solution to system (1.7) with the initial value $x_{10} > 0, x_{20} > 0$.

Assume that $k_0 > 0$ is sufficiently large such that x_{10} and x_{20} lying within $[\frac{1}{k_0}, k_0]$. For integer $k > k_0$, define the stopping times

$$(2.2)\qquad \tau_k = \inf\left\{ t \in [0, \tau_e] : x_1(t) \notin \left(\frac{1}{k}, k\right) \ or\ x_2(t) \notin \left(\frac{1}{k}, k\right) \right\}.$$

Let $\tau_\infty = \lim_{k\to\infty} \tau_k$. Then $\tau_\infty \le \infty$ a.s. To complete the proof of Lemma 2.1 we need only to prove $\tau\infty = \infty$. Assume that the statement does not hold, then there exists a constant $T > 0$ and $\epsilon \in (0, 1)$ such that $P\{\tau_\infty \le T\} > \epsilon$. Thus there is an integer $k_1 \ge k_0$ such that $P\{\tau_k \le T\} \ge \epsilon$. We define

$$(2.3)\qquad V(x_1, x_2) = (\sqrt{x_1} - 1 - 0.5 \ln x_1) + (\sqrt{x_2} - 1 - 0.5 \ln x_2).$$

If $(x_1(t), x_2(t)) \in R^2_+$, then

$$\begin{aligned} dV(x_1, x_2) \ &= \ 0.5 r_1(\sqrt{x_1} - 1)\Big[1 - \frac{x_1}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\Big] dt \\ &\quad + \frac{\sigma_1^2(2 - \sqrt{x_1})}{8}\Big[1 - \frac{x_1}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\Big]^2 dt \\ &\quad + 0.5\sigma_1(\sqrt{x_1} - 1)\Big[1 - \frac{x_1}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\Big] dW_1(t) \\ &\quad + 0.5 r_2(\sqrt{x_2} - 1)\Big[1 - \frac{x_2}{K} - \frac{\beta(x_1 - c_1)^2}{K}\Big] dt \end{aligned}$$

$$
\begin{aligned}
&+\frac{\sigma_2^2(2-\sqrt{x_2})}{8}\left[1-\frac{x_2}{K}-\frac{\beta(x_1-c_1)^2}{K}\right]^2 dt\\
&+0.5\sigma_2(\sqrt{x_2}-1)\left[1-\frac{x_2}{K}-\frac{\beta(x_1-c_1)^2}{K}\right]dW_2(t)\\
= \;& 0.5r_1\left[\sqrt{x_1}-\frac{x_1^{\frac{3}{2}}}{K}-\frac{\sqrt{x_1}\alpha(x_2-c_2)^2}{K}-1+\frac{x_1}{K}+\frac{\alpha(x_2-c_2)^2}{K}\right]dt\\
&+0.5r_2\left[\sqrt{x_2}-\frac{x_2^{\frac{3}{2}}}{K}-\frac{\sqrt{x_2}\beta(x_1-c_1)^2}{K}-1+\frac{x_2}{K}+\frac{\beta(x_1-c_1)^2}{K}\right]dt\\
&+\frac{\sigma_1^2}{8}\left[2+\frac{2x_1^2}{K^2}+\frac{2\alpha^2(x_2-c_2)^4}{K^2}-\frac{4x_1+4\alpha(x_2-c_2)^2}{K}\right.\\
&+\frac{4\alpha x_1(x_2-c_2)^2}{K^2}-\sqrt{x_1}-\frac{x_1^{\frac{3}{2}}}{K^2}-\frac{\alpha^2\sqrt{x_1}(x_2-c_2)^2}{K^2}+\frac{2x_1^{\frac{3}{2}}}{K}\\
&\left.+\frac{2\alpha\sqrt{x_1}(x_2-c_2)^2}{K}-\frac{2x_1^{\frac{3}{2}}\alpha(x_2-c_2)^2}{K^2}\right]dt\\
&+\frac{\sigma_2^2}{8}\left[2+\frac{2x_2^2}{K^2}+\frac{2\beta^2(x_1-c_1)^4}{K^2}-\frac{4x_2+4\beta(x_1-c_1)^2}{K}\right.\\
&+\frac{4\alpha x_2(x_1-c_1)^2}{K^2}-\sqrt{x_2}-\frac{x_2^{\frac{3}{2}}}{K^2}-\frac{\beta^2\sqrt{x_2}(x_1-c_1)^2}{K^2}+\frac{2x_2^{\frac{3}{2}}}{K}\\
&\left.+\frac{2\beta\sqrt{x_2}(x_1-c_1)^2}{K}-\frac{2x_2^{\frac{3}{2}}\beta(x_1-c_1)^2}{K^2}\right]dt\\
&+0.5\sigma_1(\sqrt{x_1}-1)\left[1-\frac{x_1}{K}-\frac{\alpha(x_2-c_2)^2}{K}\right]dW_1(t)\\
&+0.5\sigma_2(\sqrt{x_2}-1)\left[1-\frac{x_2}{K}-\frac{\beta(x_1-c_1)^2}{K}\right]dW_2(t)\\
\le \;& 0.5r_1\left[\sqrt{x_1}+\frac{x_1}{K}+\frac{\alpha(x_2-c_2)^2}{K}\right]dt\\
&+0.5r_2\left[\sqrt{x_2}+\frac{x_2}{K}+\frac{\beta(x_1-c_1)^2}{K}\right]dt\\
&+\frac{\sigma_1^2}{8}\left[\frac{2x_1^2}{K^2}+\frac{2\alpha^2(x_2-c_2)^4}{K^2}+\frac{4\alpha x_1(x_2-c_2)^2}{K^2}\right.\\
&\left.+\frac{2x_1^{\frac{3}{2}}}{K}+\frac{2\alpha\sqrt{x_1}(x_2-c_2)^2}{K}+2\right]dt\\
&+\frac{\sigma_2^2}{8}\left[\frac{2x_2^2}{K^2}+\frac{2\beta^2(x_1-c_1)^4}{K^2}+\frac{4\beta x_2(x_1-c_1)^2}{K^2}\right.\\
&\left.+\frac{2x_2^{\frac{3}{2}}}{K}+\frac{2\beta\sqrt{x_1}(x_1-c_1)^2}{K}+2\right]dt
\end{aligned}
$$

$$\begin{aligned}
&+0.5\sigma_1(\sqrt{x_1}-1)\left[1-\frac{x_1}{K}-\frac{\alpha(x_2-c_2)^2}{K}\right]dW_1(t)\\
&+0.5\sigma_2(\sqrt{x_2}-1)\left[1-\frac{x_2}{K}-\frac{\beta(x_1-c_1)^2}{K}\right]dW_2(t)\\
\leq\ & \chi_1 dt+\chi_2 dt\\
&+0.5\sigma_1(\sqrt{x_1}-1)\left[1-\frac{x_1}{K}-\frac{\alpha(x_2-c_2)^2}{K}\right]dW_1(t)\\
&+0.5\sigma_2(\sqrt{x_2}-1)\left[1-\frac{x_2}{K}-\frac{\beta(x_1-c_1)^2}{K}\right]dW_2(t), \qquad (2.4)
\end{aligned}$$

where χ_1 and χ_2 are positive numbers. Integrating both sides of (2.4) from 0 to $t_k \wedge T$, and taking the expectations, we have

$$EV(x_1(\tau_k\wedge T),x_2(\tau_k\wedge T))\leq V(x_{10},x_{20})+(\chi_1+\chi_2)T. \qquad (2.5)$$

Set $\Phi_k=\{\tau_k\leq T\}$, then it follows that $P(\Phi_k)\geq\epsilon$. Since for every $\varpi\in\Phi_k$, there exists some i such that $x_i(\tau_k,\varpi)$ is equal to either k or $\frac{1}{k}$ for $i=1,2$. Thus $V(x_1(\tau_k,\varpi),x_2(\tau_k,\varpi)$ is no less than

$$\min\left\{\left(\sqrt{k}-1-0.5\ln k\right),\left(\sqrt{\frac{1}{k}}-1-0.5\ln\frac{1}{k}\right)\right\}. \qquad (2.6)$$

In view of (2.5), we have

$$\begin{aligned}
&V(x_{10},x_{20})+(\chi_1+\chi_2)T\geq E[1_{\Phi_k}(\varpi)V(x_1(\tau_k),x_2(\tau_n))]\\
&\quad\geq\epsilon\min\left\{\left(\sqrt{k}-1-0.5\ln k\right),\left(\sqrt{\frac{1}{k}}-1-0.5\ln\frac{1}{k}\right)\right\},
\end{aligned} \qquad (2.7)$$

where 1_{Φ_k} stands for the indicator function of Φ_k. Let $k\to\infty$, then

$$\infty>V(x_{10},x_{20})+(\chi_1+\chi_2)T=\infty, \qquad (2.8)$$

which is a contradiction. The proof of Theorem 2.1 is complete. ■

Let $E(x_1^*,x_2^*)$ be the positive equilibrium of system (1.7), then x_1^* and x_2^* satisfy the following equations

$$\begin{cases}1-\dfrac{x_1^*}{K}-\dfrac{\alpha(x_2^*-c_2)^2}{K}=0,\\[2mm] 1-\dfrac{x_2^*}{K}-\dfrac{\beta(x_1^*-c_1)^2}{K}=0.\end{cases} \qquad (2.9)$$

Now we will give the second main result of the paper.

Theorem 2.2 *Let*

$$\begin{aligned}
\mu_1&=-\left(\frac{r_1}{K}-\frac{\sigma_1^2x_1^*}{2K}\right)+\frac{\sigma_2^2x_2^*r_2^2\beta^2(x_1^*+K-2c_2)^2}{2K^2},\\
\mu_2&=(1+\sigma_1^2x_1^*)\frac{r_1\alpha|x_2^*+K-2c_2|}{K}+(1+\sigma_2^2x_2^*)\frac{r_2\beta|x_1^*+K-2c_1|}{K},\\
\mu_3&=-\left(\frac{r_2}{K}-\frac{\sigma_2^2x_2^*}{2K}\right)+\frac{\sigma_1^2x_1^*r_1^2\alpha^2(x_2^*+K-2c_2)^2}{2K^2}.
\end{aligned}$$

If $\mu_1 < 0, 4\mu_1\mu_3 > \mu_2^2$, then the equilibrium (x_1^*, x_2^*) of system (1.7) is globally asymptotically stable, that is, for any initial value $x_{10}(0) > 0, x_{20}(0) > 0$, the solution of system (1.7) satisfies

$$\lim_{t\to+\infty} x_1(t) = x_1^*, \ \lim_{t\to+\infty} x_2(t) = x_2^*, \ a.s.$$

Proof. Define the following functions

$$V_1(x_1) = \int_0^{x_1 - x_1^*} \frac{\theta}{\theta + x_1^*} d\theta, \tag{2.10}$$

$$V_2(x_2) = \int_0^{x_2 - x_2^*} \frac{\theta}{\theta + x_2^*} d\theta. \tag{2.11}$$

By Itô's formula, we have

$$\begin{aligned}
LV_1(x_1) &= r_1(x_1 - x_1^*)\left[1 - \frac{x_1}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\right] \\
&\quad + \frac{\sigma_1^2 x_1^*}{2}\left[1 - \frac{x_1}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\right]^2 \\
&= r_1(x_1 - x_1^*)\left[\frac{1}{K}(x_1^* - x_1) + \frac{\alpha(x_2^* - c_2)^2}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\right] \\
&\quad + \frac{\sigma_1^2 x_1^*}{2}\left[\frac{1}{K}(x_1^* - x_1) + \frac{\alpha(x_2^* - c_2)^2}{K} - \frac{\alpha(x_2 - c_2)^2}{K}\right]^2 \\
&= -\frac{r_1}{K}(x_1 - x_1^*)^2 - \frac{r_1\alpha(x_2^* + x_2 - 2c_2)}{K}(x_1 - x_1^*)(x_2 - x_2^*) \\
&\quad + \frac{\sigma_1^2 x_1^*}{2}\left[\frac{1}{K}(x_1 - x_1^*)^2 + \frac{r_1^2\alpha^2(x_2^* + x_2 - 2c_2)^2}{K^2}(x_2 - x_2^*)^2\right. \\
&\quad \left. - \frac{2r_1\alpha(x_2^* + x_2 - 2c_2)}{K^2}(x_1 - x_1^*)(x_2 - x_2^*)\right] \\
&\le -\left(\frac{r_1}{K} - \frac{\sigma_1^2 x_1^*}{2K}\right)(x_1 - x_1^*)^2 + (1 + \sigma_1^2 x_1^*)\frac{r_1\alpha|x_2^* + K - 2c_2|}{K} \\
&\quad \times |x_1 - x_1^*||x_2 - x_2^*| + \frac{\sigma_1^2 x_1^* r_1^2 \alpha^2 (x_2^* + K - 2c_2)^2}{2K^2}(x_2 - x_2^*)^2, \qquad (2.12)
\end{aligned}$$

$$\begin{aligned}
LV_2(x_2) &= r_2(x_2 - x_2^*)\left[1 - \frac{x_2}{K} - \frac{\beta(x_1 - c_1)^2}{K}\right] \\
&\quad + \frac{\sigma_2^2 x_2^*}{2}\left[1 - \frac{x_2}{K} - \frac{\beta(x_1 - c_1)^2}{K}\right]^2 \\
&= r_2(x_2 - x_2^*)\left[\frac{1}{K}(x_2^* - x_2) + \frac{\beta(x_1^* - c_1)^2}{K} - \frac{\beta(x_1 - c_1)^2}{K}\right] \\
&\quad + \frac{\sigma_2^2 x_2^*}{2}\left[\frac{1}{K}(x_2^* - x_2) + \frac{\beta(x_1^* - c_1)^2}{K} - \frac{\beta(x_1 - c_1)^2}{K}\right]^2
\end{aligned}$$

$$\begin{aligned}
&= -\frac{r_2}{K}(x_2-x_2^*)^2 - \frac{r_2\beta(x_1^*+x_1-2c_1)}{K}(x_1-x_1^*)(x_2-x_2^*) \\
&\quad +\frac{\sigma_2^2 x_2^*}{2}\left[\frac{1}{K}(x_2-x_2^*)^2 + \frac{r_2^2\beta^2(x_1^*+x_1-2c_1)^2}{K^2}(x_1-x_1^*)^2\right. \\
&\quad \left. -\frac{2r_2\beta(x_1^*+x_1-2c_1)}{K^2}(x_1-x_1^*)(x_2-x_2^*)\right] \\
&\leq -\left(\frac{r_2}{K}-\frac{\sigma_2^2 x_2^*}{2K}\right)(x_2-x_2^*)^2 + (1+\sigma_2^2 x_2^*)\frac{r_2\beta|x_1^*+K-2c_1|}{K} \\
&\quad \times|x_1-x_1^*||x_2-x_2^*| + \frac{\sigma_2^2 x_2^* r_2^2\beta^2(x_1^*+K-2c_2)^2}{2K^2}(x_1-x_1^*)^2. \qquad (2.13)
\end{aligned}$$

Now we define

$$V(t) = V_1(x_1) + V_2(x_2). \qquad (2.14)$$

Then we get

$$\begin{aligned}
LV(t) &= LV_1(x_1) + LV_2(x_2) \\
&\leq -\left(\frac{r_1}{K}-\frac{\sigma_1^2 x_1^*}{2K}\right)(x_1-x_1^*)^2 + (1+\sigma_1^2 x_1^*)\frac{r_1\alpha|x_2^*+K-2c_2|}{K} \\
&\quad \times|x_1-x_1^*||x_2-x_2^*| + \frac{\sigma_1^2 x_1^* r_1^2\alpha^2(x_2^*+K-2c_2)^2}{2K^2}(x_2-x_2^*)^2 \\
&\quad -\left(\frac{r_2}{K}-\frac{\sigma_2^2 x_2^*}{2K}\right)(x_2-x_2^*)^2 + (1+\sigma_2^2 x_2^*)\frac{r_2\beta|x_1^*+K-2c_1|}{K} \\
&\quad \times|x_1-x_1^*||x_2-x_2^*| + \frac{\sigma_2^2 x_2^* r_2^2\beta^2(x_1^*+K-2c_2)^2}{2K^2}(x_1-x_1^*)^2 \\
&= \mu_1(x_1-x_1^*)^2 + \mu_2|x_1-x_1^*||x_2-x_2^*| + \mu_3(x_2-x_2^*)^2. \qquad (2.15)
\end{aligned}$$

Let $|Y-Y^*| = (|x_1-x_1^*|, |x_2-x_2^*|)^T$. By (2.15), we have

$$LV(t) \leq \frac{1}{2}|Y-Y^*|^T \begin{bmatrix} 2\varrho_1 & \varrho_2 \\ \varrho_2 & 2\varrho_3 \end{bmatrix} |Y-Y^*|. \qquad (2.16)$$

According to the conditions in Theorem 2.2, we know that $LV(t) < 0$ along all trajectories in the first quadrant except (x_1^*, x_2^*). Then $\lim_{t\to+\infty} x_1(t) = x_1^*$ and $\lim_{t\to+\infty} x_2(t) = x_2^*$. The proof is complete. ∎

3. Computer simulations

In this section, we give an example to illustrate our main results obtained in previous sections by the Milstein method [22]. Consider the following stochastic competition and cooperation model of two enterprises

$$(3.1)\qquad \begin{cases} x_{1,k+1} = x_{1,k} + r_1 x_{1,k}\left[1 - \dfrac{x_{1,k}}{K} - \dfrac{\alpha(x_{2,k}-c_2)^2}{K}\right]\Delta t \\ \qquad\qquad + \sigma_1 x_{1,k}\left[1 - \dfrac{x_{1,k}}{K} - \dfrac{\alpha(x_{2,k}-c_2)^2}{K}\right]\sqrt{\Delta t}\xi_k \\ \qquad\qquad + \dfrac{\sigma_1^2}{2} x_{1,k}\left[1 - \dfrac{x_{1,k}}{K} - \dfrac{\alpha(x_{2,k}-c_2)^2}{K}\right](\xi_k^2 - 1)\Delta t, \\ x_{2,k+1} = x_{2,k} + r_2 x_{2,k}\left[1 - \dfrac{x_{2,k}}{K} - \dfrac{\beta(x_{1,k}-c_1)^2}{K}\right]\Delta t \\ \qquad\qquad + \sigma_2 x_{2,k}\left[1 - \dfrac{x_{2,k}}{K} - \dfrac{\beta(x_{1,k}-c_1)^2}{K}\right]\sqrt{\Delta t}\eta_k \\ \qquad\qquad + \dfrac{\sigma_2^2}{2} x_{2,k}\left[1 - \dfrac{x_{2,k}}{K} - \dfrac{\beta(x_{1,k}-c_1)^2}{K}\right](\eta_k^2 - 1)\Delta t, \end{cases}$$

where ξ_k and η_k are Gaussian random variables that follow $N(0,1)$. We choose $r_1 = 0.45, r_2 = 0.74, \alpha = 0.33, \beta = 0.63, K_1 = 2.4, k_2 = 1.7$. Let $\sigma_1^2 = 0.1, \sigma_2^2 = 0.3$. Then we can easily check that all the conditions of Theorem 2.2 are satisfied. Thus we can conclude that the equilibrium (x_1^*, x_2^*) is globally asymptotically stable which is illustrated in Figure 1 and Figure 2.

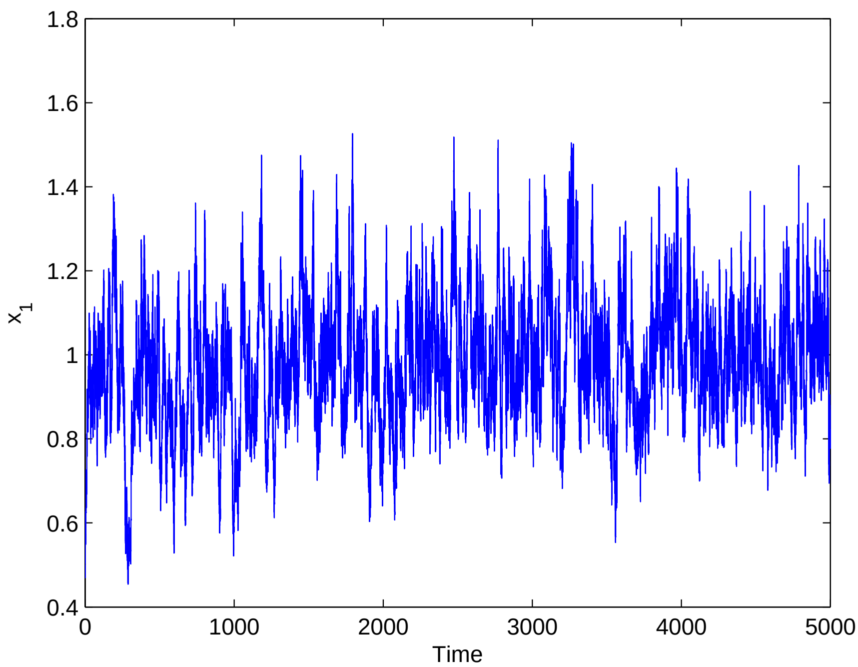

Figure 1: The solutions of system (3.1) with $\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.3$.

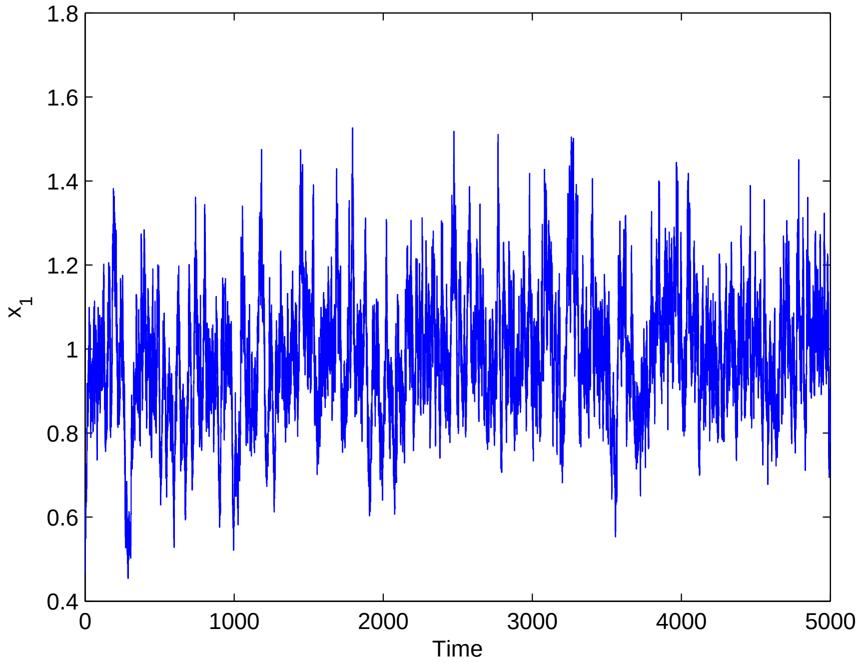

Figure 2: The solutions of system (3.1) with $\sigma_1^2 = 0.1$, $\sigma_2^2 = 0.3$.

4. Conclusions and further research

In this paper, we have investigated the global asymptotic stability of a stochastic competition and cooperation model of two enterprises. It is shown that under appropriate conditions, the stochastic competition and cooperation model has a unique global positive solutions for any initial values. Some sufficient conditions which guarantee the global stability of the stochastic model are derived. From the perspective of enterprise, the results play an prominent role in practical applications. The fact that a positive equilibrium is globally asymptotically stable implies that both enterprise could be co-exist. To the best of our knowledge, it is the first attempt to carry out such a investigation on competition and cooperation of a stochastic competition and cooperation model of two enterprises. In real society, we know that discontinuity often appears. Thus the investigation on the fractal competition and cooperation model of two enterprises has important value. This aspect will be our future work.

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