ENDOGENOUS CONTROL IN A TERNARY LOTKA-VOLTERRA MODEL AND ITS APPLICATIONS

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Abstract. This paper aims at highlighting the role of endogenous controls in Lotka-Volterra predator-prey models. Unlike other studies in which the core lies in expanding the number of conflicting species, here the stress is laid upon a control variable that does not need to be of the same nature as the species involved in the conflict. As in the case of the logistic equation Lotka-Volterra models have proved highly fruitful also outside the pure biological frame, clarifying many socio-economical and psychological evolution phenomena. A main problem lies in the change of the structural parameters, that can alter the evolution. Rather than an exterior action a model is most satisfactory in presence of an endogenous change, that should happen without any external intervention. The critical model is described in Section 2 under the name of tripartite antagonist model, where there are three actors, and three predator-prey equations involving, two at each time, the actors. The structure is that each actor once is predator and once prey in a circular scheme. In some natural cases the three variables are all biological, but in most meaningful cases the control has a different nature, usually social or behavioral. The paper highlights the particular case where the initial existence of a control, albeit at a very low level, is essential to allow the development towards equilibrium, while its absence leads to the destruction of the prey. A similar case arises when predator and prey are in temporary equilibrium, but a hidden control variable operates until a sudden change of equilibrium bursts out. Section 1 is devoted to recall some preliminary knowledge of system theory and of Lotka.Volterra classical equations. Section 3 is devoted to illustrate some actual cases where tripartite antagonist system can explain social, psychological, economic developments, even with reference to the academic world. The long lasting competition between Stoicism and Epicurism, between hedonism and Aristotle's eudaimonia fully enters in this scheme.

Keywords: evolution, tripartite antagonist model, predator-prey, ecology, eudaimonia, top-down and bottom-up controls.
AMS Mathematics Subject Classification: 34D23, 65P40, 92D15, 93C15, 93C40.

Introduction

The key word of this study is evolution, within a framework of endogenous dynamics. There are at least three cultural lines of thought: an evolution that tends towards a state of equilibrium in which transient phenomena gradually subside [1], an evolution that tends towards a final catastrophe, possibly followed by a regeneration, and a cyclic evolution with "twists and turns" in a perpetual coming and going of the story in both collective and individual terms. External phenomena cannot be ignored, but a good explanatory model aims at reducing the number of the exogenous variables that govern them. As soon as one steps away from the physical phenomena, the predictive value of the models becomes more limited (think of the econometric models), but still the ability to explain the trends and the possible structural transitions may survive. The tendency to make more realistic models adds new variables and new equations, making the models less and less controllable. On the other hand lies the tendency to seek the core of phenomena, with models that, beyond the odd accidental disturbance, make reason of their essential lines. This fact is important, since it allows the detection of situations of irreversibility and stability that may occur even in simple cases, so that the complexity of the system is not a necessary condition for their onset. The simpler models allow a mathematical treatment that is much more complete, since the mechanisms by which they operate can be discovered, whilst the more complex often end in a series of computer simulations that give no guarantee of being resolutive in situations of instability and bifurcation.

Lotka-Volterra models of the fight for life have become more and more popular because they can explain the kernel of many biological, social, economical, cultural evolution phenomena. From the first legendary papers of the 30's of last century, due to Lotka [21] and Volterra [48, 49], many developments have been performed. A complete structural analysis can be found in [37], while a beautiful experimental description is in [42]. A summary of those equations and their relations with stability forms part of chapter 5 of the book by Piccinini, Stampacchia, and Vidossich [34]. A wide book about the dynamics of population is [17]. An application to the themes of landscape is to be found in Piccinini-Taverna-Chang [35]. Up to date contributions are [25] and [16]. The present paper studies a different problem, namely not connected with the multiplication of conflicting species, but rather with the problem of controlling conflicts. The problem arose since the statement of Lotka-Volterra predator-prey model, and it was clear that it was enough to change the parameters in order to reach new equilibriums. But who could change the parameters? An exogenous change (*deus ex machina*) is obvious, and experimented since the ancient tragedy. What is required is an endogenous change, that should happen without any external intervention. Incidentally it must be observed that also the equation of exponential growth became Verhulst's (= logistic) equation transposing the control into the interior of the system. The

[1]The Hegelian thesis-antithesis-synthesis conceptual scheme enters this category when damping is moderately fast: not so fast as to prevent any antithesis, yet not so slow that multiple oscillations are required.

authors therefore considered a similar predator-prey system and inserted an endogenous control, according to a same predator-prey model, obtaining thus a tripartite antagonist system. The original theme was the evolution of touristic resorts and their limiting capabilities, and were dealt with in [31]. A similar model was applied to social sciences and fine arts in [30]. The authors decided that the model deserved to be analyzed from a mathematical point of view, highlighting the phenomena of hidden instability and checking some conditions under which they can arise.

The first section is devoted to Lotka-Volterra predator-prey system. In particular it recalls that also periodic linear systems may be interpreted as simplified Lotka-Volterra predator-prey models. The main objective is the construction of a tripartite antagonist system that glues together in a cycle three Lotka-Volterra linear systems. It will help understanding the main models of the second section since it represents a linear approximation for small non damped oscillations.

The second section is the original part where three Lotka-Volterra predator-prey systems are superimposed; formally all the variables have the same role, but it is shown that it is easy to consider two of them as conflicting analogous variables, while the third one may be considered as a control. It is no longer exogenous but becomes endogenous to the model and explains the possible outcomes according to the structural parameters and initial data. Many different situations may arise, but the most interesting are those in which in absence of control the prey vanishes, while if the control is present, albeit at a very low level, the equilibrium is finally achieved. Another interesting case is that where the prey survives (at a low level) even if the control is absent, but in presence of a control it finally jumps from an apparent equilibrium to a different final equilibrium.

The third section discusses the nature of the possible control in different real situations, distinguishing in particular the extreme cases of a top-down control (state, academy, school) or a bottom-up control (individual, public opinion, social nets), envisaging also meaningful mixed forms of control. Finally, Section 3 offers numerous examples of situations in which the tripartite antagonist model can be applied to life, behavior and well-being, both private and public.

1. From periodicity to Predator-Prey model

A first aim in this section is to show how periodic phenomena may arise and how damped oscillations lead to stability [2]. The section then recalls non linearity considering Verhulst's equation (logistic) and the two-variables predator-prey Lotka-Volterra equations.

1.1. From periodicity to stability

The simplest case is that of the non-forced harmonic oscillator, which satisfies the equation $y'' + \omega^2 y = 0$. It can be read as a model of the predator-prey type. In the case of biological balance, predator and prey numbers remain stable

[2] The experienced reader can skip the parts that he knows. He should just check the knowledge of system (10), that can form a "starred" exercise of system theory.

because the losses suffered by the prey are offset by its biological capability to reproduce, whilst the benefits gained by the predator are overshadowed by the limits of available prey.

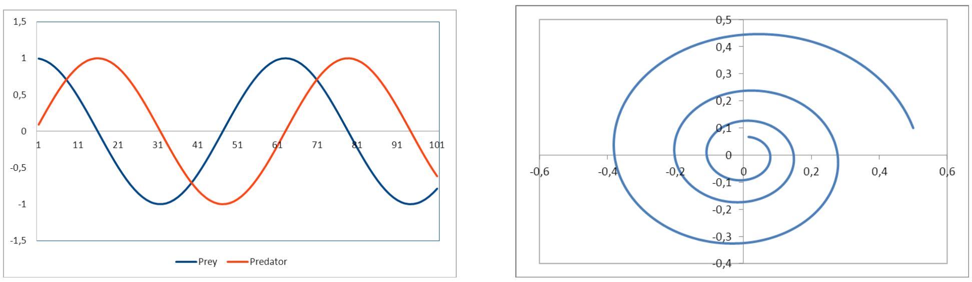
Let X be the abundance of prey, X° its equilibrium value, Y the abundance of the predator, Y° its equilibrium value. The easiest description is the following, which corresponds to the harmonic oscillator. The parameters m_{21} and m_{21} are assumed to be positive [3].

$$
\begin{aligned}
X' &= -\, m_{21}(Y - Y\circ) \\
Y' &= +\, m_{12}(X - X^\circ)
\end{aligned} \tag{1}
$$

The first equation states that prey decreases (negative derivative) when the abundance of the predator exceeds the equilibrium value ($Y > Y^\circ$) and then $Y - Y^\circ > 0$, whereas predator increases when the amount of prey exceeds the equilibrium value ($X > X^\circ$) and then $X - X^\circ > 0$. The general solution of the system is

$$
\begin{aligned}
X = &\quad X^\circ + R\sqrt{k}\cos(\omega(t-\phi)) \\
Y = &\quad Y^\circ + \tfrac{R}{\sqrt{k}}\sin(\omega(t-\phi))
\end{aligned}
$$

where $k^2 = \frac{m_{21}}{m_{12}}$ and $\omega^2 = m_{21}m_{12}$ with $m_{21}, m_{12} > 0$. It represents an ellipse with semiaxes $R\sqrt{k}$, $\frac{R}{\sqrt{k}}$ running counterclockwise at a constant angular velocity, ϕ is the phase, which depends on the initial position; R also depends on the initial position, while the parameters of the equilibrium position are exogenous. The essential requisite is that the second variable generates a negative effect on the first, while the first variable generates a positive effect on the second. When each variable exerts an effect of the same sign on the other cycles can no longer happen, and there is a trend either to equilibrium or to instability. For a general theory of systems of linear differential equations, please refer to the specific texts, such as [38] and [6].



(a) The phase shift between two periodic variables

(b) Phase space for damped oscillations

Figure 1: Periodicity and stability

An important feature of system (1) is the shift between the two variables that can be observed in Figure 1(a) (centered in $(0, 0)$ and is to be compared with the symmetric behavior of the tripartite antagonist model of Section 1.2.

[3]Lotka-Volterra equations are more complicated than the elementary harmonic oscillator, in view of their nonlinearity, but approaching the equilibrium this is a good approximation allowing the substance of the phenomenon to be grasped.

The previous systems have periodic solutions, centered around an exogenous equilibrium point, but they do not converge to it. A stabilizing action may arise from the change of the structural parameters of the system of equations, in order to bring it back to the damped oscillations around the value of final equilibrium, regardless of the initial situation. Such systems are called asymptotically stable. In this case the trajectories become spirals converging towards equilibrium. A system that is asymptotically stable can have this form

$$\begin{aligned} X' &= -\,a(X - X^\circ) - m_{21}(Y - Y^\circ) \\ Y' &= m_{12}(X - X^\circ) - b(Y - Y^\circ) \end{aligned}$$

where the positive constants a and b are the intrinsic rates at which the variables X and Y tend to shrink to their reference values X° and Y°.

Fig. 1(b) shows the spiral trajectories that arise with this type of parameters (as long as $(a - b)^2 < 4m_{12}m_{21}$).

1.2. Linear tripartite antagonist model

The models with three actors provide a more complex picture, which will become very rich in the nonlinear case of the second section. Suppose that there are three variables X, Y, and C[4] that satisfy two by two a predator-prey system. For simplicity, we set $X^\circ = Y^\circ = C^\circ = 0$. A tripartite antagonist situation is one in which each variable is once predator and once prey; adding symmetry conditions, the following three systems of equations arise (we recall that $m_{ij} > 0$)

(2a) $$X' = -2m_{12}Y \qquad Y' = 2m_{21}X$$

(2b) $$Y' = -2m_{23}C \qquad C' = 2m_{32}Y$$

(2c) $$C' = -2m_{31}X \qquad X' = 2m_{13}C$$

Each system separately would lead to the harmonic oscillator of equation (1). When the effects overlap we arrive at a system of three equations

(3) $$\begin{aligned} X' &= -m_{12}Y + m_{13}C \\ Y' &= -m_{23}C + m_{21}X \\ C' &= -m_{31}X + m_{32}Y \end{aligned}$$

Even if each equation (2) entrains periodic solutions, (3) may have non periodic unstable solutions. Let M the matrix of coefficients of the system, i.e.

(4) $$M = \begin{bmatrix} 0 & -m_{12} & m_{13} \\ m_{21} & 0 & -m_{23} \\ -m_{31} & m_{32} & 0 \end{bmatrix}$$

Let us remember that if $\boldsymbol{Z(t)} = (X(t), Y(t), C(t))$, $\boldsymbol{Z(t)} = exp(Mt)\boldsymbol{Z^\circ}$ is the solution of the system $\boldsymbol{Z'} = M\boldsymbol{Z}$, with $\boldsymbol{Z(0)} = \boldsymbol{Z^\circ}$ ([34] cap. II, 2 and II, 3, see also [20]). The characteristic equation becomes

(5) $$\lambda^3 + (m_{12}m_{21} + m_{23}m_{32} + m_{31}m_{13})\lambda + m_{12}m_{23}m_{31} - m_{13}m_{12}m_{32} = 0$$

[4] C is used for analogy to Section 2, where it denotes the control variable.

An interesting case is given by the uniformity condition, $m_{12} = m_{23} = m_{31}$ and $m_{13} = m_{21} = m_{32}$, since the uniformity condition is preserved in the exponential, for example starting from

$$M = \begin{bmatrix} 0 & -3 & 2 \\ 2 & 0 & -3 \\ -3 & 2 & 0 \end{bmatrix} \tag{6}$$

the characteristic equation (5) becomes

$$\lambda^3 + 18\lambda + 19 = 0$$

with eigenvalues -1, $\frac{1}{2} \pm i\,\frac{5\sqrt{3}}{2}$. Hence

$$exp(Mt) = \begin{bmatrix} f & g & h \\ h & f & g \\ g & h & f \end{bmatrix} \tag{7}$$

where

$$f = \exp(-t)/3 + 2/3\exp(t/2)\cos(Ht)$$
$$g = \exp(-t)/3 - 1/3\exp(t/2)\cos(Ht) - 5/2\exp(t/2)sen(Ht)/Ht$$
$$h = \exp(-t)/3 - 1/3\exp(t/2)\cos(Ht) + 5/2\exp(t/2)sen(Ht)/Ht$$

with $H = 5\sqrt{3}/2$. Compare Fig. 2

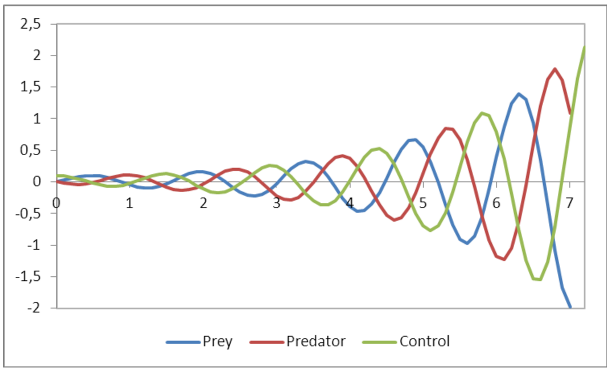


Figure 2: Time sheet of a uniform unstable system

In the symmetric situation when $m_{12}m_{23}m_{31} = m_{13}m_{12}m_{32}$, determinant is 0 and there is stability. For example, $m_{12} = m_{21} = 2$ and $m_{23} = m_{13} = m_{31} = m_{32} = 1$, lead to

$$M = \begin{bmatrix} 0 & -2 & 1 \\ 2 & 0 & -1 \\ -1 & 1 & 0 \end{bmatrix} \tag{8}$$

and $\exp(Mt)$ becomes

$$\begin{bmatrix} 1+5\cos(\sqrt{6}t) & 1-\cos(\sqrt{6}t)-2\sqrt{6}\sin(\sqrt{6}t) & 2-2\cos(\sqrt{6}t)+\sqrt{6}\sin(\sqrt{6}t) \\ 1-\cos(\sqrt{6}t)-2\sqrt{6}\sin(\sqrt{6}t) & 1+5\cos(\sqrt{6}t) & 2-2\cos(\sqrt{6}t)-\sqrt{6}\sin(\sqrt{6}t) \\ 2-2\cos(\sqrt{6}t)-\sqrt{6}\sin(\sqrt{6}t) & 2-2\cos(\sqrt{6}t)+\sqrt{6}\sin(\sqrt{6}t) & 4+2\cos(\sqrt{6}t) \end{bmatrix}$$

For the time sheet compare Fig. 3

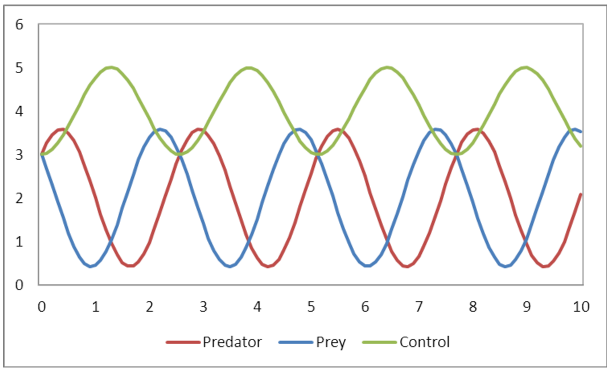


Figure 3: Time sheet of a symmetric non uniform system

A uniform symmetric system arises when $m_{12} = m_{21} = m_{23} = m_{13} = m_{31} = m_{32} = \omega$, the system is

$$\begin{aligned} X' &= \omega(-Y + C) \\ Y' &= \omega(-C + X) \\ C' &= \omega(-X + Y) \end{aligned} \tag{9}$$

The matrix M is

$$M = \begin{bmatrix} 0 & -\omega & \omega \\ \omega & 0 & -\omega \\ -\omega & \omega & 0 \end{bmatrix} \tag{10}$$

and the exponential matrix has the form (7) where

$$\begin{aligned} f &= 1 + 2/3\cos(\sqrt{3}\omega t) \\ g &= 1 - \cos(\sqrt{3}\omega t)/3 - \sqrt{3}sen(\sqrt{3}\omega t)/3 \\ h &= 1 - \cos(\sqrt{3}\omega t)/3 + \sqrt{3}sen(\sqrt{3}\omega t)/3 \end{aligned}$$

Observe the uniform shift of trajectories, as shown in Fig. 4, to be compared with Fig. 1(a).

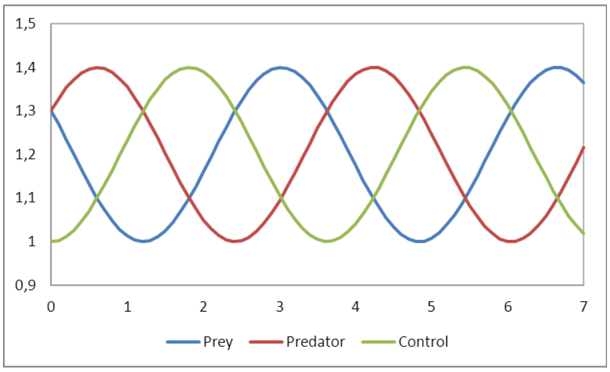


Figure 4: Time sheet of a symmetric uniform system

Some remarks for (9)

• There is a nonzero constant solution, given by any set $X = Y = C = A$, while in separate equations (2) only the constant solutions $X = Y = C = 0$ exists.

• By adding the three equations member by member it follows that

$$(X + Y + C)' = X' + Y' + C' = 0 \tag{11}$$

and multiplying the first for X, the second for Y and the third for C and summing up it follows that

$$(X^2 + Y^2 + C^2)' = 2(XX' + YY' + CC') = 0. \tag{12}$$

so that the trajectories of system (10) are circumferences that lie in a plane perpendicular to the vector $(1, 1, 1)$.

Therefore, a more geometrical representation is obtained using the three independent solutions

$$\begin{aligned}(X_1, Y_1, C_1) =&(1, 1, 1)\\(X_2, Y_2, C_2) =&(\cos(\sqrt{3}\omega t), \cos(\sqrt{3}\omega t - \frac{2\pi}{3}), \cos(\sqrt{3}\omega t - \frac{4\pi}{3}))\\(X_3, Y_3, C_3) =&(\sin(\sqrt{3}\omega t), \sin(\sqrt{3}\omega t - \frac{2\pi}{3}), \sin(\sqrt{3}\omega t - \frac{4\pi}{3}))\end{aligned}$$

by which the general solution, depending on the data A, R, ω, is generated:

$$\begin{aligned}X =&A + R\cos(\sqrt{3}\omega t - \phi),\\Y =&A + R\cos(\sqrt{3}\omega t - \frac{2\pi}{3} - \phi),\\C =&A + R\cos(\sqrt{3}\omega t - \frac{4\pi}{3} - \phi).\end{aligned}$$

1.3. From Verhulst to Lotka-Volterra equations

A significant example of a non-linear model is given by the equation of Verhulst, better known as logistic equation. Its origin is the following: the first hypothesis on the growth of a population in the absence of any constraint is that it evolves according to the exponential equation $P' = kP$, where k represents the coefficient of growth or decrease according to its sign. The solution is given by $P(t) = P(0)\exp(kt)$, which tends to infinity if $k > 0$.

The salient point of Verhulst's model is the belief that the growth rate of the population does not remain constant, but rather decreases with the increase of the population (in relation to the environment available), and may become negative in case of overcrowding. The simplest formulation implies a linear dependence descending around the equilibrium level C (called endogenous carrying capability), i.e., $k = v(C-P)$. With this modification, the exponential equation is transformed into the equation of Verhulst

$$P' = v(C - P)P \tag{13}$$

where all the parameters have already been lumped together in the constants v and C. The equation (13) has the important feature of possessing a solution of asymptotic equilibrium, given by $P = C$. The evolution governed by equation (13), if the initial data are below the threshold value C, presents a trend of sigmoid curves, increasing asymptotically to the value C, whereas if the initial data are higher than this value, the trend is decreasing asymptotically towards C_1 [5]. This equation is of considerable interest because it presents a level of stable equilibrium, that is to say that even if the system is disturbed (for example by an epidemic), it will tend to return to the equilibrium state provided that the structural parameters have not been changed. The logistic equation was originally referred to the evolution of biological species, and was then applied to many social and economic phenomena, in particular the spread of epidemics and of fashions.

The extension to multiple components present in the same area took place about a hundred years later, and is mainly due, independently, to Volterra [48, 49] and Lotka [21]. The predator-prey model in particular is the frame for the subsequent discussion. Its standard form may be written as

$$\begin{aligned} X' &= uX(C - X - hY) \\ Y' &= vY(D + kX - Y), \end{aligned} \tag{14}$$

where u and v are positive constants representing the speed of adjustment of the two components, h and k are positive constants. The logistic evolution of the X component is damaged (according to h) by the presence of the Y component (and therefore X is the "prey"), while the Y component is advantaged (according to k) by the presence of the species X (and then is the "predator"). The carrying

[5] The corresponding finite difference equation has a similar behavior when parameters are small enough, but as they increase it shows a set of periodic solutions, then it degenerates into a chaotic behavior (compare [33], ch 6.4)

capacities C and D are those that would separately hold for each component should the other component be absent [6]. The situation of equilibrium in presence of both species is given by the solution of the linear system

$$\begin{aligned} X + hY &= C \\ -kX + Y &= D \end{aligned} \tag{15}$$

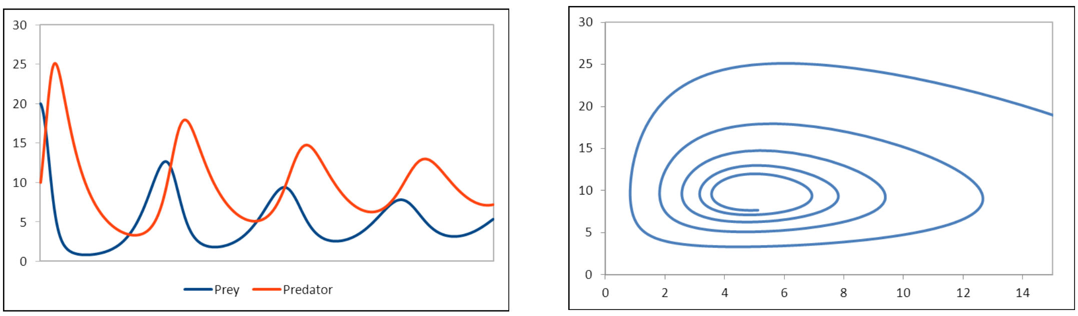
namely

$$\begin{aligned} X &= (C - hD)/(1 + hk) \\ Y &= (kC + D)/(1 + hk). \end{aligned} \tag{16}$$

The existence of equilibrium requires that the solutions (16) are both positive, otherwise the only possible solution is the disappearance of one of the two components. The evolution of the system involves oscillations around the equilibrium, damped according to the magnitude of the speed of adaptation. Also in this case, as already in the logistic equation, the values of equilibrium do not depend on the initial situation.

If one of the two variables is a candidate for disappearance, it is usually the prey, provided the survival of the predator does not depend on it alone. However, if the model predicts that D, the carrying capability of the predator, is negative then it follows that the predator component can exist only in the presence of the prey component, since the essential resource for the development of the predator is that very prey. In this case, the oscillations around equilibrium are showier and more persistent. Fig. 2 show precisely this case, and respectively represent the evolution in phase space and the laws of temporal evolution, when the values of the parameters are $u = v = 0.001$, $C = 100$, $D = -40$, $h = k = 10$. The balance is achieved when $X = 4.95...$, $Y = 9.505...$.

Figure 5(a) shows in particular the shift between the maximum of the prey and that of the predator, similar to what was seen in Fig. 1 for the linear system (1).



(a) Phase space of predator-prey model (b) Time sheet of predator-prey model

Figure 5: Predator-prey model

[6] In other presentations there is a constant before X in the first equation and Y in the second, but so the role of C and D is no longer that of carrying capacity.

A very different case arises when the coefficient k becomes negative, because then the two components have a symmetrical role, so that each component damages the other. This situation of conflict no longer presents oscillatory phenomena, even if also in this case a rich choice of possible exits arises. Refer to [34] pp. 365-368 for further details, and to [35] for some special cases of multiple equilibria.

2. Lotka-Volterra tripartite antagonist models

2.1. Open and closed chains

The models with multiple components are very articulate, and have been widely proposed over the last fifty years. In many cases, the studies have examined how the main structural features change as the parameters vary, but usually all the components are biological species, and none of them have an essentially different role. Multiple models mostly represent open chains, where a lattice hierarchical sequence is available. A new articulated and far reaching model involving human and natured dynamics has recentrly been presented by Motesharrei, Rivas, and Kalnay in [25]. In what follows, on the contrary, a closed (i.e., circular) model will be presented.

Lotka-Volterra models deal with the competition between two or more species, but usually there is no endogenous control. In this section, we construct a model with three components, where two variables correspond to biological species (prey and predator) and the third corresponds to a control that is not necessarily biological, but endogenously arises from biological evolution. To the authors this seems to be the simplest model where the role of control is interior to the system. The model is apt to describe the competition between the prey) and the predator, in presence of a control.

2.2. Non linear tripartite antagonist model

There are three actors on the stage: the prey, the predator, and the possibility of a control. X is the perceived value of the prey, Y is the the level of predator, C the control capability. We will assume that each of these variables evolves according to logistics, i.e. self-sustaining until the respective level of equilibrium $(X^\circ, Y^\circ, C^\circ)$ is attained [7]. Suppose also that the evolution of each variable is conditioned by the level of the other two, dominant with respect to one of them, inferior to the other, in a cyclic structure depending on the m_{hk} positive coefficients. It corresponds and generalizes the linear model (3). The model will be called "tripartite antagonist model". We interpret the conceptual schema:

The prey is the fundamental resource, but is continually debased by the predator, while it is protected by the ability to control that can slow the degeneration caused by the thirst for profit.

[7]Some carrying capacities may be negative, but this does not seem particularly consistent with the meaning of the model.

The predator is self-powered ("appetite comes with eating") and generally tends to draw on the resources of the prey; it is moderated by control and critical consciousness.

The control tend to spread, but is actually deterred by the excess of the prey, which dulls and trivializes it. It feeds much more forcefully on the antagonism against the ugly caused by obtuse profit, and in any case the profit tends to donate a portion of its benefit for saving a part of prey

The tripartite antagonist model takes the following (general) form:

$$\begin{aligned} X' &= uX(X^\circ - X - m_{12}Y + m_{13}C) \\ Y' &= vY(Y^\circ + m_{21}X - Y - m_{23}C) \\ C' &= wC(C^\circ - m_{31}X + m_{32}Y - C) \end{aligned} \tag{17}$$

Many different equilibrium situations may arise, but the most interesting is achieved when the master system

$$\begin{aligned} X^\circ - X^* - m_{12}Y^* + m_{13}C^* = 0 \\ Y^\circ + m_{21}X^* - Y^* - m_{23}C^* = 0 \\ C^\circ - m_{31}X^* + m_{32}Y^* - C^* = 0 \end{aligned} \tag{18}$$

has positive solutions, what implies asymptotical stability with coexistence of all the variables. The reduced systems are obtained when a variable is absent, so that the master system shrinks to a pure predator-prey model

$$X^\circ - \hat{X} - m_{12}\hat{Y} = 0, \qquad Y^\circ + m_{21}\hat{X} - \hat{Y} = 0, \qquad \hat{C} = 0 \tag{19a}$$

$$Y^\circ - \hat{Y} - m_{23}\hat{C} = 0, \qquad C^\circ + m_{32}\hat{Y} - \hat{C} = 0, \qquad \hat{X} = 0 \tag{19b}$$

$$C^\circ - m_{31}\hat{X} - \hat{C} = 0, \qquad X^\circ - \hat{X} + m_{13}\hat{C} = 0, \qquad \hat{Y} = 0 \tag{19c}$$

Positive solutions are conditionally stable solutions, that is solutions that remain stable provided the lacking variable is not allowed to become positive. The most meaningful case of the tripartite antagonist model arises when there is no conditional stability for (19a), while asymptotic stability of the master system exists, what means that without any endogenous control the prey vanishes. Remark that in some unbalanced cases, when a positive solution of the master system does not exist, one of systems (19) may have an asymptotically stable solution, otherwise this is to be looked for in one of the one-variable logistics; in any case these solutions are at least conditionally stable if the two remaining variables are constrained to 0.

Adding conditions to the structural parameters in system (17), it becomes possible to find sufficient conditions for the solution of the master system (18) to be positive. A first hypothesis is the condition of symmetry $m_{23} = m_{32} = b, m_{31} = m_{13} = e, m_{12} = m_{21} = c$. If $b = e = c(= a)$. The system is uniform and becomes conservative, namely

$$X^* + Y^* + C^* = X^\circ + Y^\circ + C^\circ$$

since $X^* = [X^\circ(1+a^2)+Y^\circ(a^2-a)+C^\circ(a^2+a)]/D, Y^* = [Y^\circ(1+a^2)+C^\circ(a^2-a)+X^\circ(a^2+a)]/D, C^* = [C^\circ(1+a^2)+X^\circ(a^2-a)+Y^\circ(a^2+a)]/D$, where $D = 1+3a^2$. The condition does not imply the existence of positive solutions as it can be seen by the example $X^\circ = 1$, $Y^\circ = 8$, $C^\circ = 1$, $a = 0,5$, where $X^* = 0$, $Y^* = 6$, $C^* = 4$. This case arises from the coexistence of an unbalanced structure of carrying capacities joined with small interactions. The positivity condition is obviously unconditionally satisfied in case of strong interrelation, that is $a \geq 1$. An important structural remark is that, on the contrary, for reduced systems (19) the positivity condition is $a < 1$.

We say that the system is balanced if $X^\circ = Y^\circ = C^\circ (= A)$. If the system is balanced and symmetric it is sub-conservative, namely it holds

$$X^* + Y^* + C^* \leq X^\circ + Y^\circ + C^\circ$$

and equality holds only for conservative systems. Remark that a non positive solution could still arise when one of the interrelations is much greater than 1 and the others are much smaller than 1. For example $c = 2, b = 0,2, e = 0,2, A = 1$ leads to $X^* = -6.3.., Y^* = 64.6.., C^* = 114.1...$ On the contrary, a symmetric system that is balanced and uniform has always positive solutions since they coincide exactly with the carrying capacity A. Remark that the speeds of adjustment (u, v, w in (17)) do not modify the equilibrium position. This simplified situation allows to detect interesting delay phenomena that arise in the study of real models.

2.3. Critical tripartite antagonist models

The main model corresponds to the linear model (10). Here $A = 1, a = 1$, time is normalized so that $u = v = w = 1$.

$$\begin{aligned} X' &= X(1 - X - Y + C) \\ Y' &= Y(1 + X - Y - C) \\ C' &= C(1 - X + Y - C) \end{aligned} \tag{20}$$

The asymptotically stable equilibrium is $X^* = Y^* = C^* = 1$. However, as in all Lotka-Volterra systems, there are solutions that provide for the elimination of some components. With the parameter values of (20), there are no final solutions with two nonzero components, while there are those with only one nonzero component. This case occurs when a component is absent since the initial time (and then remains definitively null). The model then becomes a predator-prey model, with the disappearance of the prey:

- If $X(0) = 0$, $Y(0) > 0$, $C(0) > 0$, the system arrives at $C = 1$, $X = Y = 0$. The predator is destroyed by an excess of control, and is not compensated by the prey.
- If $C(0) = 0$, $X(0) > 0$, $Y(0) > 0$, the system arrives at $Y = 1$, $X = C = 0$. The predator destroys the prey because of the absence of control. The result is destructive and entrains insane development at the expense of the prey.

- If $Y(0) = 0$, $X(0) > 0$, $C(0) > 0$, the system arrives at $X = 1$, $Y = C = 0$. A dreamy prey eliminates the need for controls in a model that is seemingly idyllic, but is born in the absence of prey (golden Age): ataraxic in the contemplation of unproductive prey.

However, if each element is present, albeit at a very small, but positive level, it will be able to bear fruit and rebalance the system toward the global situation. The presence-absence of a component is then a bifurcation point.

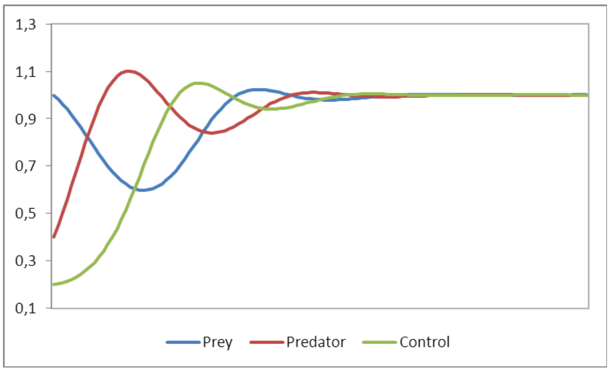
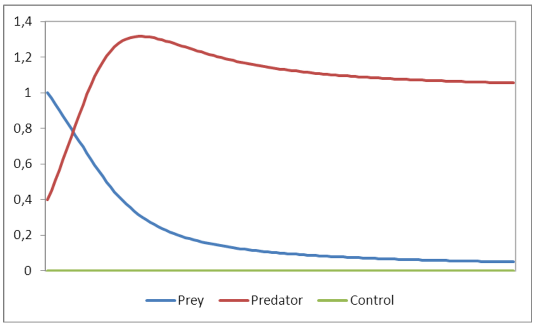
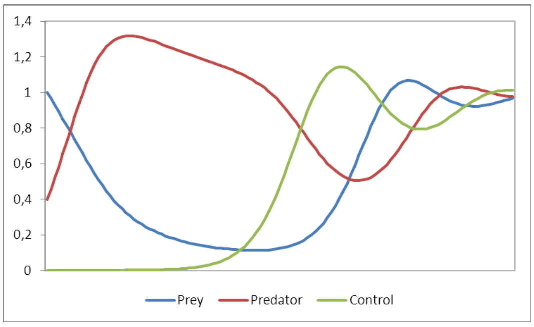


Figure 6: Evolution of the tripartite antagonist system (20)

Fig. 6 shows the evolution of a system in which the component C (Control) starts from a lower level than the other two and then rises to the equilibrium level. Fig. 7(a) instead shows the evolution in the case of total absence of the C component: X (Prey) tends to disappear, despite starting from a higher value than Y (Predator).



(a) Tripartite antagonist system with degenerate initial data (system (20), C $(0) = 0$)



(b) Tripartite antagonist delayed system ((20), $C(0) = 0.001$)

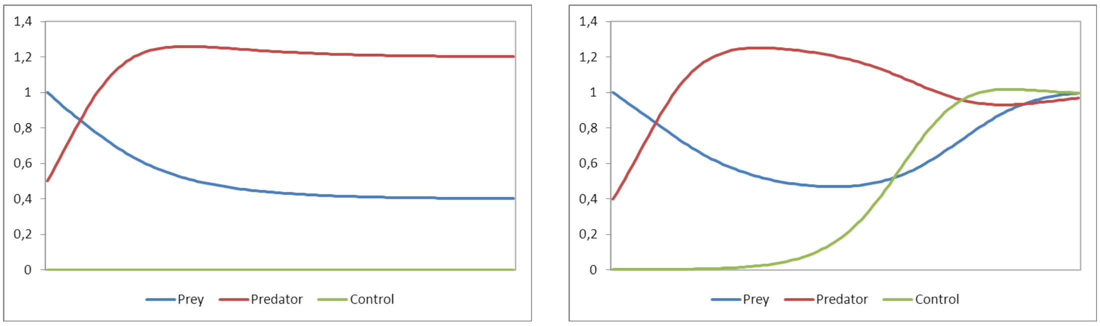
Figure 7: Degeneration or final equilibrium?

This fact seems paradoxical, since the initial presence of a minimum quantity of the variable C is sufficient in order to eventually reach the equilibrium, albeit after some fluctuations of no small importance. Fig. 7(b) shows a case in which the initial level of control is 0.001. The initial trend is similar to Fig. 7(a), but later the presence of control becomes significant and leads to equilibrium.

In the case of system (20) such extreme situations occur because of the high level of the interaction coefficients between the variables. Should coefficients be lower, the effects are not so obvious, and in particular there might be no bifurcation given by the presence-absence of a variable. It is in fact sufficient that the interaction coefficients fall below 1 to eliminate the phenomenon. An example is the following system (halved coefficients).

$$\begin{aligned} X' &= X(1 - X - 0.5Y + 0.5C) \\ Y' &= Y(1 + 0.5X - Y - 0.5C) \\ C' &= C(1 - 0.5X + 0.5Y - C) \end{aligned} \tag{21}$$

If $C(0) = 0$, the absence of control will be maintained all the time, however, the variables X and Y will continue to exist, even if the limit value $X(= 0.4)$ is much lower than that of $Y(= 1.2)$, compare Fig. 8(a). If the system proceeded from a positive value of C, though very close to 0, the limit value would be, as in the other cases, 1 for each of the variables. This apparent bifurcation at 0 should in fact be read as an extension of the time required to reach equilibrium, becoming more pronounced as the initial values are close to 0, and diverging to infinity at 0.



(a) Antagoniste stable tripartite system (21)

(b) Sudden change of equilibrium (system (21))

Figure 8: Which equilibrium?

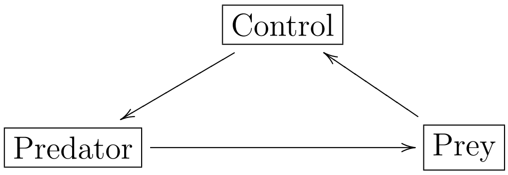
Even in model (21) the only stable equilibrium is $X = C = Y = 1$, for which balance disturbances lead back to the same final situation. All other equilibrium positions are not stable, since a small deviation from the value of unstable equilibrium may deflect towards a new equilibrium position. The situation $X = 0.4$, $C = 0$, $Y = 1.2$ is unstable, as a deviation of C towards a positive value leads to stable equilibrium, whilst the subsystem in which $C(0) = 0$ has a stable equilibrium position when it is restricted to X and Y components. This is very interesting because it means that there are cases of apparent stable equilibrium in which even a small perturbation of the exogenous control (fashion or awareness, or ecological consciousness) can lead to pronounced changes in the overall situation, albeit over the course of time.

Remark that if $C(0) > 0$, but very small, a long transition period can be expected, where the solutions initially tend to an unstable equilibrium and then suddenly pass to the stable equilibrium as shown in fig 8(b).

3. Examples of interpretation of the tripartite antagonist model

The abstract tripartite antagonist model stated in equations (17) can now be applied to more realistic situations. For brevity, we will use the symbol

(22)

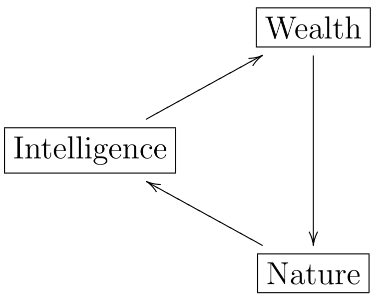


to indicate the circular chain where $X$is the predator of Y. C is the control and acts as a predator of X. Y is the predator of C, in the sense that the presence of C allows the increase of Y, while it is the lack of Y that generates the need for C, and on the contrary a strong presence of Y makes it useless. It can be summarized that the endogenous control responds positively or negatively to the unbalance between predator and prey. The examples are sorted according to the nature of the control: top-down (public control, legislation, school, rule), or bottom-up (social networks, market, ecology, individual freedom). There are cases of mixed control where the two forms coexist or succeed one another over time from private to public, such as political consensus, or from public to private through education campaigns and awareness raising. In the case of top-down control the prey is some form of social good and the predator is some form of private interest or benefit, while in the opposite case the prey is some form of individual liberty and the predator is the constriction of the rules and of the lack of privacy. We start with three mixed control systems

3.1. Mixed control

Many instances of the basic model (20) (more generally system (17)) are represented by the formula

(23)



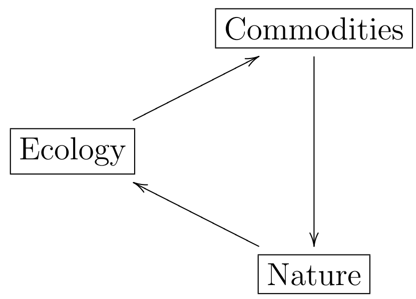
Here nature is the fundamental resource that needs to be saved. Non reproducible resources end up in their vanishing, but also reproducible resources (like agriculture and cattle breeding) require some kind of balance otherwise they reach lower levels of equilibrium. Intelligence cannot mean short period maximization of revenues and must be connected with medium term temporal horizon. Intelligence is strengthened by wealth, both as a reaction and as a perfunctory good supported by the very wealth.

Remark that under the parameters stated in (20) without intelligence nature vanishes, hence wealth will be totally artificial. The result is destructive and entrains insane development at the expense of the environment and its beauty: this is a risk well recognized by ecologists and nowadays also by many stakeholders. What is somewhat paradoxical is that in the absence of nature also wealth will suffer or vanish while the "intelligent" control will become overwhelming. The economic sustainability is destroyed by an excess of control, and is not compensated by natural and artistic resources. In the absence of wealth intelligence becomes needless, hence a dreamy natural beauty eliminates the need for controls in a model that is seemingly idyllic, but is born in the absence of productive instances (golden Age): ataraxic in the contemplation of unproductive beauty.

Provided we replace nature by beauty, the model may easily be applied with reference to resorts and also to many forms of artistic production such as literature, figurative arts, music, movies. The art, for being readable by a wide audience, must humanize, otherwise it remains closed within the ivory tower of critics. So there is a broad fuzzy band of uncertainty between art and bad taste (compare Eco [11]) , and also this band, in turn, changes with time. Often the remains of bad taste become extremely demanded, as was the case with Art Nouveau or neo-medieval reconstructions. Here beauty synthesizes art, creativity, originality; wealth distribution and sale to many users (to be motivated and convinced); intelligence critical skills and reaction to mass production: the tripartite model recommends not indulging in the mannerism of uncritical reproduction of what people like and can easily be sold, and at the same time recognizes the need for a certain amount of bad taste to stimulate innovation and creativity, since the reaction to decadence fertilizes minds more than the mere imitation of the beautiful, when devoid of novelty.

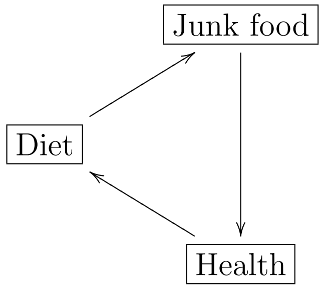
A scheme similar to (23) holds nowadays for the food. The role nature contains also biodiversity, tradition; that of wealth means production/distribution of commodities (also acquired by biotechnology), loss of biodiversity, slavery of multinational corporations; that of intelligence involves ecological private and public consciousness. The model is therefore

(24)



or its more personal variant

(25)



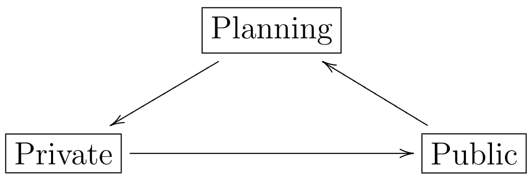
This model expounds that the difference in the speed with which different groups of consumers have reacted to the mass production of commodities is due to the dose of ecology present at the outset. The evolution towards equilibrium is endogenous, and is not only due to an imported mode. It should be noted, moreover, that the model affirms the necessity of commodities, otherwise the regression to a state of paleo-naturalness would lead to starvation as well as to the loss of ecological consciousness (as indeed it has for many thousands of years).

3.2. Top-down control

We now turn to models with top-down control. When systems are troubled by overly individualistic subsystems that tend to degenerate into self-interest and chaos [8], the key is to restore order. A decision maker (unit or team) is required. Other situations of conflict arise when decisions are taken with contrasting temporal horizons (often private prefers short period, while public can afford medium period) or when a solution with low return and high probability is preferred to a solution of high return and low probability (game theory with all its paradoxes).

The basic model is that of the town-planning scheme, where the public interest is undermined by the interests of the private sector. The control is exercised by the planning scheme.

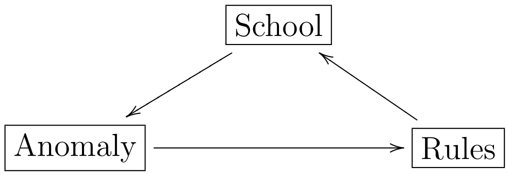
(26)



A variant of the model (26) may be found in semiotics and linguistics, where the anomaly of particularism tends to destroy the established order and the rules of analogy. Control instruments include the school, the academy, and the media: remark that nowadays Google has acquired an almost monopolistic position.

[8]A reference to Hirschmann's work ([15]) is essential, while the transitions between various states have been widely discussed in [29].

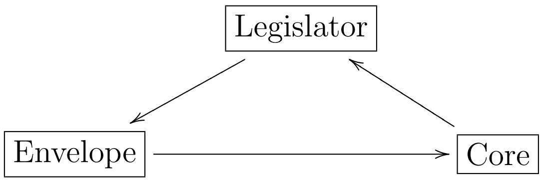
(27)



The organization of most forms of structured education obey this scheme. It can be remarked that also leisure, tourism [9], and environment discovery are covered by this scheme, where the role of the school is carried out by the various forms of guidance (book, network, organizer, personal trainer); a user's autonomous position could be envisaged in a generalized model (32). Also the alimentary education and its economic spin-off according to [9] can belong to scheme (27).

A rich model in this section is the Kernel model. The authors took the definition from the book by Lorch [20] where he explained that his is a kernel not a hull presentation, what made his book a lifelong companion for functional analysis. In any form of network, both physical and conceptual, there is the contrast between the core of the system and its developments and additions (envelope), so that some legislator [10] must govern it. The model is as follows

(28)



In particular, in the absence of a legislator the core becomes unreadable and unrecognizable due to the complexity of the accretions. In the teaching the over determination typical of most textbooks is overcome by good teachers that play the role of a legislator. In the case of networks a rearrangement was attempted, at least at the cognitive level, by Barabasi ([1], but had no consequences. In literature we can recall the wonderful (short) story of Calvino in Invisible Cities ([5]) where Ersilia, fourth among the "Town and exchanges", is characterized by a network of colored wires that expresses all the relationships in the city: when the network becomes incomprehensible the city is abandoned, and the new city begins with the core of a new network. Encyclopedic knowledge tends to destroy the structured knowledge and diagrams recall Calvino's cities. In [2] this defect is attributed to the absence of a clear conceptual scheme, according to the scientific statements of [24] and [26].

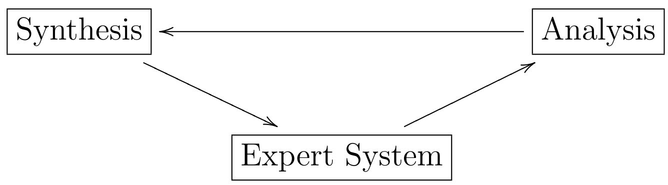
[9]The organization and falsification of touristic "authenticity" is considered in [22] where typical shows and trips *ad usum delphini* are described and their didactic necessity is explained. Ejarque in [12] follows a similar path, applying it to the artificial development of a tourist city.

[10]This is the model of Corpus Iuris due to Emperor Justinian, who, according to Dante's Divine Comedy (see [8], Par. 6, 12), remarks: "*I pruned the law of waste, excess, and sham*".

3.3. Bottom-up control

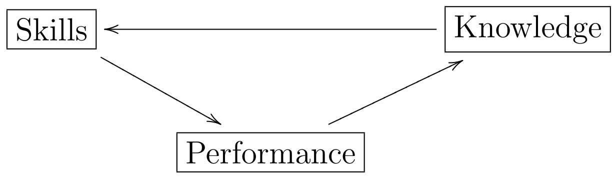
The core model (28) has some consequences as was pointed out by one of the fathers of artificial intelligence, namely the Nobel laureate H. Simon, as he recalls in his autobiography ([45]) with reference to [44]. Actually he says academy prefers teaching analysis, where it is possible to state sound basis and methods that can be learnt by the pupils, rather than proposing synthesis, where rules are fuzzy. Synthesis is left to hospital training, to projects, to auditing, in general to non structured science practical. The solution lies in artificial intelligence (today we prefer to say "expert systems"), where the need for explicit rules bursts out. Of course not every situation can be solved [11], but a good expert system that works at 95% nowadays is usually accepted, since the supervision requires much less time than starting from scratch. Typical cases are pattern and voice recognition and translation between different languages. Simon's scheme can thus be expressed as

(29)



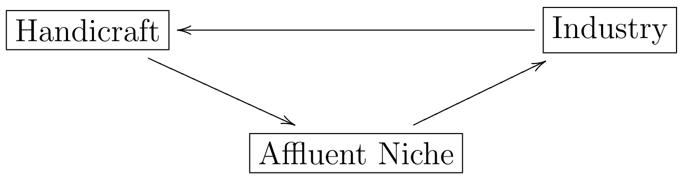
General purpose everyday activities require skills ready for use, so that theoretical knowledge is considered a hindrance rather than a help [12].

(30)



A somewhat similar effect is achieved in production, where coasty skills may be saved only in a market niche of affluent customers:

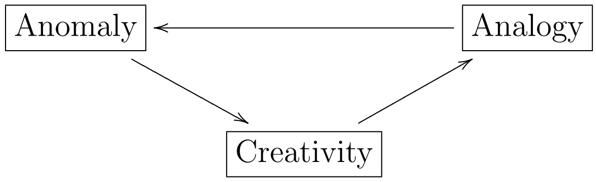
(31)



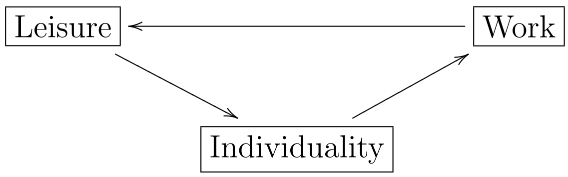
[11] This is in accordance with phenomenological psychology in comparison with cognitivism, as pointed by Vicario ([47])

[12] That is true also in mathematics. Vicario [46] reported of an experiment of Kanisza at the University of Trieste: a group of students were shown examples of solutions of some problems, a second group were taught the underlying theory. In the presence of a new similar problem the students of the first group easily succeeded while in the second group very few could understand what to do (and this was expected). But there was a third group that were shown the examples and then taught the theory. Instead of finding the best solution, the third group scored worse than the purely practical group, even if better than the theoretical group.

We come finally to the reaction of liberty against the rules that sterilize it. The model is that of subjective creation. Creativity enhances the anomaly in the conformist and well-ordered world of rule and analogy.

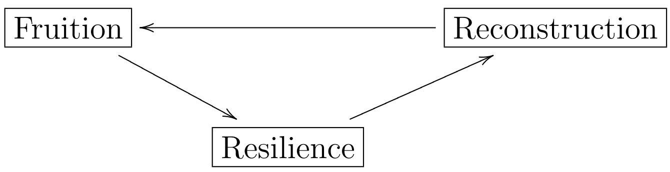
(32)


A model essential for a balanced life is the use of time and its economic assessment [13]. The productive working time destroys the time of leisure, if there is no instance of individuality to protect it.

(33)


The model also says that individual freedom can manifest itself fully only in the balance between work and leisure, in agreement with the issues of vacation time and retirement [14].

It is also interesting to remark that in case of catastrophes the resilience operates in the opposite direction with respect to the long term planning of equation (26), since the prey in this case is the immediate survival (hence a private need) rather that the reconstruction from scratch. The contrast between (26) and (34) becomes evident in retrospect, where usually any solution is criticized.

(34)


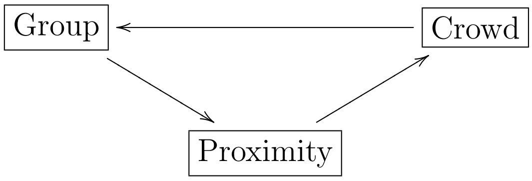
For researchers an important question concerns the optimal dimension of research groups. The tendency to huge research groups not always leads to success in pure science, while it can help in applied sciences where many different competences are required. But also in this case some clustering is required, in order to prepare interdisciplinar contacts. Proximity should avoid that groups get confused in crowds as it happens in world general congresses. In the case of scientific research proximity is not always physical, but usually it helps, just as it happens in territorial phenomena. The difficulty of defining structural proximity becomes

[13]Forerunners to this analysis were Becker and De Grazia (see [10] and [3], with important progress in [4])

[14]The characteristics of the tourist were fully analized in [23], while in [18] many economic implications were derived.

even greater in economic analysis, as it was shown in Section 3.1 of [7]. Positive effects of some special cases of clustering in random Bak Sneppen evolution models were discussed in Section 4 of [32]. The scheme is

(35)



This local frame explains in particular how some research and development structures work when they resort to brainstorming, official or informal: the members must be not too little, but also not too many, as explained in [41] and [36], and described in a glamorous example in [43]. The most striking case is perhaps the arising of Institute for Advanced Study of Princeton (compare Flexner [14]).

4. Conclusions

Paulo maiora canamus
Let us sing something nobler
Vergilius, Bucolics, 4,1

Models of Lotka-Volterra type present many practical problems of estimation of the parameters, beginning with the carrying capabilities of the single variables. In particular a tripartite model can find sharp application only in the case of biological defense of some natural species, where the three actors are homogeneous. An accurate discussion of the way of choosing and measuring parameters between non homogeneous data can be found in [25].

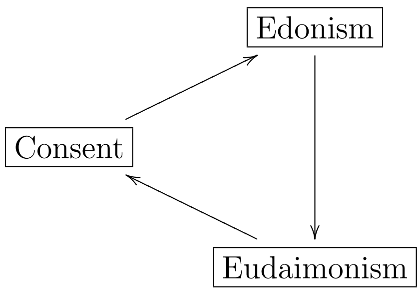
Simplified models represent somehow a synthesis according to platonic ideas of what a perfect abstract science should look for. Reality does not strictly obey to the abstract model, that expresses only a potentiality, but not actual reality. A summary of this interpretation of platonic ideas related to mathematics can be found in the Section 4 of the fundamental article by Enriques [13]. In many practical cases a model can suggest various forms of scenario analysis, that show a fuzzy panorama of possible realizations of an idea. Also in our paper the reader can expand the research to check what change of parameters (in particular dropping symmetry of equations) implies.

From a philosophical point of view it is also worth to observe that the tripartite antagonist frame can be easily referred to the long lasting competition between Epicurism and Stoicism, and their search for human happiness. While Stoicism searched it in the transformation beyond the self, also in everyday life, Epicurism followed the suggestion "lathe biosas" (live hidden, avoid public life). Both philosophical theories aim at man's well-being, although in the latter case this is through (moderate) hedonism, whereas in the former it is achieved through Aristotle's eudaimonia: the promotion of "self acceptance, positive relationships with others, personal growth, purpose in life, ability to realize everyday demands, and autonomy", as summarized by Kalvand in [19]. Strangely enough, in the modern age, hedonism and its limits are well studied by psychologists, whilst the heavier eudaimonia (that entrains also prestige, self-realization, power) has started to be really recognized only in the last few years. Some essential references for it are the recent [40] and [39].

The approach to the psychological competition between eudaimonism and hedonism is essentially the same of the master model (23), where intelligence is replaced by the search for socio-cultural consent. This consent actually is a mixed control, since no longer stems only from authority or religion, but is founded on public opinion through information and nets. Recall that a first strong privatization of control arose with the diffusion of photography in the "roaring Twenties" when the classical pair "Private vices, public virtues" could no longer save the privacy of vices. More generally happiness is not confined to personal well-being alone, but relies also on the surrounding world, both at the macro and micro levels: city and landscape, nature, art, friends, hobbies, sport. The competition between personal and public happiness involves also this world: its solution, in terms of achieving a non-traumatic equilibrium, requires a control that can be top-down (legislator), or bottom-up (social network, market).

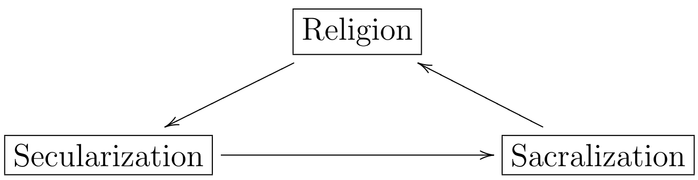
The model can be represented as

(36)



When consent arises from religion or authority the control may be considered top-down. In this case the role of hedonism is taken by a general, skeptical, secularization, while the role of eudaimonism is assumed by sharper forms of sacralization, so that the scheme becomes

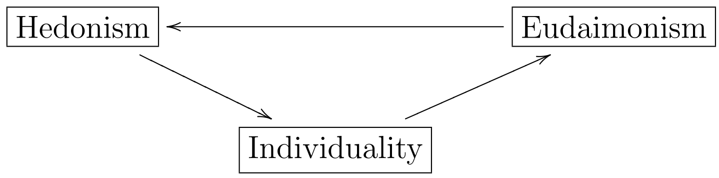
(37)



This may hold also in more profane forms of beliefs. An important feature of the model is that control may start from very small groups and then burst out, while in some cases the absence of any control leads to the vanishing of the prey (in this case sacralization). On the other side, the absence of secularization may lead to reduce the interior strength of the religion, confining it to a pure sequence of formal rules and rites.

Attention is required in using scheme (36) and specially its more severe form (37) since it should be used for lighthearted persons and pupils, whilst for severe persons the scheme ought to be turned into a bottom-up reverse control of type (33) so that it may become hedonism-preserving, otherwise private life could actually suffer exceeding restrictions. Also public evolution might be hindered, as in the case of European austerity, since the excess of constraints can lead to stagnation. The defense must start from the bottom, so we have the scheme

(38)



A form of this reaction is described in the book of Ordine [OR] where the schemes (32) and (33) are superimposed. The author proves the utility of the useless, both in fine arts/literature and in science, since unconditioned ideas are usually required at the basis of new creations.

In particular, there is a contrast between the freedom of artistic creation and the preservation of tradition or nature. On this basis, some renowned monuments such as the Sacra di San Michele, which crowns the mountains at the entrance of the Susa Valley near Turin, would never have been built. In the modern age it would not have been allowed to construct the Villa Malaparte in Capri, the stunning work of A. Libera that seems to timelessly complement the rocky seascape. The fishing and sailor villages of the Amalfi Coast or Cinque Terre would never have developed. To the natural landscape would be admitted, albeit reluctantly, only ports, dams, chair lifts or cable car stations, and only a few works of major artistic and cultural value.

A systemic view, with attention to all the participants, present and future, becomes essential although all too often there is a tendency of the stakeholders to shine a light only on the little stretch of the route that most moves them personally, and offers the greatest return of personal and political prestige.

The great risk of such a shorteyed view is that catastrophes can suddenly arise. The last example of Section 2 (Fig. 8(b)) shows clearly what can happen. At the beginning it seems that there exists only the predator-prey component, and that it is stable, apart some small deviations that can be attributed to random perturbations. Some hidden forces that act as a bottom-up controls can develop up to a non controllable level before the global predator-prey system reaches a perturbation, say of 10% of the equilibrium value; in the example 10% deviation is reached when the control has already reached about 50% of its capacity level, so that it becomes very costly to eradicate it. Remark also that its growth at the moment of the discovery is at the highest speed. Of course a faster detection is possible when the attention level is put at 5% of deviation from the norm or less. But random phenomena and a path that seems to approximate the equilibrium level may make the early discovery difficult. This happens in medicine, in territorial geology, in climate control, in social and political revolutions.

References

[1] Barabasi, A.L., *Linked. The new Science of Networks*, 2002. Italian Transl. di B. Antonelli d'Oulx: *Link. La scienza delle reti*, Einaudi Torino, 2004.

[2] Baroni, M.R., *Psicologia ambientale*, Il Mulino Bologna, 1998.

[3] Becker, G.S., *A Theory of the Allocation of Time*, Economic Journal, 75 (1965), 493-508.

[4] Becker, G.S., *The Endogenous Determination of Time Preference*, Quarterly Journal of Economics, 102 (1997), 729-758.

[5] Calvino, I., *Le città invisibili*, Einaudi, Torino (1972). Translated by W. Weaver: *Invisible Cities*, Secker and Warburg, 1974.

[6] Chen, C.T., *Linear SystemTheory and Design*, Oxford University Press, NY, 1995.

[7] Chang, T.F.M., Piccinini, L.C., Iseppi, L., Lepellere, M.A., *The black-box of economic interdependence in the process of structural change: EU and EA on the stage*, Italian Journal of Pure and Applied Mathematics, 31 (2013), 285-306.

[8] Alighieri, D., Translation by J. Ciardi: *The Paradiso*, Signet Classics, New York, 1970-2001.

[9] Droli, M., Chang, T.F.M., Iseppi, L., Piccinini, L.C., *Managing trade contacts in HotRest intermediate markets: a resource-based view analysis in EU countries*, Tourism Economics, 20 (2014), 757-778.

[10] De Grazia, S., *Of Time, Work and Leisure*, The twentieth Century Fund, New York, 1962.

[11] Eco, U., *Apocalittici e integrati*, RCS-Bompiani, 1964.

[12] Ejarque, J., *La destinazione turistica di successo*, Hoepli Milano, 2003.

[13] Enriques, F., *Matematiche e teoria della conoscenza*, Scientia, 11 (1912), 1-17.

[14] Flexner, A., *The usefulness of Useless Knowledge*, Harper's Magazine, 1939, 544-552.

[15] Hirschman, A.O., *Shifting Involvements. Private Interest and Public Action*, Princeton University Press, 1982.

[16] Hsu, S.B., Ruan, S., Yang, T.H., *Analysis of three species Lotka-Volterra Food Web models with omnivory*, J. Math. Anal. Appl., 426 (2015), 659-687

[17] Iannelli, M., Pugliese, A., *An Introduction to Mathematical Dynamics – Along the Trails of Volterra and Lotka*, Springer Verlag Unitext, vol. 79, Heidelberg, 2014.

[18] Jansen-Verbeke, M., *Leisure shopping, a magic concept for the tourism industry?*, Tourism Management, 12 (1991), 9-14.

[19] Kalvans, E., *Psychological Well-being Content in Today's psychological Theories*, The coll. of Scientific Papers PSPI 2012, Rezekne Augstskola, (2012), 9-22.

[20] Lorch, E.R., *Spectral Theory*, Oxford University Press, New York, 1962.

[21] Lotka, A.J., *Elements of Physical Biology*, Williams and Wilkins, 1925.

[22] Mac Cannel, D., *Staged authenticity: arrangement of social space in tourist settings*, American Journal of Sociology, (1973), 586-603.

[23] Mac Cannel, D., *The tourist, a new theory of the leisure class*, Shocken, New York, 1989.

[24] Mandler, J.M., *Stories, scripts and scenes: aspects of schema theories*, N.J. Hillsdale, Erlbaum, 1984.

[25] Motesharrei, S., Rivas, J., Kalnay, E., *Human and nature dynamics (HANDY): Modeling inequality and use of resources in the collapse or sustainability of societies*, Ecol. Econ., 101, 90-102.

[26] Neisser, U., *Cognition and reality*, Freeman, San Francisco, 1976.

[27] Newcomb, R.M., *Planning the Past: historical landscape resources and recreation*, Dawson ed., Folkestone, 1979.

[28] Ordine, N., *L'utilité de l'inutile Paris*, Les belles letters, 2013.

[29] Piccinini, L.C., Chang, T.F.M., *Il mosaico come sintesi di libertà individuali in competizione*, Agribusiness Paesaggio & Ambiente 11, 109-118.

[30] Piccinini, L.C., Chang, T.F.M., Iseppi, L., *From Pleasantville to new Pleasantville: a Tripartite Model for the transition*, Agribusiness Paesaggio & Ambiente, 16 (2013), 104-119.

[31] Piccinini, L.C., Chang, T.F.M., Taverna, M., *Corsi e ricorsi nell'evoluzione del mosaico paesistico-culturale*, Topscape-Paysage (Overview), 9 (2012), 1-20 ISSN 2279-7610.

[32] Piccinini, L.C., Lepellere, M.A., Chang, T.F.M., Iseppi, L., *Partitioned frames in discrete Bak-Sneppen models*, Italian Journal of Pure and Applied Mathematics, 33 (2014), 461-488.

[33] Prigogyne, I., Stengers, I., *La nouvelle alliance*, Mtamorphose de la science Gallimard, Paris (1979).

[34] Piccinini, L.C., Stampacchia, G., Vidossich, G., *Ordinary differential equations in Rn. Problems and methods*, Springer Verlag, New York (1984).

[35] Piccinini, L.C., Taverna, M., Chang, T.F.M., *Ruolo dell'attività agricola nell'evoluzione del paesaggio*, Agribusiness Paesaggio & Ambiente, 3 (1999), 119-136.

[36] Quinn, J.B., *Managing Innovation: Controlled Chaos*, Harvard Business Review, 63 (3) (1985), 73-80.

[37] Rosenzweig, M.L., Mac Arthurr, R.H., *Graphical Representation and Stability Conditions of Predator-prey Interactions*, American Naturalist, XCVII, 895 (1963), 209-223.

[38] Ruberti, A., Isidori, A., *Teoria dei sistemi*, Boringhieri, Torino, 1979.

[39] Ruff, C.D., Singer, B.H., *Know Thyself and Become what you are: a Eudaimonic Approach to Psychological Well-being*, Journal of Happiness Studies, 9 (2008), 13-39.

[40] Ryan, R.M., Deci, E.L., *On Happiness and Human Potentials: a Review of Research on Hedonic and Eudaimonic Well-Being*, Annual Review Psychology, 52 (2001), 141-166.

[41] Sawyer, R.K., *Group Genius. The creative Power of Collaboration*, Basic Books, 2007.

[42] Schaller, G.B., *The Serengeti Lion: A study of Predator-Prey relations*, University of Chicago Press, 2009.

[43] Semler, R., *Maverick: The Success Story Behind the World's most Unusual Workplace*, New York, Warner Business Books, 1993.

[44] Simon, H.A., *The sciences of Artificial*, MIT Press, Cambridge MA, 1969.

[45] Simon, H.A., *Models for my life*, Basic Books of Harper Collins Publishers Inc. Ch., 16 (1991).

[46] Vicario, G.B., private communication to L. Piccinini (1984).

[47] Vicario, G.B., *A proposito di fenomenologia sperimentale*, in Tra percezione e arte (a cura di A. Cavedon e L. Zanuttini), Il poligrafo Padova, 1997, 13-28.

[48] Volterra, V., *Variazioni e fluttuazioni del numero di individui in specie animali conviventi*, Mem. Accad. Lincei Roma, 2 (1926), 31-113.

[49] Volterra, V., *Leçons sur la théorie mathématique de la lutte pour la vie*, (redigées par M. Brelot) Paris, 1930.

Accepted: 31.10.2015