A NOTE ON SPECIAL MATRICES

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Abstract. The word "matrix" comes from the Latin word for "womb" because of the way that the matrix acts as a womb for the data that it holds. The first known example of the use matrices was found in a Chinese text called Nine Chapters of the Mathematical Art, which is thought to have originated somewhere between 300 B.C. and 200 A.D. The modern method of matrix solution was developed by a German mathematician and scientist Carl Friedrich Gauss. There are many different types of matrices used in different modern career fields. We introduce and discuss the different types of matrices that play important roles in various fields.
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1. Introduction

The history of matrices goes back to ancient times! Around 4000 years ago, the Babylon knew how to solve a simple 2 2 system of linear equations with two unknowns. But the term "matrix" was not applied to the concept until 1850. The origins of mathematical matrices lie with the study of systems of simultaneous linear equations. An important Chinese text from between 300 BC and AD 200, Nine Chapters of the Mathematical Art [1], gives the first known example of the use of matrix methods to solve simultaneous equations [2]. In the seventh chapter, the concept of a determinant first appears, nearly two millennia before its supposed invention by the Japanese mathematician Seki Kowa [3] in

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1683 or his German contemporary Gottfried Leibnitz [4]. More uses of matrix-like arrangements of numbers appear in chapter eight, ”Methods of rectangular arrays,” in which a method is given for solving simultaneous equations using a counting board that is mathematically identical to the modern matrix method of solution outlined by Carl Friedrich Gauss (1777–1855) [5], also known as Gaussian elimination.

In the beginning of 19th century, Gauss introduced a procedure to be used for solving a system of linear equations. His work mainly dealt with the linear equations and had yet to bring the idea of matrices. He dealt with equations of different numbers and variables and also the traditional works of Euler, Leibnitz and Cramer. Gauss work is now termed as Gauss Elimination method. In 1848, James Joseph Sylvester [6] introduced the term ”matrix”, the Latin word for womb, as a name for an array of numbers. He used womb because he viewed a matrix as a generator of determinants [7]. On the other part, matrix multiplication or matrix algebra came from the work of Arthur Cayley in 1855. Cayley's efforts were published in two papers, one in 1850 and the other in 1858. His works introduced the idea of the identity matrix as well as the inverse of a square matrix [8]. The elevation of the matrix from mere tool to important mathematical theory owes a lot to the work of female mathematician Olga Taussky Todd (1906–1995) [9], who began by using matrices to analyze vibrations on airplanes during World War II. Later on, she became the ”mother” of matrix theory.

2. Matrices

2.1 Definition. A rectangular arrangement of mn numbers, in m rows and n columns and enclosed within a bracket is called a matrix. We shall denote the matrices by capital letters like A, B, C etc.

$$A = \begin{pmatrix} A_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots & \vdots \\ A_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = (a_{ij})_{m\times n}$$

A is a matrix of order $m \times n$. A matrix is just not a collection of elements but every element has assigned a definite position in a particular row and column.

2.2 Fundamental matrices

- **Square matrix:** A matrix which has equal number of rows and columns is called a square matrix.
- **Diagonal matrix:** A square matrix $A = (a_{ij})_{n\times n}$ is called a diagonal matrix, if each of its non-diagonal elements is zero.
- **Identity matrix:** A diagonal matrix whose diagonal elements are equal to 1 is called identity matrix.

- **Upper Triangular matrix:** A square matrix is said to be an Upper triangular matrix, if $a_{ij} = 0$ for $i > j$.
- **Lower Triangular matrix:** A square matrix is said to be a Lower triangular matrix, if $a_{ij} = 0$ for $i < j$.
- **Symmetric matrix:** A square matrix $A = (a_{ij})_{n\times n}$ is said to be a Symmetric matrix, if $a_{ij} = a_{ji}$ for all i and j.
- **Skew - Symmetric matrix:** A square matrix $A = (a_{ij})_{n\times n}$ is said to be a Skew–Symmetric matrix, if $a_{ij} = -a_{ji}$ for all i and j.
- **Zero matrix:** A matrix whose all the elements are zero is called a Zero matrix, and a zero matrix of order $n \times m$ is denoted by $O_{n\times m}$.
- **Row matrix:** A matrix which has only one row is called a Row matrix or Row vector.
- **Column matrix:** A matrix which has only one column is called a Column matrix or Column vector.

3. Special matrices in Mathematics

Matrices have numerous applications, both in mathematics and other sciences. We will list the matrices which have applications in the field of mathematics. The matrices are given in alphabetical order with special matrices under each category, if applicable.

3.1 Alternate matrix

An alternate matrix is a matrix with a particular structure, in which successive columns have a particular function applied to their entries. Examples of alternate matrices include Vandermonde matrices and Moore matrices. Alternate matrices are used in coding theory in the construction of alternate codes.

3.1.1 Vandermonde matrix

A Vandermonde matrix [10], [11] of order n is of the form

$$V = \begin{bmatrix} 1 & a_1 & a_1^2 & \cdots & a_1^{n-1} \\ 1 & a_2 & a_2^2 & \cdots & a_2^{n-1} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & a_n & a_n^2 & \cdots & a_n^{n-1} \end{bmatrix}$$

A Vandermonde matrix is also called an alternate matrix. Sometimes, the transpose of an alternate matrix is known as the Vandermonde matrix. A Vandermonde matrix is a matrix that arises in the Lagrange interpolating polynomials, polynomial least squares fitting, and the reconstruction of a statistical distribution from the distribution's moments.

3.1.2 Moore matrix

A Moore matrix [12] is a matrix of the form

$$M = \begin{bmatrix} x_1 & x_1^q & \cdots & x_1^{q^{n-1}} \\ x_2 & x_2^q & \cdots & x_2^{q^{n-1}} \\ x_3 & x_3^q & \cdots & x_3^{q^{n-1}} \\ \vdots & \vdots & \ddots & \vdots \\ x_m & x_m^q & \cdots & x_m^{q^{n-1}} \end{bmatrix}$$

It is a matrix defined over a finite field. This matrix was introduced by E.H. Moore in the year 1896. The Moore matrix has successive powers of the Frobenius automorphism applied to the first column. Therefore, it is an $m \times n$ matrix.

3.2 Anti-diagonal matrix

Anti-diagonal is the diagonal of a matrix starting from the lower left corner to the upper right corner of the matrix. An anti-diagonal matrix is a matrix where all the entries are zero except the anti-diagonal ($\nearrow$). Example of Anti-diagonal matrix is

$$A = \begin{bmatrix} 0 & 0 & 0 & a \\ 0 & 0 & b & 0 \\ 0 & c & 0 & 0 \\ d & 0 & 0 & 0 \end{bmatrix}$$

3.3 Arrowhead matrix

An arrowhead matrix [13, 14] is a square matrix which has zeros in all entries except the first row, first column, and the main diagonal. The general form of an Arrowhead matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & 0 & 0 \\ a_{31} & 0 & a_{33} & 0 \\ a_{41} & 0 & 0 & a_{44} \end{bmatrix}$$

Real symmetric arrowhead matrices are used in some algorithms for finding of eigenvalues and eigenvectors. Arrowhead matrices are important for the computation of the eigenvalues, via divide and conquer approaches.

3.4 Band matrix

A band matrix [15] is a sparse matrix whose non-zero entries are confined to a diagonal *band*, comprising the main diagonal and zero or more diagonals on either side. The general form of a Band matrix is

$$A = \begin{bmatrix} a_{11} & a_{12} & 0 & 0 & 0 \\ a_{21} & a_{22} & a_{23} & 0 & 0 \\ 0 & a_{32} & a_{33} & a_{34} & 0 \\ 0 & 0 & a_{43} & a_{44} & a_{45} \\ 0 & 0 & 0 & a_{54} & a_{55} \end{bmatrix}$$

Let us consider an $n \times n$ matrix $A = (a_{ij})$. If all the elements of the matrix are zero outside a diagonally bordered band whose range is determined by the constants p and q as follows:

$$a_{ij} = 0 \text{ if } j < i - p \text{ or } j > i + q, \ \ p, q \geq 0,$$

then the quantities p and q are called the lower and upper bandwidth, respectively. The bandwidth of the matrix is the maximum of p and q, that is, it is the number r such that

$$a_{ij} = 0 \text{ if } |i - j| > r.$$

A matrix is called a band matrix or banded matrix if its bandwidth is reasonably small. A band matrix with $p = q = 0$ is a diagonal matrix. A band matrix with $p = q = 1$ is a tridiagonal matrix. When $p = q = 2$, it is a pentadiagonal matrix. If $p = 0$, $q = n - 1$, we get an upper triangular matrix. Similarly, for $p = n - 1$, $q = 0$, we obtain a lower triangular matrix. In numerical analysis, matrices from finite element or finite difference problems are often banded.

3.5 Binary matrix

A Binary matrix [16] or a Boolean matrix is a matrix with entries being either 0 or 1. Such a matrix can be used to represent a binary relation between a pair of finite sets. Such type of matrices are also called as Logical matrix, Relation matrix, or (0,1) matrix. Some of the (0,1) matrices are discussed below.

3.5.1 Adjacency matrix

An adjacency matrix [17], also called the connection matrix, is a matrix with rows and columns representing which vertices (or nodes) of a graph are adjacent to which other vertices. Another matrix representation for a graph is the incidence matrix. In the special case of a finite simple graph, an Adjacency matrix is a (0,1)-matrix with zeros on its diagonal. For an undirected graph, the adjacency matrix is symmetric. The adjacency matrix of a complete graph contains all ones except along the diagonal where there are only zeros. The adjacency matrix of an empty graph is a zero matrix. The *biadjacency matrix* of a simple, undirected bipartite graph is a (0,1)-matrix.

3.5.2 Design matrix

A design matrix [18], [19] is a matrix which contains explanatory variables, and is used in statistical models like the general linear model.

3.5.3 Incidence matrix

The incidence matrix [20] of a graph has one row for each vertex of the graph and one column for each edge of the graph. If an edge runs from node a to node b, the row corresponding to that edge has $-$ in column a and 1 in column b and all other entries in that row are 0. However, some authors define the incidence matrix to be the transpose of this, with a column for each vertex and a row for each edge. The physicist Kirchhoff (1847) was the first to define the incidence matrix. Incidence matrices are mostly used in graph theory. In graph theory an undirected graph G has two kinds of incidence matrices: unoriented and oriented.

3.5.4 Permutation matrix

A permutation matrix [15] is a square matrix obtained from the same size identity matrix by a permutation of rows. Every row and column has a single 1 with 0s everywhere else. There are $n!$ permutation matrices of size n.

The permutation matrices of order two are given by

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and of order three are given by

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$$

A permutation matrix is nonsingular, and the determinant is always ± 1. In addition, a permutation matrix A satisfies

$$AA^\top = I,$$

where $A^\top$ is a transpose and I is the identity matrix.

3.5.5 Exchange matrix

The exchange matrix is an anti-diagonal matrix in which all the entries in the anti-diagonal are 1 and all other elements are zero. It is also a special case of permutation matrix. In other words, it is a 'row-reversed' or 'column-reversed' version of the identity matrix. The exchange matrix is also called the reversal matrix, backward identity, or standard involutory permutation.

3.6 Bi-Symmetric matrix

A bisymmetric matrix [21] is a square matrix which is symmetric about both of its main diagonals. More precisely, an $n \times n$ matrix B is bisymmetric if it satisfies both $B = B^{\top}$ and $BE = EB$ where E is the $n \times n$ exchange matrix. For example:

$$\begin{bmatrix} a & b & c & d & e \\ b^f & g & h & d & \\ c & g & i & g & c \\ d & h & g & f & b \\ e & d & c & b & a \end{bmatrix}$$

Bisymmetric matrices are both symmetric centrosymmetric and symmetric persymmetric.

3.6.1 Centrosymmetric matrix

A matrix which is symmetric about its center is called a centrosymmetric matrix. More precisely, an $m \times m$ matrix $A = [A_{ij}]$ is centrosymmetric when its entries satisfy

$$A_{i,j} = A_{m-i+1,m-j+1} \text{ for } 1 \leq i,\ j \leq m.$$

If E denotes the $m \times m$ matrix with all the counterdiagonal elements as 1 and 0 elsewhere, then a matrix A is centrosymmetric if and only if $AE = EA$. The matrix E is sometimes named as the exchange matrix.

3.6.2 Persymmetric matrix

A persymmetric matrix [21] is a square matrix which is symmetric in the northeast-to-southwest diagonal or a square matrix such that the values on each line perpendicular to the main diagonal are the same for a given line.

3.7 Block matrix

A block matrix [22] is a matrix which is obtained using smaller matrices, called blocks. For example,

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix},$$

where A, B, C and D are matrices, is a block matrix. In the above example

$$A = \begin{bmatrix} 0 & 3 \\ 3 & 0 \end{bmatrix},\ B = \begin{bmatrix} 6 & 6 & 6 \\ 6 & 6 & 6 \end{bmatrix},\ C = \begin{bmatrix} 7 & 7 \\ 7 & 7 \\ 7 & 7 \end{bmatrix},\ D = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 2 & 0 \\ 2 & 0 & 2 \end{bmatrix}$$

therefore, it can be written as the matrix

$$\begin{bmatrix} 0 & 3 & 6 & 6 & 6 \\ 3 & 0 & 6 & 6 & 6 \\ 7 & 7 & 2 & 0 & 2 \\ 7 & 7 & 0 & 2 & 0 \\ 7 & 7 & 2 & 0 & 2 \end{bmatrix},$$

Block matrices are also called partitioned matrices.

3.8 Cauchy matrix

Cauchy matrix [23] is an $m \times n$ matrix if its entries a_{ij} are of the form

$$a_{ij} = \frac{1}{\alpha_i - \beta_j}, \text{ or } a_{ij} = \frac{1}{\alpha_i + \beta_j}, \ \alpha_i \mp \beta_j \neq 0, \ 1 \leq i \leq m, \ 1 \leq j \leq n,$$

where α_i and β_j are elements which are distinct and not repeated. Every submatrix of a Cauchy matrix is a Cauchy matrix. The Hilbert matrix is a special case of the Cauchy matrix, with $x_i - y_j = i + j - 1$.

3.9 Conference matrix

Conference matrix or a C-matrix [24] is a square matrix C with diagonal entries 0 and off-diagonal entries ± 1, such that the product of C and its transpose is the multiple of the identity matrix, that is,

$$C^\top C = (n-1)I, \text{ where } n \text{ is the order of } C.$$

Conference matrix is either symmetric or antisymmetric. Sometimes, the Conference matrix or a C-matrix is defined as the matrix, which requires a single 0 in each row and column but not necessarily on the diagonal [25], [26].

The two symmetric C- matrices of order 2 are

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and two antisymmetric C-matrices of order 2 are

$$\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

3.10 Complex matrix

Complex matrix is a matrix whose entries are complex numbers.

3.11 Copositive matrix

A Copositive matrix [27] is a real $n \times n$ square matrix A, if

$$x^\top A x \geq 0$$

for all non-negative n-vectors x. The collection of real positive-definite matrices is the subset of copositive matrix.

3.12 Diagonal matrix

A Diagonal matrix [28] is a square matrix D, in which all the entries other than the main diagonal are zero. The diagonal entries themselves may be zero or may not be. Thus, the matrix $D = (d_{ij}$ with n columns and n rows is diagonal if

$$d_{ij} = 0 \text{ if } i \neq j, \ \forall\, i, j \in \{1, 2, ..., n\}.$$

Sometimes, the term *diagonal matrix* refers to a rectangular diagonal matrix, which is an $m \times n$ matrix with only the entries of $d_{i,j}$ will be non-zero. For example,

$$\begin{bmatrix} 8 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -5 \\ 0 & 0 & 0 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 7 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & -6 & 0 & 0 \end{bmatrix}.$$

Any square diagonal matrix is also a symmetric matrix. A diagonal matrix is also defined as a matrix which is both upper- and lower-triangular. The identity matrix and square zero matrix are diagonal matrices. A one-dimensional matrix is always a diagonal matrix. The different types of diagonal matrices are discussed below.

3.12.1 Bidiagonal matrix

A bidiagonal matrix [29] is a matrix with non-zero entries along the main diagonal and *either* the diagonal above or the diagonal below the main diagonal. A bidiagonal matrix has exactly two non- zero diagonals in the matrix.

When the entries above the main diagonal are non-zero, the matrix is called an upper bidiagonal matrix and when the entries below the main diagonal are non-zero, the matrix is called lower bidiagonal matrix.

3.12.2 Tridiagonal matrix

A tridiagonal matrix is a matrix which has non-zero elements along the main diagonal and the diagonals immediately above and below the main diagonal.

3.12.3 Pentadiagonal matrix

A pentadiagonal matrix is a matrix which has non-zero elements along the main diagonal and two diagonals immediately above and below the main diagonal.

3.12.4 Block diagonal matrix

A block diagonal matrix [30] or a diagonal block matrix, is a square diagonal matrix in which the diagonal elements are square matrices of any size (possibly even 1×1), and the off-diagonal elements are 0. Therefore, a block diagonal matrix is a block matrix in which the blocks off the diagonal are the zero matrices, and the diagonal matrices are square.

3.12.5 Block tridiagonal matrix

A block tridiagonal matrix [31] is another special block matrix, which is also a square matrix, having square matrices (blocks) in the main diagonal, lower diagonal and upper diagonal, with all other blocks being zero matrices.

3.13 Diagonally Dominant matrix

A matrix A of order $n \times n$ is said to be row diagonally dominant (rdd) [32] if

$$|a_{ii}| \geq \sum_{j \neq i} |a_{ij}| \text{ for all } i,$$

where a_{ij} denotes the entry in the i^{th} row and j^{th} column. If a strict inequality ($>$) is used, then this matrix is called strictly diagonally dominant. It is said to be column diagonally dominant (cdd), if $A^{\top}$ is row diagonally dominant. In other words, a matrix A of order $n \times n$ is said to be diagonally dominant for every row of the matrix, the magnitude of the diagonal entry in a row is larger than or equal to the sum of the magnitudes of all the other non-diagonal entries in that row.

3.14. Echelon matrix

A matrix that has undergone Gaussian elimination is said to be in row echelon form [33] or, more properly, "reduced echelon form" or "row-reduced echelon form." Such a matrix has the following characteristics:

- All zero rows are at the bottom of the matrix
- The leading entry of each nonzero row after the first occurs to the right of the leading entry of the previous row.
- The leading entry in any nonzero row is 1.
- All entries in the column above and below a leading 1 are zero.

Another common definition of echelon form only requires zeros below the leading ones, while the above definition also requires them above the leading ones.

3.15 Elementary matrix

An elementary matrix [34] is a matrix which can be obtained from the identity matrix by one single elementary row operation. Left multiplication (pre-multiplication) by an elementary matrix represents elementary row operations, while right multiplication (post-multiplication) represents elementary column operations. The acronym "ERO" is commonly used for "elementary row operations".

3.16 Equivalent matrix

An $m \times n$ matrix A is said to be equivalent [35] to a matrix $m \times n$ matrix B iff $B = PAQ$, for any non-singular matrices P and Q of order $m \times m$ and $n \times n$ respectively.

3.17 Frobenius matrix

A Frobenius matrix is a special kind of square matrix from numerical mathematics. A matrix is a Frobenius matrix if it has the following three properties:

- all entries on the main diagonal are ones,
- the entries below the main diagonal of at most one column are arbitrary,
- every other entry is zero.

Example for Frobenius matrix is

$$A = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & a_{32} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 0 & a_{n2} & 0 & \cdots & 1 \end{bmatrix}$$

Frobenius matrices are named after Ferdinand Georg Frobenius. An alternative name for this class of matrices is Gauss transformation, after Carl Friedrich Gauss [15].

3.18 Hadamard matrix and Complex Hadamard matrix

A Hadamard matrix [36], named after the French mathematician Jacques Hadamard, is a square matrix whose entries are either +1 or −1 and whose rows are mutually orthogonal.

A complex Hadamard matrix is any complex $N \times N$ matrix H satisfying the following two conditions:

- unimodularity (the modulus of each entry is unity)
- orthogonality: $HH^{\theta} = NI$, where H^{θ} is the Hermitian transpose of H and I is the identity matrix.

3.19 Hankel matrix

A square matrix with constant skew diagonals is called a Hankel matrix [37]. In other words, a Hankel matrix is a matrix in which the $(i, j)^{\text{th}}$ entry depends only on the sum $i + j$. Such matrices are sometimes known as persymmetric matrices or, in older literature, orthosymmetric matrices.

3.20 Hermitian and skew–Hermitian matrix

A Hermitian matrix (or self-adjoint matrix) [38] is a square matrix with complex entries that is equal to its own conjugate transpose-that is, the element in the i^{th} row and j^{th} column is equal to the complex conjugate of the element in the j^{th} row and i^{th} column, for all indices i and j:

$$a_{ij} = \overline{a_{ji}} \quad \text{or} \quad A = \overline{A^{\top}} \text{ in matrix form.}$$

Hermitian matrices are named after Charles Hermite.

A square matrix with complex entries is said to be skew-Hermitian or anti-hermitian, if its conjugate transpose is equal to its negative. That is, the matrix A is skew-Hermitian if it satisfies the relation

$$a_{ij} = \overline{-a_{ji}} \quad \text{or} \quad A^{\top} = \overline{-A} \text{ in matrix form.}$$

3.21 Hessenberg matrix

A Hessenberg matrix [39] is a matrix of the form

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & \cdots & a_{1(n-1)} & a_{1n} \\ a_{21} & a_{22} & a_{23} & \cdots & a_{2(n-1)} & a_{2n} \\ 0 & a_{32} & a_{33} & \cdots & a_{3(n-1)} & a_{3n} \\ 0 & 0 & a_{43} & \cdots & a_{4(n-1)} & a_{4n} \\ 0 & 0 & 0 & \cdots & a_{5(n-1)} & a_{5n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & a_{(n-1)(n-1)} & a_{(n-1)n} \\ 0 & 0 & 0 & \cdots & a_{n(n-1)} & a_{nn} \end{bmatrix}$$

Hessenberg matrices were given by a German engineer Karl Hessenberg (1904–1959), whose dissertation investigated the computation of eigenvalues and eigenvectors of linear operators.

3.22 Hollow matrix

A hollow matrix [40] is a square matrix whose diagonal elements are all zero. The real skew-symmetric matrix is an example of a hollow matrix. That is, any square matrix which takes the form

$$\begin{bmatrix} 0 & & & & \\ & 0 & & & \\ & & 0 & & \\ & & & \ddots & \\ & & & & 0 \end{bmatrix}$$

is a hollow matrix.

3.23 Integer matrix

A matrix whose all entries are integers is called an integer matrix. Binary matrices, the zero matrix, the unit matrix, and the adjacency matrices used in graph theory, are examples of integer matrices.

3.24 Metzler matrix

A matrix is called Metzler [41] or quasi-positive or essentially nonnegative if all of its elements are non-negative except those elements on the main diagonal, which are unconstrained. That is, a Metzler matrix is any matrix A which satisfies the following condition:

$$a_{ij} \geq 0;\ i \neq j.$$

3.25 Monomial matrix

A generalized permutation matrix (or monomial matrix) is a matrix with the same nonzero pattern as a permutation matrix. That is, entry in each row and each column has exactly one nonzero element. In a permutation matrix, the nonzero entry is 1, whereas in a generalized permutation matrix the nonzero entry can be any nonzero value.

3.26 Non - negative matrix

A nonnegative matrix is a matrix in which all the elements are greater than or equal to zero. That is,

$$x_{ij} \geq 0,\ \forall i, j.$$

3.27 Polynomial matrix

A polynomial which has matrices as its coefficients is called a polynomial matrix [42], [43]. An n^{th} order matrix polynomial in a variable x is given by

$$Q(x) = A_0 + A_1 x + A_2 x^2 + \cdots + A_n x^n,$$

where each A is a square matrix.

3.28 Positive matrix

A positive matrix is a matrix in which all the elements are greater than zero. The set of positive matrices is a subset of all non-negative matrices.

3.29 Signature matrix

A matrix, whose diagonal elements are either $+1$ or -1, is called a signature matrix [44].

3.30 Sparse matrix

A matrix in which most of its elements are zero is called a sparse matrix. Also, if most of the elements are nonzero, then the matrix is said to be dense.

3.31 Stochastic matrix

A stochastic matrix [45] is a matrix used to describe the transitions of a Markov chain. Each entry of this matrix is a nonnegative real number and represents a probability. It is also termed as probability matrix, transition matrix, substitution matrix, or Markov matrix. The other types of stochastic matrices are:

- A doubly stochastic matrix is a square matrix of nonnegative real numbers with the sum of each row and column as 1.

- A left stochastic matrix is a real square matrix, in which the sum of each column is 1.
- A right stochastic matrix is a real square matrix, in which the sum of each row is 1.

3.32 Sylvester matrix

Let $P(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$ and $Q(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0$ be two polynomials of degree n and m respectively. The Sylvester matrix [46] of order $(n+m) \times (n+m)$ corresponding to these polynomials is obtained by writing the coefficients of $P(x)$ from the leftmost corner of the first row followed by zeros, then shifting down one row and one column to the right and writing the coefficients starting from that position. This is continued until the coefficients of $P(x)$ touches the last position of the right hand side.The process is then repeated for the coefficients of $Q(x)$. The Sylvester matrix for the above polynomials $P(x) = a_2x^2 + a_1x + a_0$ and $Q(x) = b_3x^3 + b_2x^2 + b_1x + b_0$ is given as follows:

$$S = \begin{bmatrix} a_2 & a_1 & a_0 & 0 & 0 \\ 0 & a_2 & a_1 & a_0 & 0 \\ 0 & 0 & a_2 & a_1 & a_0 \\ b_3 & b_2 & b_1 & b_0 & 0 \\ 0 & b_3 & b_2 & b_1 & b_0 \end{bmatrix}$$

3.33 Toeplitz matrix

Any nn matrix in which the negative sloping diagonal elements are constants is called a Toeplitz matrix [47]. A toeplitz matrix is of the form

$$T = \begin{bmatrix} x_0 & x_{-1} & x_{-2} & \cdots & x_{-(n-1)} \\ x_1 & x_0 & x_{-1} & \ddots & \ddots \\ x_2 & x_1 & x_0 & \ddots & \ddots \\ \ddots & \ddots & \ddots & \ddots & \ddots \\ x_{n-1} & \ddots & \ddots & \ddots & x_0 \end{bmatrix}$$

3.34 Unitary matrix

A matrix U is said to be unitary [48], if $U^*U = I$, where U^* is the conjugate transpose of U and I is the identity matrix.

3.35 Walsh matrix

A square matrix with entries being +1 or −1, with the condition that the dot product of any two distinct rows or columns being zero is called a Walsh matrix [49]. The dimension of the matrix will be the powers of 2.

4. Conclusion

In this paper, we have discussed different types of matrices, which have great applications in the field of Mathematics and sciences. There are also other matrices that have important role in science, Engineering and other fields.

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