THE INDEX OF A SPECIAL BIPARTITE GRAPH[1]

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Abstract. The Harary index of a graph is defined as the sum of reciprocals of distances between all pairs of vertices of the graph. In this paper we provide an upper bound of the Harary index in the class of all connected n-vertex bipartite graphs with a given matching number q. We characterize the unique graph with the maximum Harary index in the class of all connected n-vertex bipartite graphs with a given matching number q.
Keywords: Harary index; matching number; bipartite graphs.

1. Introduction

Let G be a simple graph with vertex set $V(G)$ and edge set $E(G)$. The distance between two vertices u and v of G, denoted by $d_G(u,v)$, is defined as the minimum length of the paths between u and v in G. The Harary index of a graph G, denoted by $H(G)$, has been introduced independently by Plavšić et al. [21] and by Ivanciuc et al. [19] in 1993 for the characterization of molecular graphs. It has been named in honor of Professor Frank Harary on the occasion of his 70th birthday. The Harary index $H(G)$ is defined as the sum of reciprocals of distances between all pairs of vertices of the graph G, i.e.,

$$H(G) = \sum_{u,v \in V(G)} \frac{1}{d_G(u,v)}\,.$$

Mathematical properties and applications of the Harary index are reported in [15], [17], [20]. Note that in any disconnected graph G, the distance is infinite between any two vertices from two distinct components. Therefore, its reciprocal can be viewed as 0. Thus, we can define validly the Harary index of disconnected graph G as follows:

$$H(G) = \sum_{i=1}^{k} H(G_i),$$

where $G_1, G_2, ..., G_k$ are the components of G.

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Another distance-based topological index of a graph G is the Wiener index, denoted by $W(G)$. As an oldest topological index, the Wiener index of a graph G, first introduced by Wiener [22] in 1947, was defined as

$$W(G) = \sum_{u,v \in V(G)} d_G(u,v).$$

The motivation for introduction of the Harary index was pragmatic – the aim was to design a distance index differing from the Wiener index in that the contributions to it from the distant atoms in a molecule should be much smaller than from near atoms, since in many instances the distant atoms influence each other much less than near atoms.

Let $\gamma(G,k)$ be the number of vertex pairs of the graph G that are at distance k. Then

$$H(G) = \sum_{k \geq 1} \frac{1}{k}\gamma(G,k). \tag{1.1}$$

It will be convenient to determine the exact value by equation (1.1) for some graphs with the simple structure (e.g., the graphs with small diameter), but in general it is very difficult to give the exact value of $\gamma(G,k)$. So it is very useful to provide upper or lower bounds for the Harary index; see, e.g., [1], [5], [16]. In addition, the extremal Harary index of a given class of graphs has also been studied extensively; see, e.g., [4], [6], [11]–[15].

In this paper, we provide an upper bound of the Harary index in the class of all connected n-vertex bipartite graphs with a given matching number q. We characterize the unique graph with the maximum Harary index in the class of all connected n-vertex bipartite graphs with a given matching number q.

In this paper, we only consider connected, simple and undirected graphs. Let $G = (V_G, E_G)$ be a graph with $u, v \in V_G$. Then $G - v$, $G - uv$ denote the graph obtained from G by deleting vertex $v \in V_G$, or edge $uv \in E_G$, respectively (this notation is naturally extended if more than one vertex or edge is deleted). Similarly, $G + uv$ is obtained from G by adding an edge $uv \notin E_G$. For $v \in V_G$, let $N_G(v)$ denote the set of all the adjacent vertices of v in G and $d(v) = |N_G(v)|$, the degree of v in G. In particular, let $\Delta(G) = \max\{d(x) \mid x \in V_G\}$ and $\delta(G) = \min\{d(x) \mid x \in V_G\}$. For convenience, let $N_G[u] = N_G(u) \cup \{u\}$. The distance $d(u,v)$ between vertices u and v in G is defined as the length of a shortest path between them. The diameter of G is the maximal distance between any two vertices of G. $D_G(u)$ denotes the sum of all distances from u in G.

Recall that G is called k-connected if $|G| > k$ and $G - X$ is connected for every set $X \in V_G$ with $|X| < k$. The greatest integer k such that G is k-connected is the connectivity $\kappa(G)$ of G. Thus, $\kappa(G) = 0$ if and only if G is disconnected or K_1, and $\kappa(K_n) = n - 1$ for all $n \geq 1$.

A bipartite graph G is a simple graph, whose vertex set V_G can be partitioned into two disjoint subsets V_1 and V_2 such that every edge of G joins a vertex of V_1 with a vertex of V_2. A bipartite graph in which every two vertices from different partition classes are adjacent is called complete, which is denoted by $K_{m,n}$, where $m = |V_1|$, $n = |V_2|$.

A vertex (edge) independent set of a graph G is a set of vertices (edges) such that any two distinct vertices (edges) of the set are not adjacent (incident on a common vertex). The vertex (edge) independence number of G, denoted by $\alpha(G)$ $(\alpha'(G))$, is the maximum of the cardinalities of all vertex (edge) independent sets. A vertex (edge) cover of a graph G is a set of vertices (edges) such that each edge (vertex) of G is incident with at least one vertex (edge) of the set. The vertex (edge) cover number of G, denoted by $\beta(G)$ $(\beta'(G))$, is the minimum of the cardinalities of all vertex (edge) covers. For a connected graph G of order n, its matching number $\alpha'(G)$ satisfies $1 \le \alpha'(G) \le [\frac{n}{2}]$. When we consider an edge cover of a graph, we always assume that the graph contains no isolated vertex. It is known that for a graph G of order n, $\alpha(G) + \beta(G) = n$; and if in addition G has no isolated vertex, then $\alpha'(G) + \beta'(G) = n$. For a bipartite graph G, one has $\alpha;(G) = \beta(G)$, and $\alpha(G) = \beta'(G)$.

Let B_n^q be the class of all bipartite graphs of order n with matching number q. In this paper we study the quantity $H(G)$ in the case of n-vertex bipartite graphs, which is an important class of graphs in graph theory. Based on the structure of bipartite graphs, sharp bounds on $H(G)$ among B_n^q are determined. The corresponding extremal graphs are identified, respectively.

2. Maximum index among B_n^q

In this section, we determine the sharp bound on the transmission of all n-vertex bipartite graphs with matching q. The unique corresponding extremal graph is identified.

Lemma 2.1. ([12]). *Let G be a graph with $u, v \in V(G)$. If $uv \notin E(G)$, then $H(G) < H(G + uv)$. If $uv \in E(G)$, then $H(G) < H(G - uv)$.*

Theorem 2.2. *Let G be in B_n^q . Then $H(G) \le \frac{1}{4}n^2 - \frac{1}{2}q^2 + \frac{1}{2}nq - \frac{1}{4}n$ with equality if and only if $G \cong K_{q,n-q}$.*

Proof. It is routine to check that

$$\begin{aligned} H(K_{q,n-q}) &= \frac{1}{2}\left[(n-q)q + \frac{1}{2}(q-1)q + (n-q)q + \frac{1}{2}(n-q-1)(n-q)\right] \\ &= \frac{1}{2}\left[2nq - \frac{3}{2}q^2 - \frac{1}{2}q + \frac{1}{2}n^2 + \frac{1}{2}q^2 - nq - \frac{1}{2}n + \frac{1}{2}q\right] \\ &= \frac{1}{2}\left[\frac{1}{2}n^2 - q^2 + nq - \frac{1}{2}n\right] = \frac{1}{4}n^2 - \frac{1}{2}q^2 + \frac{1}{2}nq - \frac{1}{4}n \end{aligned}$$

So, in what follows, we show that $K_{q,n-q}$ is the unique graph in B_n^q with the maximum transmission.

Choose G in B_n^q such that its transmission is as large as possible. If $q = \left[\frac{n}{2}\right]$, by Lemma 2.1 the extremal graph is just $K_{[\frac{n}{2}],[\frac{n}{2}]}$, as desired. So, in what follows, we consider $q < \left[\frac{n}{2}\right]$ ·

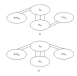

Figure 1

Let (U, W) be the bipartition of the vertex set of G such that $|W| \geq |U| \geq q$, and let M be a maximal matching of G. By Lemma 2.1, the sum of all distances of a graph decreases with addition of edges, so if $|U| = q$, then the extremal graph is $G \cong K_{q,n-q}$. So, we assume that $|U| > q$ in what follows. Let U_M, W_M be the sets of vertices of U, W which are incident to the edges of M, respectively. Therefore, $|U_M| = |W_M| = q$. Note that G contains no edges between the vertices of $U \setminus U_M$ and the vertices of $W \setminus W_M$, otherwise any such edge may be united with M to produce a matching of cardinality greater than that of M, violating the maximality of M.

Adding all possible edges between the vertices of U_M and W_M, U_M and $W \setminus W_M$, $U \setminus U_M$ and W_M, we get a graph G' with $H(G') > H(G)$. Note that the matching number of G' is at least $k+1$. Hence, $G' \notin B_n^k$ and $G \cong G'$. Based on G', we construct a new graph, say G'', which is obtained from G' by deleting all the edges between $U \setminus U_M$ and W_M, and adding all the edges between $U \setminus U_M$ and W_M. G'' is depicted in Fig. 1. It is routine to check that $G'' \cong K_{k,n-k}$.

Let $|U \setminus U_M| = n_1$, $|W \setminus W_M| = n_2$. Suppose $n_1 \leq n_2$. We partition $V_{G'} = V_{G''}$ into $U_M \cup W_M \cup (U \setminus U_{<M}) \cup (W \setminus W_M)$ as shown in Fig. 1. By direct calculation, for all $x \in W \setminus W_M$ (resp. $y \in U_M$, $z \in W_M$, $w \in U \setminus U_M$), one has

$$\begin{aligned}
D_G(x) &= q + \frac{1}{2}q + \frac{1}{3}n_1 + \frac{1}{2}(n_2 - 1) = \frac{3}{2}q + \frac{1}{3}n_1 + \frac{1}{2}n_2 - \frac{1}{2} \\
D_{G'}(x) &= q + \frac{1}{2}(q + n_1) + \frac{1}{2}(n_2 - 1) = \frac{3}{2}q + \frac{1}{2}n_1 + \frac{1}{2}n_2 - \frac{1}{2} \\
D_G(y) &= n_2 + q + \frac{1}{2}n_1 + \frac{1}{2}(q - 1) = \frac{3}{2}q + \frac{1}{2}n_1 + n_2 - \frac{1}{2} \\
D_{G'}(y) &= n_1 + n_2 + q + \frac{1}{2}(q - 1) = \frac{3}{2}q + n_1 + n_2 - \frac{1}{2} \\
D_G(z) &= q + \frac{1}{2}n_2 + n_1 + \frac{1}{2}(q - 1) = \frac{3}{2}q + \frac{1}{2}n_2 + n_1 - \frac{1}{2} \\
D_{G'}(z) &= q + \frac{1}{2}(n_2 + n_1) + \frac{1}{2}(q - 1) = \frac{3}{2}q + \frac{1}{2}n_2 + \frac{1}{2}n_1 - \frac{1}{2} \\
D_G(w) &= q + \frac{1}{2}q + \frac{1}{3}n_2 + \frac{1}{2}(n_1 - 1) = \frac{3}{2}q + \frac{1}{2}n_1 + \frac{1}{3}n_2 - \frac{1}{2} \\
D_{G'}(w) &= q + \frac{1}{2}(q + n_2) + \frac{1}{2}(n_1 - 1) = \frac{3}{2}q + \frac{1}{2}n_1 + \frac{1}{2}n_2 - \frac{1}{2}
\end{aligned}$$

$$
\begin{aligned}
H(G')-H(G) &= \frac{1}{2}\left(\sum D_{G'}(x) - \sum D_G(x)\right) \\
&= \frac{1}{2}\Bigg(\sum_{x\in W\setminus W_m} D_{G'}(x) - \sum_{x\in W\setminus W_m} D_G(x) + \sum_{y\in U_m} D_{G'}(y) - \sum_{y\in U_m} D_G(y) \\
&\quad + \sum_{z\in W_m} D_{G'}(z) - \sum_{z\in W_m} D_G(z) + \sum_{w\in U\setminus U_m} D_{G'}(w) - \sum_{w\in U\setminus U_m} D_G(w)\Bigg) \\
&= \frac{1}{2}\Bigg[n_2\left(\frac{3}{2}q + \frac{1}{2}n_1 + \frac{1}{2}n_2 - \frac{1}{2}\right) - n_2\left(\frac{3}{2}q + \frac{1}{3}n_1 + \frac{1}{2}n_2 - \frac{1}{2}\right) \\
&\quad + q\left(\frac{3}{2}q + n_1 + n_2 - \frac{1}{2}\right) - q\left(\frac{3}{2}q + \frac{1}{2}n_1 + n_2 - \frac{1}{2}\right) \\
&\quad + q\left(\frac{3}{2}q + \frac{1}{2}n_2 + \frac{1}{2}n_1 - \frac{1}{2}\right) - q\left(\frac{3}{2}q + \frac{1}{2}n_2 + n_1 - \frac{1}{2}\right) \\
&\quad - n_1\left(\frac{3}{2}q + \frac{1}{2}n_1 + \frac{1}{2}n_2 - \frac{1}{2}\right) - n_1\left(\frac{3}{2}q + \frac{1}{2}n_1 + \frac{1}{3}n_2 - \frac{1}{2}\right)\Bigg] \\
&= \frac{1}{6}n_1 n_2 > 0
\end{aligned}
$$

This completes the proof. ∎

Acknowledgments. The research is partially supported by National Science Foundation of China (11401008, 61472003,61272153 and 61402011) and Natural Science Foundation of Anhui Provincial Education Department (KJ2014A064).

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Accepted: 05.10.2015