FIXED POINTS IN INTUITIONISTIC MENGER SPACE

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Abstract. The purpose of this paper is to use the new concept of absorbing mappings in intuitionistic Menger space and prove fixed point theorems, without appeal to continuity. The results thus obtained, generalizes and extends the result of Rashwan and Hedar [5] in intuitionistic Menger space. We also cited an example in support of our result.

Keywords and phrases: intuitionistic Menger space; common fixed points; compatible mappings; absorbing Mappings.

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1. Introduction

There have been a number of generalizations of metric space. One such generalization is Menger space initiated by Menger [3]. It is a probabilistic generalization in which we assign to any two points x and y, a distribution function $F_{x,y}$. Schweizer and Sklar [6], [7] studied this concept and gave some fundamental results on this space. Sehgal and Bharucha-Reid [8] obtained a generalization of Banach Contraction Principle on a complete Menger space which is a milestone in developing fixed point theory in Menger space.

Kutukcu et al. [2] introduced the notion of intuitionistic Menger spaces with the help of t-norms and t-conorms as a generalization of Menger space due to Menger [3]. Using the concept of weakly compatible maps in intuitionistic Menger space, Pant et. al. [4] proved a common fixed point theorem for six self maps without appeal to continuity.

In this paper, we prove some common fixed point theorems for absorbing mappings in Intuitionistic Menger spaces without using the condition of continuity. We extend and generalize the result Rashwan and Hedar [5]. We support our theorem by providing a suitable example.

2. Preliminaries

Definition 2.1. [4] A binary operation $* : [0,1] \times [0,1] \to [0,1]$ is called a t-norm if it satisfies the following conditions:

(1) $*$ is commutative and associative,
(2) $*$ is continuous,
(3) $a * 1 = a$, for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Definition 2.2. [1] A binary operation $\diamondsuit : [0,1] \times [0,1] \to [0,1]$ is called a t-conorm if it satisfies the following conditions:

(1) $\diamondsuit$ is commutative and associative.
(2) $\diamondsuit$ is continuous,
(3) $a \diamondsuit 0 = a$, for all $a \in [0,1]$
(4) $a \diamondsuit b \leq c \diamondsuit d$ whenever $a \leq c$ and $b \leq d$, for all $a, b, c, d \in [0,1]$.

Remark 2.1. [4] The concept of triangular norms (t-norms) and triangular conorms (t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersection and union respectively. These concepts were originally introduced by Menger [3] in his study of statistical metric spaces.

Definition 2.3. [4] A distance distribution function is a function $\mathbf{F} : R \to R^+$ which is left continuous on R, non-decreasing and $\inf_{t \in R} F(t) = 0$, $\sup_{t \in R} F(t) = 1$. We will denote by D the family of all distance distribution function and by H a special of D defined by

$$H(t) = \begin{cases} 0, & \text{if } t \leq 0 \\ 1, & \text{if } t > 0. \end{cases}$$

If X is a non-empty set, $\boldsymbol{F} : X \times X \to D$ is called a probabilistic distance on X and $F(x, y)$ is usually denoted by $F_{x,y}$.

Definition 2.4. [4] A non-distance distribution function is a function $\boldsymbol{L} : R \to R^+$ which is right continuous on R, non-increasing and $\inf_{t \in R} L(t) = 1$, $\sup_{t \in R} L(t) = 0$. We will denote by E the family of all distance distribution function and by G a special of E defined by

$$G(t) = \begin{cases} 1, & \text{if } t \leq 0 \\ 0, & \text{if } t > 0 \end{cases}.$$

If X is a non-empty set, $\boldsymbol{L} : X \times X \to E$ is called a probabilistic non-distance on X and $L(x, y)$ is usually denoted by $L_{x,y}$.

Definition 2.5. [4] A 5-tuple $(X, \boldsymbol{F}, \boldsymbol{L}, *, \Diamond)$ is said to be an intuitionistic Menger space if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is continuous t-conorm, $\boldsymbol{F}$ is a probabilistic distance and $\boldsymbol{L}$ is a probabilistic non-distance on X satisfying the following conditions :
for all $x, y, z \in X$ and $t, s \geq 0$,

(1) $F_{x,y}(t) + L_{x,y}(t) \leq 1$,
(2) $F_{x,y}(0) = 0$,
(3) $F_{x,y}(t) = H(t)$ if and only if $x = y$,
(4) $F_{x,y}(t) = F_{y,x}(t)$,
(5) if $F_{x,y}(t) = 1$ and $F_{y,z}(s) = 1$, then $F_{x,z}(t + s) = 1$,
(6) $F_{x,z}(t + s) \geq F_{x,y}(t) * F_{y,z}(s)$,
(7) $L_{x,y}(0) = 1$,
(8) $L_{x,y}(t) = G(t)$ if and only if $x = y$,
(9) $L_{x,y}(t) = L_{y,x}(t)$,
(10) if $L_{x,y}(t) = 0$ and $L_{y,z}(s) = 0$, then $L_{x,z}(t + s) = 0$,
(11) $L_{x,z}(t + s) \leq L_{x,y}(t) \Diamond L_{y,z}(s)$.

The functions $F_{x,y}(t)$ and $L_{x,y}(t)$ denote the degree of nearness and degree of non-nearness between x and y with respect to t, respectively.

Example 2.1. [4] Let (X, d) be a usual metric space. Then the metric d induces a distance distribution function $\boldsymbol{F}$ defined by $F_{x,y}(t) = H(t - d(x, y))$ and non-distance distribution function $\boldsymbol{L}$ defined by $L_{x,y}(t) = G(t - d(x, y))$ for all $x, y \in X$ and $t \geq 0$. Then $(X, \boldsymbol{F}, \boldsymbol{L})$ is an intuitionistic probabilistic metric space. We call this intuitionistic probabilistic metric space induced by a metric d, the induced intuitionistic probabilistic metric space. If t-norm $*$ is $a * b = \min\{a, b\}$ and t-conorm $\Diamond$ is $a \Diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ then $(X, \boldsymbol{F}, \boldsymbol{L}, *, \Diamond)$ is an intuitionistic Menger space.

Definition 2.6. Self mappings A and S of an intuitionistic Menger space $(X, \boldsymbol{F}, \boldsymbol{L}, *, \Diamond)$ are said to be compatible if

$$\lim_{n\to\infty} F_{ASx_n,SAx_n}(t) = 1 \text{ and } \lim_{n\to\infty} L_{ASx_n,SAx_n}(t) = 0$$

whenever there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = z$, $z \in X$.

Definition 2.7. Self mappings A and S of an intuitionistic Menger space $(X, \boldsymbol{F}, \boldsymbol{L}, *, \Diamond)$ are said to be reciprocal continuous if

$$\lim_{n\to\infty} ASx_n = At \text{ and } \lim_{n\to\infty} SAx_n = St$$

for some $t \in X$ whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} Ax_n = \lim_{n\to\infty} Sx_n = t$, $t \in X$.

Lemma 2.1. [4] Let $(X, \boldsymbol{F}, \boldsymbol{L}, *, \Diamond)$ be an intuitionistic Menger space with $t*t \geq t$ and $(1-t) \Diamond (1-t) \leq (1-t)$ and for all $x, y \in X$, $t > 0$ and if for a number $k \in (0,1)$

$$F_{x,y}(kt) \geq F_{x,y}(t) \text{ and } L_{x,y}(kt) \leq L_{x,y}(t).$$

Then x = y.

Definition 2.8. Suppose A and S be two self mappings of an intuitionistic Menger space $(X, \boldsymbol{F}, \boldsymbol{L}, *, \Diamond)$, then A is called S-absorbing if there exists a positive integer $R > 0$ such that $F_{Sx,SAx}(t) \geq F_{Sx,Ax}(t/R)$ and $L_{Sx,SAx}(t) \leq L_{Sx,Ax}(t/R)$ for all $x \in X$.

Similarly, S is called A-absorbing if there exists a positive integer $R > 0$ such that $F_{Ax,ASx}(t) \geq F_{Ax,Sx}(t/R)$ and $L_{Sx,SAx}(t) \leq L_{Sx,Ax}(t/R)$ for all $x \in X$.

Map A is called pointwise S-absorbing if for given $x \in X$ there exists a positive integer $R > 0$ such that $F_{Sx,SAx}(t) \geq F_{Sx,Ax}(t/R)$ and $L_{Sx,SAx}(t) \leq L_{Sx,Ax}(t/R)$ for all $x \in X$.

Similarly, we can define pointwise A-absorbing map.

Example 2.2. Let $X = [2, 20]$ for each $t \in (0, \infty)$ and $x, y \in X$. Define $(\boldsymbol{F}, \boldsymbol{L})$ by

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases}$$

$$L_{x,y}(t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, & \text{if } t > 0 \\ 1, & \text{if } t = 0 \end{cases}$$

Define self maps A and B as follows:

$$A(x) = \begin{cases} 6, & \text{if } 2 \leq x \leq 5 \text{ and } x = 6 \\ 10, & \text{if } x > 6 \\ \frac{x-1}{2}, & \text{if } 5 < x < 6 \end{cases}$$

$$B(x) = \begin{cases} 2, & \text{if } 2 \leq x \leq 5 \\ \frac{x+1}{3}, & \text{if } x > 5 \end{cases}$$

Now, consider a sequence $x_n = 5 + \frac{1}{2n}$ for $n = 1, 2, 3, ...$ then it is easy to see that both pairs (A, B) and (B, A) are not-compatible but A is B-absorbing and B is A-absorbing.

Remark 2.2. In view of above example, it is clear that absorbing maps are more general than that of compatible maps.

Lemma 2.2. [4] Let $(X, \boldsymbol{F}, \boldsymbol{L}, *, \Diamond)$ be an intuitionistic Menger space with $t*t \geq t$ and $(1-t) \Diamond (1-t) \leq (1-t)$ and for all $x, y \in X,\ t > 0$. If there exists a constant $k \in (0,1)$ such that for $n = 1, 2, ...$

$$F_{y_{n+2},y_{n+1}}(kt) \geq F_{y_{n+1},y_n}(t) \text{ and } L_{y_{n+2},y_{n+1}}(kt) \leq L_{y_{n+1},y_n}(t).$$

Then $\{y_n\}$ is a Cauchy sequence in X.

Lemma 2.3. [4] In an intuitionistic Menger space $(X, \boldsymbol{F}, \boldsymbol{L}, *, \Diamond)$, $F_{x,y}(.)$ is non-decreasing and $L_{x,y}(.)$ is non-increasing for all $x, y \in X$.

3. Main result

Theorem 3.1. *Let P be pointwise S-absorbing and Q be pointwise T-absorbing self maps on a complete intuitionistic Menger space $(X, \mathbf{F}, \mathbf{L}, *, \Diamond)$ with continuous t-norm $*$ and continuous t-conorm $\Diamond$ defined by $a * b = \min\{a, b\}$ and $a \Diamond b = \max\{a, b\}$ for all $a, b \in [0,1]$ satisfying the following conditions:*

(i) $P(X) \subseteq T(X),\ \ Q(X) \subseteq S(X)$;
(ii) *there exists $k \in (0,1)$ such that for every $x, y \in X$ and $t > 0$*

$$F_{Px,Qy}(kt) \geq \min\Big\{F_{Sx,Ty}(t), F_{Px,Sx}(t), F_{Qy,Ty}(t), F_{Px,Ty}(t)\Big\}$$
$$L_{Px,Qy}(kt) \leq \max\Big\{L_{Sx,Ty}(t), L_{Px,Sx}(t), L_{Qy,Ty}(t), L_{Px,Ty}(t)\Big\};$$

(iii) *for all $x, y \in X$, $F_{x,y}(t) = 1$ and $L_{x,y}(t) = 0$.*

If the pair (P, S) is reciprocally continuous compatible maps then P, Q, S and T have a unique common fixed point in X.

Proof. Let x_0 be any arbitrary point in X, construct a sequence $\{y_n\} \in X$ such that $y_{2n-1} = Tx_{2n-1} = Px_{2n-2}$ and $y_{2n} = Sx_{2n} = Qx_{2n-1}, n = 1, 2, 3, ...$ This can be done by virtue of (i).
By using contractive condition, we obtain

$$\begin{aligned} F_{y_{2n+1},y_{2n+2}}(kt) &= F_{Px_{2n},Qx_{2n+1}}(kt) \\ &\geq \min\{F_{Sx_{2n},Tx_{2n+1}}(t), F_{Px_{2n},Sx_{2n}}(t), F_{Qx_{2n+1},Tx_{2n+1}}(t), F_{Px_{2n},Tx_{2n+1}}(t)\} \\ &\geq \min\{F_{y_{2n},y_{2n+1}}(t), F_{y_{2n+1},y_{2n}}(t), F_{y_{2n+2},y_{2n+1}}(t), 1\} \end{aligned}$$

$$\begin{aligned} L_{y_{2n+1},y_{2n+2}}(kt) &= L_{Px_{2n},Qx_{2n+1}}(kt) \\ &\leq \max\{L_{Sx_{2n},Tx_{2n+1}}(t), L_{Px_{2n},Sx_{2n}}(t), L_{Qx_{2n+1},Tx_{2n+1}}(t), L_{Px_{2n},Tx_{2n+1}}(t)\} \\ &\leq \max\{L_{y_{2n},y_{2n+1}}(t), L_{y_{2n+1},y_{2n}}(t), L_{y_{2n+2},y_{2n+1}}(t), 1\} \end{aligned}$$

which implies,

$$F_{y_{2n+1},y_{2n+2}}(kt) \geq F_{y_{2n},y_{2n+1}}(t)$$
$$L_{y_{2n+1},y_{2n+2}}(kt) \leq L_{y_{2n},y_{2n+1}}(t).$$

In general,

$$\begin{aligned} F_{y_n,y_{n+1}}(kt) &\geq F_{y_{n-1},y_n}(t) \\ L_{y_n,y_{n+1}}(kt) &\leq L_{y_{n-1},y_n}(t). \end{aligned} \tag{1}$$

To prove $\{y_n\}$ is a Cauchy sequence, we have to show

$$F_{y_n,y_{n+1}}(t) \to 1 \text{ and } L_{y_n,y_{n+1}}(t) \to 0.$$

For this, from (iii) we have

$$F_{y_n,y_{n+1}}(t) \geq F_{y_{n-1},y_n}(t/k) \geq F_{y_{n-2},y_{n-1}}(t/k^2) \geq ... \geq F_{y_0,y_1}(t/k^n) \to 1 \text{ as } n \to \infty,$$
$$L_{y_n,y_{n+1}}(t) \leq L_{y_{n-1},y_n}(t/k) \leq L_{y_{n-2},y_{n-1}}(t/k^2) \leq ... \leq L_{y_0,y_1}(t/k^n) \to 0 \text{ as } n \to \infty.$$

For $p \in N$, by (1) we have

$$\begin{aligned} F_{y_n,y_{n+p}}(t) &\geq F_{y_n,y_{n+1}}((1-k)t) * F_{y_{n+1},y_{n+p}}(kt) \\ &\geq F_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) * F_{y_{n+1},y_{n+2}}(t) * F_{y_{n+2},y_{n+p}}((k-1)t) \\ &\geq F_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) * F_{y_0,y_1}\left(\frac{t}{k^n}\right) * F_{y_{n+2},y_{n+3}}(t) * F_{y_{n+3},y_{n+p}}((k-2)t) \\ &\geq F_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) * F_{y_0,y_1}\left(\frac{t}{k^n}\right) * F_{y_0,y_1}\left(\frac{(1-k)t}{k^{n+2}}\right) * \cdots * F_{y_0,y_1}\left(\frac{(k-p)t}{k^{n+p+1}}\right) \end{aligned}$$

and

$$\begin{aligned} L_{y_n,y_{n+p}}(t) &\leq L_{y_n,y_{n+1}}((1-k)t) \diamondsuit L_{y_{n+1},y_{n+p}}(kt) \\ &\leq L_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) \diamondsuit L_{y_{n+1},y_{n+2}}(t) \diamondsuit L_{y_{n+2},y_{n+p}}((k-1)t) \\ &\leq L_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) \diamondsuit L_{y_0,y_1}\left(\frac{t}{k^n}\right) \diamondsuit L_{y_{n+2},y_{n+3}}(t) \diamondsuit L_{y_{n+3},y_{n+p}}((k-2)t) \\ &\leq L_{y_0,y_1}\left(\frac{(1-k)t}{k^n}\right) \diamondsuit L_{y_0,y_1}\left(\frac{t}{k^n}\right) \diamondsuit L_{y_0,y_1}\left(\frac{(1-k)t}{k^{n+2}}\right) \\ &\qquad \diamondsuit \ldots \diamondsuit L_{y_0,y_1}\left(\frac{(k-p)t}{k^{n+p+1}}\right). \end{aligned}$$

Thus, $F_{y_n,y_{n+p}}(t) \to 1$ and $L_{y_n,y_{n+p}}(t) \to 0$ for all $t > 0$.

Therefore, $\{y_n\}$ is a Cauchy sequence in X.

But $(X, \boldsymbol{F}, \boldsymbol{L}, *, \diamondsuit)$ is complete, so there exists a point z (say) in X such that $\{y_n\} \to z$.

Also, using (i), we have

$$\{Px_{2n-2}\},\ \{Tx_{2n-1}\},\ \{Sx_{2n}\},\ \{Qx_{2n-1}\} \to z.$$

Since the pair (P, S) is reciprocally continuous, then we have

$$\lim_{n\to\infty} PSx_{2n} = Pz \ \text{ and } \ \lim_{n\to\infty} SPx_{2n} = Sz.$$

Compatibility of P and S yields

$$\lim_{n\to\infty} F_{PSx_{2n},SPx_{2n}}(t) = 1 \ \text{ and}$$
$$\lim_{n\to\infty} L_{PSx_{2n},SPx_{2n}}(t) = 0,$$

i.e.,

$$F_{Pz,Sz}(t) = 1 \ \text{ and}$$
$$L_{Pz,Sz}(t) = 0.$$

Hence $Pz = Sz$.

Since $P(X) \subseteq T(X)$ then there exists a point u in X such that $Pz = Tu$.

Now, by the contractive condition (ii), we get

$$\begin{aligned} F_{Pz,Qu}(kt) &\geq \min\Big\{F_{Sz,Tu}(t), F_{Pz,Sz}(t), F_{Qu,Tu}(t), F_{Pz,Tu}(t)\Big\} \\ &\geq \min\Big\{F_{Pz,Pz}(t), F_{Pz,Pz}(t), F_{Qu,Pz}(t), F_{Pz,Pz}(t)\Big\} \\ &> F_{Pz,Qu}(t) \end{aligned}$$

and

$$\begin{aligned} L_{Pz,Qu}(kt) &\leq \max\Big\{L_{Sz,Tu}(t), L_{Pz,Sz}(t), L_{Qu,Tu}(t), L_{Pz,Tu}(t)\Big\} \\ &\leq \max\Big\{L_{Pz,Pz}(t), L_{Pz,Pz}(t), L_{Qu,Pz}(t), L_{Pz,Pz}(t)\Big\} \\ &< L_{Pz,Qu}(t) \end{aligned}$$

i.e., $Pz = Qu$.

Thus, $Pz = Sz = Qu = Tu$. Since P is pointwise S-absorbing then for $R > 0$, we have

$$F_{Sz,SPz}(t) \geq F_{Sz,Pz}(t/R) = 1 \ \text{ and } \ L_{Sz,SPz}(t) \leq L_{Sz,Pz}(t/R) = 0,$$

i.e., $Pz = SPz = Sz$.

Now, by the contractive condition, we have

$$\begin{aligned} F_{Pz,PPz}(t) &= F_{PPz,Qu}(t) \\ &\geq \min\Big\{F_{SPz,Tu}(t), F_{PPz,Su}(t), F_{Qu,Tu}(t), F_{PPz,Tu}(t)\Big\} \\ &= \min\Big\{F_{Pz,Pz}(t), F_{PPz,Pz}(t), F_{Qu,Qu}(t), F_{PPz,Pz}(t)\Big\} \\ &= F_{PPz,Pz}(t) \end{aligned}$$

$$\begin{aligned} L_{Pz,PPz}(t) &= L_{PPz,Qu}(t) \\ &\leq \max\Big\{L_{SPz,Tu}(t), L_{PPz,Su}(t), L_{Qu,Tu}(t), L_{PPz,Tu}(t)\Big\} \\ &= \max\Big\{L_{Pz,Pz}(t), L_{PPz,Pz}(t), L_{Qu,Qu}(t), L_{PPz,Pz}(t)\Big\} \\ &= L_{PPz,Pz}(t) \end{aligned}$$

i.e., $PPz = Pz = SPz$.

Therefore, Pz is a common fixed point of P and S.

Similarly, Q is pointwise T-absorbing, therefore, we have

$$F_{Tu,TQu}(t) \geq F_{Tu,Qu}(t/R) = 1$$
$$\text{and } L_{Tu,TQu}(t) \leq F_{Tu,Qu}(t/R) = 0$$

i.e., $Tu = TQu = Qu$.

Now, by the contractive condition, we have

$$\begin{aligned} F_{QQu,Qu}(t) &= F_{Pz,QQu}(t) \\ &\geq \min\Big\{F_{Sz,TQu}(t), F_{Pz,Su}(t), F_{QQu,TQu}(t), F_{Pz,TQu}(t)\Big\} \\ &= \min\Big\{F_{Sz,Qu}(t), F_{Pz,Pz}(t), F_{QQu,Qu}(t), F_{Pz,Qu}(t)\Big\} \\ &= F_{QQu,Qu}(t) \end{aligned}$$

$$\begin{aligned} L_{QQu,Qu}(t) &= L_{Pz,QQu}(t) \\ &\leq \max\Big\{L_{Sz,TQu}(t), L_{Pz,Su}(t), L_{QQu,TQu}(t), L_{Pz,TQu}(t)\Big\} \\ &= \max\Big\{L_{Sz,Qu}(t), L_{Pz,Pz}(t), L_{QQu,Qu}(t), L_{Pz,Qu}(t)\Big\} \\ &= L_{QQu,Qu}(t) \end{aligned}$$

i.e., $QQu = Qu = TQu$.

Hence $Qu = Pz$ is a common fixed point of P, Q, S and T.

Uniqueness of Pz can easily follow from the contractive condition.

The proof is similar when Q and P are assumed compatible and reciprocally continuous.

Now, we give an example which supports our result.

Example 3.1. Let $X = [2, 20]$ with metric defined by $d(x, y) = |x - y|$ and for each $t \in [0, 1]$ defined

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & \text{if } t > 0 \\ 0, & \text{if } t = 0 \end{cases}$$

$$L_{x,y}(t) = \begin{cases} \frac{|x-y|}{t+|x-y|}, & \text{if } t > 0 \\ 1, & \text{if } t = 0 \end{cases}$$

for all $x, y \in X$. Clearly $(X, \boldsymbol{F}, \boldsymbol{L}, *, \Diamond)$ is an intuitionistic Menger space where $*$ is defined by $t * t \geq t$ and is defined by $(1 - t) \Diamond (1 - t) \leq (1 - t)$.

Define mappings $P, Q, R, S : X \to X$ by

$$P(X) = \begin{cases} 2 & \text{if } x = 2 \\ 3 & \text{if } x > 2 \end{cases}$$

$$S(X) = \begin{cases} 2 & \text{if } x = 2 \\ 6 & \text{if } x > 2 \end{cases}$$

$$Q(X) = \begin{cases} 2 & \text{if } x = 2 \text{ or } x > 5 \\ 6 & \text{if } 2 < x \leq 5 \end{cases}$$

$$T(X) = \begin{cases} 2 & \text{if } 2 \leq x \leq 5 \\ x - 3 & \text{if } x > 5. \end{cases}$$

Then P, Q, S and T satisfy all the conditions of above theorem with $k \in (0, 1)$ and have a unique common fixed point $x = 2$. Here P and S are reciprocally continuous compatible maps. But neither P nor S is continuous even at fixed point $x = 2$.

The mappings Q and T are non-compatible but Q is pointwise T-absorbing. To see Q and T are non-compatible, let us consider the sequence $\{x_n\}$ in X defined by

$$x_n = 5 + (1/n), \quad n \geq 1.$$

Then $\{Tx_n\} \to 2, \{Qx_n\} \to 2, \{TQx_n\} \to 2$ and $\{QTx_n\} \to 6$.

Hence Q and T are non-compatible.

Now, we prove the result by assuming the range of one of the mappings P, Q, S or T to be a complete subspace of X.

Theorem 3.2. *Let P be pointwise S-absorbing and Q be pointwise T-absorbing self maps on a complete intuitionistic Menger space $(X, \mathbf{F}, \mathbf{L}, *, \Diamond)$ with continuous t-norm $*$ and continuous t-conorm $\Diamond$ defined by $a * b = \min\{a, b\}$ and $a \Diamond b = \max\{a, b\}$ for all $a, b \in [0, 1]$ satisfying the following conditions:*

(a) $P(X) \subseteq T(X)$, $Q(X) \subseteq S(X)$;
(b) *there exists* $k \in (0,1)$ *such that for every* $x, y \in X$ *and* $t > 0$

$$F_{Px,Qy}(kt) \geq \min\left\{F_{Sx,Ty}(t), F_{Px,Sx}(t), F_{Qy,Ty}(t), F_{Px,Ty}(t)\right\}$$
$$L_{Px,Qy}(kt) \leq \max\left\{L_{Sx,Ty}(t), L_{Px,Sx}(t), L_{Qy,Ty}(t), L_{Px,Ty}(t)\right\} \text{ and}$$

(c) *for all* $x, y \in X$, $\lim_{n\to\infty} F_{x,y}(t) = 1$ *and* $\lim_{n\to\infty} L_{x,y}(t) = 0$.

If the range of one of the mapping P, Q, S *or* T *be a complete subspace of* X *then* P, Q, S *and* T *have a unique common fixed point in* X.

Proof. Let x_0 be any arbitrary point in X, construct a sequence $\{y_n\} \in X$ such that

(3.2.1) $$y_{2n-1} = Tx_{2n-1} = Px_{2n-2} \text{ and } y_{2n} = Sx_{2n} = Qx_{2n+1},\ n = 1, 2, 3, \ldots.$$

This can be done by virtue of (a) and by using the same technique of above theorem, we can show that $\{y_n\}$ is a Cauchy sequence.

Let $S(X)$ the range of X be a complete metric subspace then there exists a point Su such that $\lim_{n\to\infty} Sx_{2n} = Su$.

By (3.2.1), we get

$$Qx_{2n+1} \to Su,\ Px_{2n-2} \to Su,\ Tx_{2n-1} \to Su \text{ and } \{y_n\} \to Su \text{ as } n \to \infty.$$

By using contractive condition, we obtain

$$F_{Pu,Qx_{2n+1}}(kt) \geq \min\left\{F_{Su,Tx_{2n+1}}(t), F_{Pu,Su}(t), F_{Qx_{2n+1},Tx_{2n+1}}(t), F_{Pu,Tx_{2n+1}}(t)\right\}$$
$$L_{Pu,Qx_{2n+1}}(kt) \leq \max\left\{L_{Su,Tx_{2n+1}}(t), L_{Pu,Su}(t), L_{Qx_{2n+1},Tx_{2n+1}}(t), L_{Pu,Tx_{2n+1}}(t)\right\}.$$

Letting $n \to \infty$, we get

$$F_{Pu,Su}(kt) \geq \min\left\{F_{Su,Su}(t), F_{Pu,Su}(t), F_{Su,Su}(t), F_{Pu,Su}(t)\right\}$$
$$L_{Pu,Su}(kt) \leq \max\left\{L_{Su,Su}(t), L_{Pu,Su}(t), L_{Su,Su}(t), L_{Pu,Su}(t)\right\}$$

i.e., $Pu = Su$.

Since $P(X) \subseteq T(X)$ then there exists $w \in X$ such that $Su = Tw$.

Again, by using the contractive condition, we get

$$F_{Pu,Qw}(kt) \geq \min\left\{F_{Su,Tw}(t), F_{Pu,Su}(t), F_{Qw,Tw}(t), F_{Pu,Tw}(t)\right\}$$
$$L_{Pu,Qw}(kt) \leq \max\left\{L_{Su,Tw}(t), L_{Pu,Su}(t), L_{Qw,Tw}(t), L_{Pu,Tw}(t)\right\}$$

i.e., $Pu = Su = Qw = Tw$.

Since P is pointwise S-absorbing, then we have

$$F_{Su,SPu}(t) \geq F_{Su,Qu}(t/R)$$
$$L_{Su,SPu}(t) \leq L_{Su,Qu}(t/R)$$

i.e., $Su = SPu = SSu$ and, similarly, Q is pointwise T-absorbing then we have

$$F_{Tw,TQw}(t) \geq F_{Tw,Qw}(t/R)$$
$$L_{Tw,TQw}(t) \leq L_{Tw,Qw}(t/R)$$

i.e., $Tw = TQw = QQw$.

Thus, $Su(= Tw)$ is a common fixed point of P, Q, S and T.

Uniqueness of common fixed point follows from the contractive condition. The proof is similar when $T(X)$, the range of T, is assumed to be a complete subspace of X. Moreover, since $P(X) \subseteq T(X)$ and $Q(X) \subseteq S(X)$.

The proof follows on similar lines when either the range of P or range of Q is assumed complete. This completes the proof of the theorem.

Conclusion. As compatible mappings are absorbing mappings but reverse implication not follows. So, our Theorem 3.1 generalizes and extends the result of Rashwan and Hedar [5] in the sense that the mappings involved are absorbing mappings and we establish the theorems without appeal to continuity.

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