FILTER THEORY ON HYPER BE-ALGEBRAS

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Abstract. In this paper, we focus on investigating some types of hyper filters on hyper BE-algebras and discuss the relations among them. Also we construct quotient hyper BE-algebras and deliver some related results.

Keywords: hyper BE-algebra; hyper filter; upper sets; (positive) implicative hyper filter; quotient hyper BE-algebra.

1. Introduction

The hyper structure theory was introduced by Marty [6], at the 8th Congress of Scandinavian Mathematicians. Since then hyper structure theory has been intensively researched [1],[3],[4],[10],[11]. Recently, Radfar applied the hyper theory to the BE-algebras [5],[8],[9], and introduced the notion of a hyper BE-algebras [7], which is a generalization of a dual hyper K-algebra [2],[11].

The other part of this paper is organized as follows: in Section 2, we recall notions of hyper BE-algebras. In Section 3, we mainly introduce the notion of upper sets in hyper BE-algebras and give some important theorems about hyper filters. In Section 4, we introduce the notion of (positive) implicative hyper filters on hyper BE-algebras and discuss some relations among various hyper filters.

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Also, we give some characterizations of (positive) implicative hyper filters. In Section 5, we construct the quotient hyper BE-algebras and give out some related results.

2. Preliminaries

Definition 2.1 [5] An algebra $(X;*,1)$ of type $(2,0)$ is called a BE-algebra if it satisfies the following conditions: for any $x,y,z\in X$,
(BE1) $x*x=1$;
(BE2) $x*1=1$;
(BE3) $1*x=x$;
(BE4) $x*(y*z)=y*(x*z)$.

In any BE algebra, the relation $\leq$ can be defined as follows: $x\leq y$ iff $x*y=1$ for all $x,y\in X$.

Definition 2.2 [7] Let H be a nonempty set and $\circ : H\times H\rightarrow P^*(H)$ be a hyperoperation. Then $(H;\circ,1)$ is called a hyper BE-algebra provided it satisfies the following axioms:
(HBE1) $x<1$ and $x<x$;
(HBE2) $x\circ(y\circ z)=y\circ(x\circ z)$;
(HBE3) $x\in 1\circ x$;
(HBE4) $1<x$ implies $x=1$,
for all $x,y\in H$, where the relation $<$ is defined by $x<y\Leftrightarrow 1\in x\circ y$. For any two nonempty subsets A and B of H, $A<B$ means that there exist $a\in A$ and $b\in B$ such that $a<b$ and $A\circ B:=\bigcup_{a\in A,b\in B}a\circ b$.

In addition, we write $A\ll B$ if for any $a\in A$, there exists $b\in B$ such that $a<b$.

Example 2.3 [7] Define the hyperoperation " $\circ$ " on R as follows:

$$x\circ y=\begin{cases}\{\text{y}\} & \text{if } x=1\\ R & \text{otherwise.}\end{cases}$$

Then $(R;\circ,1)$ is a hyper BE-algebra.

Example 2.4 [7] Let $H=\{1,a,b\}$. Define two hyperoperations " $\circ_1$ " and " $\circ_2$ " on H as follows:

$\circ_1$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1,a,b\}$	$\{b\}$
b	$\{1\}$	$\{a,b\}$	$\{1,b\}$

$\circ_2$	1	a	b
1	$\{1\}$	$\{a,b\}$	$\{b\}$
a	$\{1\}$	$\{1,a,b\}$	$\{b\}$
b	$\{1,b\}$	$\{1,a,b\}$	$\{1,a,b\}$

Then $(H;\circ_1,1)$ and $(H;\circ_2,1)$ are hyper BE-algebras.

Example 2.5 [7] Let $H = \{1, a, b, c\}$. Define the hyperoperation " $\circ$ " on H as follows:

$\circ$	1	a	b	c
1	$\{1\}$	$\{a\}$	$\{b\}$	$\{c\}$
a	$\{1\}$	$\{1\}$	$\{a\}$	$\{b, c\}$
b	$\{1\}$	$\{1\}$	$\{1\}$	$\{1\}$
c	$\{1\}$	$\{1\}$	$\{a\}$	$\{1, b, c\}$

Then $(H; \circ, 1)$ is a hyper BE-algebra.

Proposition 2.6 [7] *Let* $(H; \circ, 1)$ *be a hyper BE-algebra. Then*
(1) $A \circ (B \circ C) = B \circ (A \circ C)$*;*
(2) $A < A$*;*
(3) $1 < A$ *implies* $1 \in A$*;*
(4) $x < y \circ x$*, ;*
(5) $x < y \circ z$ *implies* $y < x \circ z$*;*
(6) $x < (x \circ y) \circ y$*;*
(7) $z \in x \circ y$ *implies* $x < z \circ y$*;*
(8) $y \in 1 \circ x$ *implies* $y < x$*, for all* $x, y, z \in H$ *and* $A, B, C \subseteq H$*.*

Proposition 2.7 *Let* $(H; \circ, 1)$ *be a hyper BE-algebra. Then*
(1) $A \ll B \circ A$*;*
(2) $A < B$ *iff* $1 \in A \circ B$*;*
(3) $A \subseteq 1 \circ A$*;*
(4) $A \subseteq B$ *implies* $A < B$ *and* $A \ll B$*;*
(5) $A \ll B$ *and* $1 \in A$ *imply* $1 \in B$*, for all* $A, B, C \subseteq H$*.*

Proof. (3) Since $1 \circ A = \bigcup_{a \in A} 1 \circ a$ and $a \in 1 \circ a$, we have $A \subseteq 1 \circ A$.

(4) Let $x \in A$. Then $x \in B$. Hence $1 \in x \circ x$, which implies $1 \in A \circ B$. Using (4), the proof is finished.

(5) Since $A \ll B$ and $1 \in A$, then there exists $b \in B$ such that $1 < b$. By (HBE4), $b = 1$ and so $1 \in B$.

Definition 2.8 [7] A hyper BE-algebra $(H; \circ, 1)$ is called

1. row hyper BE-algebra (for briefly, R-hyper BE-algebra), if $1 \circ x = \{x\}$, for all $x \in H$;
2. column hyper BE-algebra (for briefly, C-hyper BE-algebra), if $x \circ 1 = \{1\}$, for all $x \in H$;
3. diagonal hyper BE-algebra (for briefly, D-hyper BE-algebra), if $x \circ x = \{1\}$, for all $x \in H$;
4. RC-hyper BE-algebra, if H is both R-hyper BE-algebra and C-hyper BE-algebra;
5. RCD-hyper BE-algebra, if H is R-hyper BE-algebra,C-hyper BE-algebra and D-hyper BE-algebra.

3. Hyper subalgebras and (weak) hyper filters

Definition 3.1 [7] A nonempty subset S of a hyper BE-algebra H is said to be a hyper subalgebra of H, if $x \circ y \subseteq S$, for all $x, y \in S$.

Theorem 3.2 *Let $(H_1; \circ_1, 1_1)$ and $(H_2; \circ_2, 1_2)$ be two hyper BE-algebras and $H = H_1 \times H_2$. We define a hyperoperation $\circ$ on H: $(x_1, y_1) \circ (x_2, y_2) = (x_1 \circ x_2, y_1 \circ y_2)$ for all $(x_1, y_1), (x_2, y_2) \in H$, where for $A \subseteq H_1$ and $B \subseteq H_2$, we define $(A, B) := \{(x, y) : x \in A, y \in B\}$ and $1 := (1_1, 1_2)$. Then $(H; \circ, 1)$ is a hyper BE-algebra and we call it the product of hyper BE-algebras H_1 and H_2.*

Proof. Let $(x, y) \in H$. Since $(x, y) \circ (1_1, 1_2) = (x \circ_1 1_1, y \circ_2 1_2)$ and $1_1 \in x \circ_1 1_1$, $1_2 \in y \circ_2 1_2$, we have $1 \in (x, y) \circ 1$. That is, $(x, y) < 1$. The proof of $(x, y) < (x, y)$ is obtained by $x < x$ and $y < y$. Therefore (HBE1) holds.

Let $x = (x_1, y_1), y = (x_2, y_2), z = (x_3, y_3) \in H$. Then

$$\begin{aligned}
x \circ (y \circ z) &= (x_1, y_1) \circ (x_2 \circ x_3, y_2 \circ y_3) \\
&= \bigcup\{(x_1, y_1) \circ (a, b) : a \in x_2 \circ_1 x_3, b \in y_2 \circ_2 y_3\} \\
&= \bigcup\{(x_1 \circ_1 a, y_1 \circ_2 b) : a \in x_2 \circ_1 x_3, b \in y_2 \circ_2 y_3\} \\
&= (x_1 \circ_1 (x_2 \circ x_3), y_1 \circ_2 (y_2 \circ y_3)) \\
&= \bigcup\{(c, d) : c \in x_1 \circ_1 (x_2 \circ x_3), d \in y_1 \circ_2 (y_2 \circ y_3)\} \\
&= \bigcup\{(c, d) : c \in x_2 \circ_1 (x_1 \circ x_3), d \in y_2 \circ_2 (y_1 \circ y_3)\} \\
&= (x_2 \circ_1 (x_1 \circ x_3), y_2 \circ_2 (y_1 \circ y_3)) \\
&= y \circ (x \circ z).
\end{aligned}$$

So (HBE2) holds.

Let $(x, y) \in H$. Then $x \in 1_1 \circ_1 x, y \in 1_2 \circ_2 y$.

Note that $(1_1, 1_2) \circ (x, y) = (1_1 \circ_1 x, 1_2 \circ y)$, so that $(x, y) \in (1_1 \circ_1 x, 1_2 \circ y)$. That is, $(x, y) \in 1 \circ (x, y)$ and thus (HBE3) holds.

Let $(x, y) \in H$ and $(1_1, 1_2) < (x, y)$. Then $1_1 < x, 1_2 < y$ and this shows $x = 1_1, y = 1_2$. So we have $(x, y) = (1_1, 1_2) = 1$. It implies that (HBE4) holds and the proof is completed.

Definition 3.3 [7] Let F be a nonempty subset of hyper BE-algebra H and $1 \in F$. Then F is called

(1) a weak hyper filter of H if $x \circ y \subseteq F$ and $x \in F$ imply $y \in F$, for all $x, y \in H$;
(2) a hyper filter of H if $x \circ y \approx F$ and $x \in F$ imply $y \in F$, where $x \circ y \approx F$ means that $x \circ y \cap F \neq \emptyset$, for all $x, y \in H$.

Remark 3.4

(1) Every hyper filter of a hyper BE-algebra H is a weak hyper filter of H, but the converse needn't hold in general (see [7]).
(2) $\{1\}$ is not a hyper subalgebra of H. If H satisfies $1 \circ 1 = \{1\}$, then $\{1\}$ is a hyper subalgebra of H.
(3) If H is a R-hyper (C-hyper, D-hyper) BE-algebra, then $\{1\}$ is a hyper subalgebra of H.

(4) $\{1\}$ is a hyper filter. Furthermore, $\{1\}$ is a weak hyper filter by (1). In fact, Let $x \in \{1\}$ and $x \circ y \approx \{1\}$ i.e. $x \circ y \cap \{1\} \neq \emptyset$ for any $x, y \in H$. Then $x = 1$ and so $1 \circ y \cap 1 \neq \emptyset$. This implies $1 \in 1 \circ y$. Hence $1 < y$, which shows $y = 1$ by (HBE4). It follows that $\{1\}$ is a hyper filter.

(5) A hyper filter F is called proper if $F \neq H$; A hyper filter F is called non-trival if $F \neq \{1\}$ and $F \neq H$.

Theorem 3.5 [7] *Let F be a hyper filter of a hyper BE-algebra H. For all $x, y \in F$, if $x \in F$ and $x < y$, then $y \in F$.*

Theorem 3.6 *Let F be a nonempty subset of a hyper BE-algebra H. Then F is a hyper filter of H if and only if*

(1) $1 \in F$;

(2) $F < x \circ y$ and $x \in F$ imply $y \in F$, for all $x, y \in H$.

Proof. Assume that F is a hyper filter of H. Clearly, (1) holds. To prove (2), we let $x \in F$ and $F < x \circ y$. Then there exist $a \in F$ and $b \in x \circ y$ such that $a < b$. Again, $a \in F$, we get $b \in F$ by Theorem 3.5. This implies that $x \circ y \cap F \neq \emptyset$ and so $y \in F$. Conversely, assume that (1) and (2) hold. Now let $x \in F$ and $x \circ y \approx F$, for any $y \in F$. Then there exists $a \in H$ such that $a \in F$ and $a \in x \circ y$, which imply that $F < x \circ y$. Combing $x \in F$, by the assumption, we conclude $y \in F$. Therefore F is a hyper filter of H.

Theorem 3.7 *Let $\{F_i : i \in I\}$ be a family of nonempty subsets of a hyper BE-algebra H.*

(1) If F_i is a (weak) hyper filter of H for all $i \in I$, then $\bigcap F_i$ is a (weak) hyper filter of H.

(2) If $\{F_i : i \in I\}$ is a chain of hyper filters of H for all $i \in I$, then $\bigcup F_i$ is a hyper filter of H.

Proof. (1) Suppose that F_i is a hyper filter of H, for all $i \in I$. Clearly, $1 \in \bigcap F_i$. Let $\bigcap F_i < x \circ y$ and $x \in \bigcap F_i$ for any $x, y \in H$. From $\bigcap F_i \subseteq F_i$, it follows that $F_i < x \circ y$ for each $i \in I$. Note that $x \in F_i$ and F_i is a hyper filter, we get $y \in F_i$ for each $i \in I$. This implies that $y \in \bigcap F_i$ and so $\bigcap F_i$ is a hyper filter of H by Theorem 3.6. The proof the other case is similar.

(2) Suppose that $\{F_i : i \in I\}$ is a chain of hyper filters of H for all $i \in I$, and $x \in \bigcup F_i$ and $x \circ y \approx \bigcup F_i$, i.e., $x \circ y \cap \bigcup F_i \neq \emptyset$. Then there exist $m, n \in I$ such that $x \in F_m$ and $x \circ y \cap F_n \neq \emptyset$. The rest proof is similar to the above case. Therefore $\bigcup F_i$ is a hyper filter of H.

Theorem 3.8 *Let $(H_1 \times H_2; \circ, (1_1, 1_2))$ be the product of hyper BE-algebras $(H_1; \circ_1, 1_1)$ and $(H_2; \circ_2, 1_2)$. If F_1 and F_2 are (weak) hyper filters of H_1 and H_2, respectively, then $F_1 \times F_2$ is a (weak) hyper filter of $H_1 \times H_2$.*

Proof. Assume that F_1 and F_2 are hyper filters of H_1 and H_2, respectively. Since $1_1 \in F_1, 1_2 \in F_2$, then $(1_1, 1_2) \in F_1 \times F_2$. For any $(x_1, y_1), (x_2, y_2) \in H_1 \times H_2$, let $(x_1, y_1) \in F_1 \times F_2$ and $F_1 \times F_2 < (x_1, y_1) \circ (x_2, y_2)$. Then $F_1 \times F_2 < (x_1 \circ_1 x_2, y_1 \circ_2 y_2)$.

It follows that $F_1 < x_1 \circ_1 x_2$, $F_2 < y_1 \circ_2 y_2$. Since $x_1 \in F_1$ and $y_1 \in F_2$, we get that $x_2 \in F_1$ and $y_2 \in F_2$, which implies $(x_2, y_2) \in F_1 \times F_2$. This shows that $F_1 \times F_2$ is a hyper filter by Theorem 3.6. The proof of the other case is similar.

In any hyper BE-algebra H, let us denote by $[S)$ the least hyper filter of H containing S, called the hyper filter generated by S, where S is a nonempty subset of H. In particular, if $S = \{a\}$, we write $[\{a\}) = [a)$, called the principal hyper filter generated by the element a in H. In addition , we use $[F \cup \{x\})$ to denote the hyper filter generated by F and x, where $x \in H - F$. The following are some results about the generated hyper filters.

Theorem 3.9 *Let S be a nonempty subset of a hyper BE-algebra H. Then*

$$[S) \supseteq \{x \in H : 1 \in a_n \circ (\cdots (a_2 \circ (a_1 \circ x)) \cdots) \text{ for some } a_1, a_2, ..., a_n \in S\}.$$

Proof. Denote the right side of the result by B and let $x \in B$. Then there exist $a_1, a_2, ..., a_n \in S$ such that $1 \in a_n \circ (\cdots (a_2 \circ (a_1 \circ x)) \cdots)$. Hence $a_n \circ (\cdots (a_2 \circ (a_1 \circ x)) \cdots) \cap [S) \neq \emptyset$. Since $a_n \in S \subseteq [S)$ and $[S)$ is a hyper filter, we get $a_{n-1} \circ (\cdots (a_2 \circ (a_1 \circ x)) \cdots) \cap [S) \neq \emptyset$. Repeating the above argument, we verify $x \in [S)$ and thus $[S) \supseteq B$. This proof is finished.

Theorem 3.10 *Let F be a hyper filter of a hyper BE-algebra H and $a \in H - F$. Then*

$$[F \cup \{a\}) \supseteq \{x \in H : a \circ x \cap F \neq \emptyset\}.$$

Proof. Denote $B = \{x \in H : a \circ x \cap F \neq \emptyset\}$ and let $x \in B$. Then $a \circ x \cap F \neq \emptyset$, $a \in H - F$. Since $F \subseteq [F \cup \{a\})$, we have $a \circ x \cap [F \cup \{a\}) \neq \emptyset$. Considering that $a \in [F \cup \{a\})$ and $[F \cup \{a\})$ is a hyper filter, it follows that $x \in [F \cup \{a\})$, proving this theorem.

Corollary 3.11 *Let H be a hyper BE-algebra and $a \in H$. Then*

$$[a) \supseteq \{x \in H : 1 \in a \circ x\}.$$

In the following, we introduce the notion of a upper set of a hyper BE-algebra H, and get some results about the upper set of H. Moreover, by means of the upper set the hyper filter of H will be characterized well.

Definition 3.12 Let $(H; \circ, 1)$ be a hyper BE-algebra and let $x, y \in H$. We define an upper set of x and y by $\{z \in H : 1 \in x \circ (y \circ z)\}$, which is denoted by $U(x, y)$.

Proposition 3.13 *Let $(H; \circ, 1)$ be a hyper BE-algebra. Then $U(x, y)$ has the following properties:*
(1) $1, x, y \in U(x, y)$;
(2) $U(1, 1) = \{1\}$;
(3) $U(x, y) = U(y, x)$.

Proof. (1) Note that $1 \in y \circ 1$ and $x \circ (y \circ 1) = \bigcup\{x \circ b : b \in y \circ 1\}$, so that $1 \in x \circ (y \circ 1)$ and thus $1 \in U(x, y)$. By Proposition 2.3 (4), $x \in U(x, y)$ holds. By (HBE1), $1 \in y \circ y$ and $1 \in x \circ 1$. Again by $x \circ (y \circ y) = \bigcup\{x \circ b : b \in y \circ y\}$, it implies that $1 \in x \circ (y \circ y)$. This shows that $y \in U(x, y)$.

(2) Clearly, $\{1\} \subseteq U(1,1)$ by (1). Let $z \in U(1,1)$. Then $1 \in 1 \circ (1 \circ z)$. This means that there exists $b \in 1 \circ z$ such that $1 \in 1 \circ b$. By (HBE4), $b = 1$. This shows that there is unique element 1 such that $1 \in 1 \circ z$. Again using (HBE2), we get $z = 1$.

(3) By (HBE2).

From Remark 3.4 we see $U(1,1) = \{1\}$ is a hyper filter, but the following examples shows that $U(x,y)$ need not be a (weak) hyper filter of H in general.

Example 3.14 In Example 2.4, $(H; \circ_1, 1)$ is a hyper BE-algebra. One can calculate that the upper set $U(1,a) = \{1,a\}$ is a (weak) hyper filter; The upper set $U(1,b) = \{1,b\}$ is a weak hyper filter, but it is not a hyper filter, since $b \circ a \approx U(1,b)$, $b \in U(1,b)$, but $a \notin U(1,b)$.

Example 3.15 Let $(H; \circ, 1)$ be a hyper BE-algebra from the Example 2.5. It is easy to check that the upper set $U(1,c) = \{1,a,c\}$ is not a weak hyper filter, since $c \circ b = \{a\} \subseteq U(1,c)$, $c \in U(1,c)$, but $b \notin U(1,c)$. Of course, $U(1,c)$ is not also a hyper filter.

Theorem 3.16 *Let H be a hyper BE-algebra with a bottom element* 0 *and $x \in H$. Then*

$$U(x,0) = H = U(0,x).$$

Proof. Clearly, $U(x,y) \subseteq H$. Now Let $z \in H$. Since $1 \in 0 \circ z$, then by (HBE1) $1 \in x \circ (0 \circ z)$, which implies $z \in U(x,0)$. It follows that $U(x,0) = H$. Again using Proposition 3.13 (3), this proof is finished.

Theorem 3.17 *Let F be a nonempty subset of a hyper BE-algebra H. Then F is a hyper filter of H if and only if $U(x,y) \subseteq F$ for all $x,y \in F$.*

Proof. Assume that F is a hyper filter and $x,y \in F$. If $z \in U(x,y)$, then $1 \in x \circ (y \circ z)$. Since $1 \in F$, we have $F < x \circ (y \circ z)$. Hence there exists $a \in y \circ z$ such that $F < x \circ a$ and so $a \in F$. It shows that $F < y \circ z$ and $z \in F$. Therefore $U(x,y) \subseteq F$. Conversely, Let $U(x,y) \subseteq F$ for all $x,y \in F$. First, $1 \in U(x,y)$ by Proposition 3.13 (1), we have $1 \in F$. If $F < a \circ b$ and $a \in F$, then there exist $u \in F$ and $v \in a \circ b$ such that $1 \in u \circ v \subseteq u \circ (a \circ b)$. Hence $b \in U(u,a) \subseteq F$. This shows that F is a hyper filter by applying Theorem 3.6.

Theorem 3.18 *Let F be a hyper filter of a hyper BE-algebra H. Then*

$$F = \bigcup \{U(x,y) : x,y \in F\}.$$

Proof. Assume that F be a hyper filter of H. If $z \in \bigcup\{U(x,y) : x,y \in F\}$, then there exist $u,v \in F$ such that $z \in U(u,v)$. By Theorem 3.17, we get $z \in F$ and so $\bigcup\{U(x,y) : x,y \in F\} \subseteq F$. Now let $z \in F$. Then $1 \in z \circ z \subseteq z \circ (1 \circ z)$ by (HBE1) and (HBE3), which implies $z \in U(z,1)$. Thus $F \subseteq \bigcup\{U(z,1) : z \in F\} \subseteq \bigcup\{U(x,y) : x,y \in F\}$. We complete this proof.

Corollary 3.19 *Let F be a hyper filter of a hyper BE-algebra H. Then*

$$F = \bigcup\{U(z,1) : z \in F\}.$$

Proof. By the proof of Theorem 3.18, $F \subseteq \bigcup\{U(z,1) : z \in F\}$. Now we prove the converse holds. Let $z \in \bigcup\{U(z,1) : z \in F\}$. Then there exists $a \in F$ such that $z \in U(a,1)$, i.e., $1 \in a \circ (1 \circ z)$. Thus there exists $b \in 1 \circ z$ such that $a < b$. Since $a \in F$ and F is a hyper filter, we have $b \in F$ by Theorem 3.5. It follows that $1 \circ z \cap F \neq \emptyset$. Again applying that F is a hyper filter and $1 \in F$, we conclude that $z \in F$ and therefore this proof is finished.

4. (Positive) Implicative hyper filters

Definition 4.1 Let $(H; \circ, 1)$ be a hyper BE-algebra and F be a nonempty subset of H containing 1. Then for all $x, y, z \in H$, F is called:

(1) an implicative hyper filter of H if $x \circ (y \circ z) \cap F \neq \emptyset$ and $x \circ y \cap F \neq \emptyset$ imply $x \circ z \cap F \neq \emptyset$.

(2) a positive implicative hyper filter of H if $x \circ ((y \circ z) \circ y) \cap F \neq \emptyset$ and $x \in F$ imply $y \in F$.

Example 4.2 Let $H = \{1, a, b\}$. Define the hyperoperation "$\circ$" on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1\}$	$\{b\}$
b	$\{1\}$	$\{1, a\}$	$\{1\}$

Then $(H; \circ, 1)$ is a hyper BE-algebra[7]. We can check that $F = \{1, a\}$ is a hyper filter. Moreover, F is both positive implicative and implicative.

Example 4.3 Let $(H; \circ_1, 1)$ be a hyper BE-algebra given by Example 2.4. One can easily check that $F_1 = \{1, a\}$ is not a positive implicative hyper filter, since $1 \circ_1 ((b \circ_1 a) \circ_1 b) \cap F \neq \emptyset$ and $1 \in F_1$, but $b \notin F_1$; $F_2 = \{1, b\}$ is not an implicative hyper filter, since $1 \circ_1 (b \circ_1 a) \cap F_2 \neq \emptyset$, $1 \circ_1 b \cap F_2 \neq \emptyset$, but $1 \circ_1 a \cap F_2 = \emptyset$.

In the following, we discuss about the relations among various hyper filters.

Theorem 4.4 *Let $(H; \circ, 1)$ be a hyper BE-algebra and F is a nonempty subset of H.*

(1) *If F is a positive implicative hyper filter of H, then F is a hyper filter of H;*

(2) *If H is a R-hyper BE-algebra and F is an implicative hyper filter of H, then F is a hyper filter of H.*

Proof. (1) Clearly, $1 \in F$. Assume that $x \circ y \approx F$ and $x \in F$. This implies that $x \circ y \cap F \neq \emptyset$. Then there exists $a \in H$ such that $a \in x \circ y$ and $a \in F$. Hence by (HBE1), (HBE2) and Proposition 2.3 (1), we get that $a \in (y \circ y) \circ (x \circ y) = x \circ ((y \circ y) \circ y)$. Thus $x \circ ((y \circ y) \circ y) \cap F \neq \emptyset$. Since $x \in F$ and F is a positive implicative hyper filter of H, then $y \in F$ and so F is a hyper filter of H.

(2) Clearly, $1 \in F$. Let $x \circ y \approx F$ and $x \in F$. Then $x \circ y \subseteq 1 \circ (x \circ y)$ by Proposition 2.4 (3) and $x \circ y \cap F \neq \emptyset$. Hence $1 \circ (x \circ y) \cap F \neq \emptyset$. In addition, since H is a R-hyper BE-algebra, then $1 \circ x = x \in F$. This shows $1 \circ x \cap F \neq \emptyset$. Since F is an implicative hyper filter, we conclude $1 \circ y \cap F \neq \emptyset$. Therefore $y = 1 \circ y \in F$ and so this proof is completed.

In the following, we give out some remarks about Theorem 4.4.

Remark 4.5

(1) About the example of Theorem 4.4, one can see Example 4.2.

(2) The converse of Theorem 4.4 (1) is not true in general. Let $(H; \circ_1, 1)$ be a hyper BE-algebra given by Example 2.4. we can easily check that $F_1 = \{1, a\}$ is a hyper filter, but F_1 is not a positive implicative hyper filter, since $1 \circ_1 ((b \circ_1 a) \circ_1 b) \cap F \neq \emptyset$ and $1 \in F$, but $b \notin F_1$.

(3) The converse of Theorem 4.4 (2) is not true in general. Let $(H; \circ, 1)$ be a hyper BE-algebra given by Example 2.5. we know $\{1\}$ is a hyper filter by Remark 3.4, but $\{1\}$ is not an implicative hyper filter, since $a \circ (a \circ b) \cap \{1\} \neq \emptyset$ and $a \circ a \cap \{1\} \neq \emptyset$, but $a \circ b \cap \{1\} = \emptyset$.

(4) The condition that H is R-hyper BE-algebra is not necessary. Let $(H; \circ_2, 1)$ be a hyper BE-algebra given by Example 2.4. Clearly, $(H; \circ_2, 1)$ is not a R hyper BE-algebra. But $F = \{1, a\}$ is a hyper filter. Moreover, F is an implicative hyper filter.

(5) In general, An implicative hyper filter of a hyper BE-algebra is not a hyper filter. Let $H = \{1, a, b\}$ and define the hyperoperation "$\circ$" on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a, b\}$	$\{b\}$
a	$\{1\}$	$\{1, a\}$	$\{1, b\}$
b	$\{1\}$	$\{1, a, b\}$	$\{1\}$

Then $(H; \circ, 1)$ is a C-hyper BE-algebra[7]. One can verify that $F = \{1, b\}$ is an implicative hyper filter. But F is not a hyper filter by Theorem 3.17, since the upper set $U(1, b) = \{1, a, b\} \nsubseteq F$.

Definition 4.6 Let $(H; \circ, 1)$ be a hyper BE-algebra and S be a nonempty subset of H. Then S is said to be $S_\circ$−reflexive if $x \circ y \cap S \neq \emptyset$ implies $x \circ y \subseteq S$ for all $x, y \in H$.

Definition 4.7 Let $(H; \circ, 1)$ be a hyper BE-algebra. Then H is said to be transitive if the following holds:

(HT) $y \circ z \ll (x \circ y) \circ (x \circ z)$ and $x \circ y \ll (y \circ z) \circ (x \circ z)$ for all $x, y, z \in H$.

Example 4.8 Let $H = \{1, a, b, c\}$ be a hyper BE-algebra defined by Example 2.5. Then H is not transitive. In fact, neither $b \circ c \ll (a \circ b) \circ (a \circ c)$ nor $a \circ b \ll (b \circ c) \circ (a \circ c)$ holds.

Example 4.9 Let $(H;\circ_1,1)$ be a hyper BE-algebra given by Example 2.4. Then we can check that $H=\{1,a,b\}$ is transitive.

Proposition 4.10 *Let $(H;\circ,1)$ be a transitive hyper BE-algebra. Then $y\circ A\ll(x\circ y)\circ(x\circ A)$ and $x\circ A\ll(A\circ z)\circ(x\circ z)$ for all $x,y,z\in H$.*

Proof. Straightforward by (HT).

Lemma 4.11 *Let $(H;\circ,1)$ be a hyper BE-algebra and F be a hyper filter of H. Then for any nonempty subsets A,B of H, if $A\cap F\neq\emptyset$ and $A\ll B$, then $B\cap F\neq\emptyset$.*

Proof. Since $A\cap F\neq\emptyset$, then there exists $c\in H$ such that $c\in A$ and $c\in F$. Again as $A\ll B$, for the above $c\in F$ it follows that there exists $b\in B$ such that $c<b$. Since F is a hyper filter, we have $b\in F$ by Theorem 3.5 and so $B\cap F\neq\emptyset$.

Theorem 4.12 *Let $(H;\circ,1)$ be a transitive hyper BE-algebra and F be a $F_\circ$-reflexive nonempty subset of H. Then F is a positive implicative hyper filter implies F is an implicative hyper filter.*

Proof. Let $x\circ(y\circ z)\cap F\neq\emptyset$ and $x\circ y\cap F\neq\emptyset$. First, we have $x\circ(y\circ z)=y\circ(x\circ z)\ll(x\circ y)\circ(x\circ(x\circ z))$ by (HBE2) and Proposition 4.10. Since $x\circ(y\circ z)\cap F\neq\emptyset$, then $(x\circ y)\circ(x\circ(x\circ z))\cap F\neq\emptyset$ by Lemma 4.11. Hence there exist $a\in x\circ y\subseteq F$ by the $S_\circ-$reflexivity of F and $b\in x\circ(x\circ z)$ such that $a\circ b\cap F\neq\emptyset$. Since $a\in F$ and F is a hyper filter by Theorem 4.4, we get $b\in F$. This means $x\circ(x\circ z)\cap F\neq\emptyset$. Since H is transitive, we get $x\circ(x\circ z)\ll((x\circ z)\circ z)\circ(x\circ z)$. Hence by Lemma 4.11 $((x\circ z)\circ z)\circ(x\circ z)\cap F\neq\emptyset$. As $((x\circ z)\circ z)\circ(x\circ z)\subseteq 1\circ(((x\circ z)\circ z)\circ(x\circ z))$ by Proposition 2.4 (3), it follows that $1\circ(((x\circ z)\circ z)\circ(x\circ z))\cap F\neq\emptyset$. Then there exists $c\in x\circ z$ such that $1\circ((c\circ z)\circ c)\cap F\neq\emptyset$, combining $1\in F$ and F is a positive implicative filter we obtain $c\in F$. This shows $x\circ z\cap F\neq\emptyset$. This finishes the proof.

Example 4.13 Let $H=\{1,a,b\}$ be a hyper BE-algebra given by Example 4.2. Then it is not difficult to check that H is transitive and $F=\{1,a\}$ is $F_\circ$-reflexive. Moreover, F is both implicative and positive implicative.

Note that In Theorem 4.12, the condition $F_\circ$-reflexivity is not necessary. In Remark 4.5 (5), we can easily check that $H=\{1,a,b\}$ is transitive, and $\{1,a\}$ is neither implicative nor positive implicative. However, $\{1,a\}$ is not $F_\circ$-reflexive, since $a\circ b\cap\{1,a\}\neq\emptyset$ doesn't imply $a\circ b\subseteq\{1,a\}$.

Now, we verify some results and give equivalent characterizations about the implicative and positive implicative hyper filter in H.

Theorem 4.14 *Let F be a hyper filter of a hyper BE-algebra H. Then the following conditions are equivalent:*
(1) *F is a positive implicative hyper filter;*
(2) *for all $x,y\in H$, $(x\circ y)\circ x\cap F\neq\emptyset$ implies $x\in F$;*
(3) *for all $x,y,u\in H$, $(x\circ y)\circ(u\circ x)\cap F\neq\emptyset$ and $u\in F$ imply $x\in F$.*

Proof. (1) $\Rightarrow$ (2) Assume that F is a positive implicative hyper filter of H. Let $(x \circ y) \circ x \cap F \neq \emptyset$ for any $x, y \in H$. Then there exists $a \in H$ such that $a \in (x \circ y) \circ x$ and $a \in F$. Since $a \in 1 \circ a \subseteq 1 \circ ((x \circ y) \circ x)$, combing $1 \in F$ and the definition of a positive implicative hyper filter, it follows that $x \in F$.

(2) $\Rightarrow$ (3) Assume that (2) holds. Let $(x \circ y) \circ (u \circ x) \cap F \neq \emptyset$ and $u \in F$ for any $x, y, u \in H$. Then there exists $a \in H$ such that $a \in (x \circ y) \circ (u \circ x)$ and $a \in F$. Hence $u \circ ((x \circ y) \circ x) = (x \circ y) \circ (u \circ x) \cap F \neq \emptyset$. Combining $u \in F$ and F is a hyper filter, we get $(x \circ y) \circ x \cap F \neq \emptyset$ and thus $x \in F$ by (2).

(3) $\Rightarrow$ (1) Assume that (3) holds. Let $x \circ ((y \circ z) \circ y) \cap F \neq \emptyset$ and $x \in F$ for all $x, y, z \in H$. Since $(y \circ z) \circ (x \circ y) = x \circ ((y \circ z) \circ y)$, then $(y \circ z) \circ (x \circ y) \cap F \neq \emptyset$. Considering $x \in F$ and (3) holds, it follows that $y \in F$. This complete the proof.

Theorem 4.15 *Let F be a hyper filter of a hyper BE-algebra H. Then the following are equivalent:*
(1) *for all $x, y \in H$, $x \circ (x \circ y) \cap F \neq \emptyset$ implies $x \circ y \cap F \neq \emptyset$;*
(2) *for all $x, y, z \in H$, $z \circ (x \circ (x \circ y)) \cap F \neq \emptyset$ and $z \in F$ imply $x \circ y \cap F \neq \emptyset$.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. Let $z \circ (x \circ (x \circ y)) \cap F \neq \emptyset$ and $z \in F$ for any $x, y, z \in H$. Since F is a hyper filter, then $x \circ (x \circ y) \cap F \neq \emptyset$. Therefore $x \circ y \cap F \neq \emptyset$ by (2).

(2) $\Rightarrow$ (1) Assume that (2) holds. Let $x \circ (x \circ y) \cap F \neq \emptyset$ for any $x, y \in H$. Since $x \circ (x \circ y) \subseteq 1 \circ (x \circ (x \circ y))$ by Proposition 2.4 (3), then $1 \circ (x \circ (x \circ y)) \cap F \neq \emptyset$. Combining $1 \in F$, it implies that $x \circ y \cap F \neq \emptyset$ by (2). This proof is completed.

Theorem 4.16 *Let F be a hyper filter of a hyper BE-algebra H. If F is implicative, then*
(1) *for all $x, y \in H$, $x \circ (x \circ y) \cap F \neq \emptyset$ implies $x \circ y \cap F \neq \emptyset$;*
(2) *for all $x, y, z \in H$, $z \circ (x \circ (x \circ y)) \cap F \neq \emptyset$ and $z \in F$ imply $x \circ y \cap F \neq \emptyset$.*

Proof. By Theorem 4.15, we only need to prove (1) holds. Assume that F is an implicative hyper filter. Let $x \circ (x \circ y) \cap F \neq \emptyset$ for any $x, y \in H$. Since $1 \in x \circ x$ by (HBE1) and so $x \circ x \cap F \neq \emptyset$. Combing the definition of an implicative hyper filter, we obtain $x \circ y \cap F \neq \emptyset$ and the condition (1) follows.

Theorem 4.17 *Let $(H; \circ, 1)$ be a transitive hyper BE-algebra, and F be a $F_\circ$-reflexive non-empty subset containing* 1 *of H. If for all $x, y, z \in H$, $z \circ (x \circ (x \circ y)) \cap F \neq \emptyset$ and $z \in F$ imply $x \circ y \cap F \neq \emptyset$, then F is an implicative hyper filter of H.*

Proof. Let $x \circ (y \circ z) \cap F \neq \emptyset$ and $x \circ y \cap F \neq \emptyset$ for any $x, y, z \in H$. Since H is transitive and so by Proposition 4.10 we have $x \circ (y \circ z) = y \circ (x \circ z) \ll (x \circ y) \circ (x \circ (x \circ z))$. Since $x \circ (y \circ z) \cap F \neq \emptyset$, From Lemma 4.11 we obtain $(x \circ y) \circ (x \circ (x \circ z)) \cap F \neq \emptyset$. As F is $S_\circ-$reflexive, this shows $x \circ y \subseteq F$. Hence there exists $a \in x \circ y$, i.e., $a \in F$ such that $a \circ (x \circ (x \circ z)) \cap F \neq \emptyset$. Therefore, by the hypothesis, $x \circ z \cap F \neq \emptyset$ holds, which implies F is an implicative hyper filter.

From Theorem 4.15, 4.16 and 4.17 we can obtain the following important corollary.

Corollary 4.18 *Let $(H;\circ,1)$ be a transitive hyper BE-algebra, and F be a $F_\circ$-reflexive nonempty subset containing 1 of H. Then the following are equivalent:*
(1) *F is an implicative hyper filter;*
(2) *for all $x,y\in H$, $x\circ(x\circ y)\cap F\neq\emptyset$ implies $x\circ y\cap F\neq\emptyset$;*
(3) *for all $x,y,z\in H$, $z\circ(x\circ(x\circ y))\cap F\neq\emptyset$ and $z\in F$ imply $x\circ y\cap F\neq\emptyset$.*

Definition 4.19 A hyper BE-algebra $(H;\circ,1)$ is called RD-hyper BE-algebra, if H is both R-hyper BE-algebra and D-hyper BE-algebra.

Clearly, in Example 4.2, $(H;\circ,1)$ is a RD-hyper BE-algebra.

Lemma 4.20 *Let $(H;\circ,1)$ be a RD-hyper BE-algebra, and F be a hyper filter of H. For all $x,y,z\in H$, If $x\circ(y\circ z)\cap F\neq\emptyset$ implies $(x\circ y)\circ(x\circ z)\cap F\neq\emptyset$, then $z\circ(x\circ(x\circ y))\cap F\neq\emptyset$ and $z\in F$ imply $x\circ y\cap F\neq\emptyset$.*

Proof. Let $z\circ(x\circ(x\circ y))\cap F\neq\emptyset$ and $z\in F$ for any $x,y,z\in H$. Since $x\circ(x\circ(z\circ y))=z\circ(x\circ(x\circ y))$ by (HBE2), then $x\circ(x\circ(z\circ y))\cap F\neq\emptyset$. By the hypothesis, it follows that $(x\circ x)\circ(x\circ(z\circ y))\cap F\neq\emptyset$. Since $z\circ(x\circ y)=x\circ(z\circ y)=(x\circ x)\circ(x\circ(z\circ y))$. Thus $z\circ(x\circ y)\cap F\neq\emptyset$. Again as F be a hyper filter, it follows that $x\circ y\cap F\neq\emptyset$ and so this proof is finished.

By Theorem 4.18 and Lemma 4.20, the following theorem can be obtained immediately.

Theorem 4.21 *Let $(H;\circ,1)$ be a transitive RD-hyper BE-algebra, and F be a $F_\circ$-reflexive hyper filter of H. If for all $x,y,z\in H$, $x\circ(y\circ z)\cap F\neq\emptyset$ implies $(x\circ y)\circ(x\circ z)\cap F\neq\emptyset$, then F is an implicative hyper filter of H.*

In the following, we give an equivalent characterization of an implicative hyper filter of a hyper BE-algebra.

Theorem 4.22 *Let F be a nonempty subset of a hyper BE-algebra H. Then F is an implicative hyper filter of H if and only if $\{x\in H : a\circ x\cap F\neq\emptyset\}$ is a hyper filter of H for all $a\in H$.*

Proof. Suppose F is an implicative hyper filter and $F_a=\{x\in H : a\circ x\cap F\neq\emptyset\}$. Clearly, $1\in F_a$. Let $u\circ v\cap F_a\neq\emptyset$ and $u\in F_a$ for any $u,v\in F$. Then $a\circ(u\circ v)\cap F\neq\emptyset$ and $a\circ u\cap F\neq\emptyset$. Since F is an implicative hyper filter, we have $a\circ v\cap F\neq\emptyset$. This implies that $v\in F_a$. Thus F_a is a hyper filter. Conversely, assume that for all $a\in H$, $F_a=\{x\in H : a\circ x\cap F\neq\emptyset\}$ is a hyper filter of H. Let $m\circ(n\circ p)\cap F\neq\emptyset$ and $m\circ n\cap F\neq\emptyset$, for any $m,n,p\in H$. Then $n\circ p\cap F_m\neq\emptyset$ and $n\in F_m$. Since F_m is a hyper filter, then $p\in F_m$. That is, $m\circ n\cap F\neq\emptyset$. This shows that F is an implicative hyper filter.

Corollary 4.23 *Let F be an implicative hyper filter of a hyper BE-algebra H and $a\in H-F$. Then the hyper filter generated by F and a can be represented as $[F\cup\{a\})=\{x\in H : a\circ x\cap F\neq\emptyset\}$.*

Proof. According to Theorem 3.10, we have $[F \cup \{a\}) \supseteq \{x \in H : a \circ x \cap F \neq \emptyset\}$. Hence we need to prove $[F \cup \{a\}) \subseteq \{x \in H : a \circ x \cap F \neq \emptyset\}$. In fact, we only need to verify that $\{x \in H : a \circ x \cap F \neq \emptyset\}$ is a hyper filter containing $F \cup \{a\}$. Since $1 \in a \circ a$, then $a \circ a \cap F \neq \emptyset$, i.e., $a \in \{x \in H : a \circ x \cap F \neq \emptyset\}$. For each $x \in F$, we have $x \cap F \neq \emptyset\}$. Again as $x \ll a \circ x$ by Proposition 2.3 (4), it follows that $a \circ x \cap F \neq \emptyset$ from Lemma 4.11, which shows $F \subseteq \{x \in H : a \circ x \cap F \neq \emptyset\}$. By Theorem 4.22, we can know that $\{x \in H : a \circ x \cap F \neq \emptyset\}$ is a hyper filter. This proof is completed.

Corollary 4.24 *Let $(H; \circ, 1)$ be a hyper BE-algebra and let $a \in H$. Then $\{1\}$ is an implicative hyper filter of H if and only if $[a) = \{x \in H : 1 \in a \circ x\}$.*

In fact, if $H = \{1, a, b\}$ is a hyper BE-algebra given by Example 4.2, then $\{1\}$ is an implicative hyper filter and $[a) = \{a, 1\}$, $[b) = H$.

5. Quotient hyper BE-algebras

Definition 5.1 Let H be a hyper BE-algebra and θ be an equivalence relation on H.

(1) for any $A, B \subseteq H$, $A\theta B$ means for all $a \in A$ there exists $b \in B$ such that $a\theta b$ and for all $b \in B$ there exists $a \in A$ such that $a\theta b$;

(2) θ is called a congruence relation if for all $x, y, u, v \in H$, $x\theta y$ and $u\theta v$ imply $x \circ u\theta y \circ v$.

Let θ be a congruence relation on a hyper BE-algebra H. Denote $H/\theta = \{[x] : x \in H\}$, where $[x] = \{y \in H : y\theta x\}$. $\bar{\circ}$ and $<$ on H/F are defined by $[x]\bar{\circ}[y] = \{[a] : a \in x \circ y\}$ and $[x] < [y]$ iff $[1] \in [x]\bar{\circ}[y]$ for any $[x], [y] \in H/\theta$, respectively. Clearly, $x < y$ implies that $[x] < [y]$.

Theorem 5.2 *Let θ be a congruence relation on a hyper BE-algebra H. Then $(H/\theta; \bar{\circ}, [1])$ is a hyper BE-algebra, which is called as a quotient hyper BE-algebra with respect to θ.*

Proof First, we prove that $\bar{\circ}$ is well defined. Assume that $[x_1] = [x_2]$ and $[y_1] = [y_2]$, $x_1, x_2, y_1, y_2 \in H$. Then $x_1\theta x_2$ and $y_1\theta y_2$. Thus $x_1 \circ y_1\theta x_2 \circ y_2$. Let $[a] \in [x_1]\bar{\circ}[y_1]$ for some $a \in x_1 \circ y_1$. Then $[a] = [b]$, i.e., $a\theta b$ for some $b \in x_1 \circ y_1$. In addition, since $x_1 \circ y_1\theta x_2 \circ y_2$, then there exists $c \in x_2 \circ y_2$ such that $b\theta c$. Hence $a\theta c$ and so $[a] = [c] \in [x_2]\bar{\circ}[y_2]$. This shows that $[x_1]\bar{\circ}[y_1] \subseteq [x_2]\bar{\circ}[y_2]$. Similarly, we have $[x_2]\bar{\circ}[y_2] \subseteq [x_1]\bar{\circ}[y_1]$. It follows that $[x_1]\bar{\circ}[y_1] = [x_2]\bar{\circ}[y_2]$. Therefore $\bar{\circ}$ is well defined. Next, let $x, y, z \in H$.

(HBE1) Since $x < 1$ and $x < x$, we have $[x] < [1]$ and $[x] < [x]$.

(HBE2) Let $[a] \in [x]\bar{\circ}([y]\bar{\circ}[z])$. Then there exists $b \in y \circ z$, $[b] \in [y]\bar{}[z]$ such that $[a]_F = [x]\bar{}[b]$. Further, there exists $c \in x \circ b$, $[c] \in [x]\bar{}[b]$ such that $[a] = [c]$. Since $c \in x \circ b \subseteq x \circ (y \circ z) = y \circ (x \circ z)$, we have $[c] \in [y]\bar{\circ}([x]\bar{\circ}[z])$. Thus $[a] \in [y]\bar{\circ}([x]\bar{\circ}[z])$. This implies that $[x]\bar{\circ}([y]\bar{\circ}[z]) \subseteq [y]\bar{\circ}([x]\bar{\circ}[z])$. Similarly, we have $[y]\bar{\circ}([x]\bar{\circ}[z]) \subseteq [y]\bar{\circ}([x]\bar{\circ}[z])$. Therefore $[x]\bar{\circ}([y]\bar{\circ}[z]) = [y]\bar{\circ}([x]\bar{\circ}[z])$.

(HBE3) Since $x \in 1 \circ x$, we have $[x] \in [1]\bar{\circ}[x]$.

(HBE4) Let $[1] < [x]$. Then $[1] \in [1]\bar{\circ}[x]$, where $1 \in 1 \circ x$ and so $x = 1$. Hence $[x] = [1]$.

Definition 5.3 Let H be a hyper BE-algebra and F be a hyper filter of H. Define a binary relation θ_F as follows: $x\theta_F y$ iff $x \circ y \cap F \neq \emptyset$ and $y \circ x \cap F \neq \emptyset$.

Lemma 5.4 *Let H be a hyper BE-algebra and F be an implicative hyper filter of H. Then θ_F is an equivalence relation on H.*

Proof. Clearly, θ_F satisfies the reflectivity and the symmetry. Let $x, y, z \in H$ such that $x\theta_F y$ and $y\theta_F z$. Then $x \circ y \cap F \neq \emptyset$ and $y \circ z \cap F \neq \emptyset$. Hence by Proposition 2.7, $y \circ z \ll x \circ (y \circ z)$ and Lemma 4.11, we have $x \circ (y \circ z) \cap F \neq \emptyset$. Since F is an implicative hyper filter and $x \circ y \cap F \neq \emptyset$, we get that $x \circ z \cap F \neq \emptyset$. Similarly, $z \circ x \cap F \neq \emptyset$. Thus, the transitivity holds, which shows that θ_F is an equivalence relation.

Proposition 5.5 *Let H be a hyper BE-algebra and F be an implicative hyper filter of H. Then $[1]_F$ is a hyper filter of H and $F = [1]_F$.*

Proof. Let $x \in [1]_F$ and $x \circ y \cap [1]_F \neq \emptyset$ for any $x, y \in H$. Then $1 \circ x \cap F \neq \emptyset$ and $1 \circ (x \circ y) \cap F \neq \emptyset$. Since F is implicative, then $1 \circ y \cap F \neq \emptyset$. Again, $y \circ 1 \cap F \neq \emptyset$. This implies that $y\theta_F 1$, that is, $y \in [1]_F$. It follows that $[1]_F$ is a hyper filter. The following proves $F = [1]_F$. Let $x \in F$. Since $x \in 1 \circ x$ and $1 \in x \circ 1$, we have $1 \circ x \cap F \neq \emptyset$ and $x \circ 1 \cap F \neq \emptyset$. Thus $x\theta_F 1$, i.e., $x \in [1]_F$ and so $F \subseteq [1]_F$. On the other hand, let $x \in [1]_F$. Then $x\theta_F 1$ and so $1 \circ x \cap F \neq \emptyset$. Considering F is a hyper filter, we get $x \in F$. Hence $[1]_F \subseteq F$. This shows that $F = [1]_F$.

Definition 5.6 Let H be a hyper BE-algebra and F be an implicative hyper filter of H. Then F is called normal if the relation θ_F preserves the operation $\circ$, that is, $x\theta_F y$ and $u\theta_F v$ implies that $(x \circ u)\theta_F(y \circ v)$ for all $x, y, u, v \in H$.

Note that in the above definition F is a normal hyper filter if and only if the relation θ_F is a congruence relation. In this condition, we denote H/θ_F by $H/F = \{[x]_F : x \in H\}$.

Example 5.7 Let $H = \{1, a, b\}$ be a hyper BE-algebra given by Example 4.2. Then $F = \{1, a\}$ is a hyper filter. Moreover, F is an implicative hyper filter of H. One can check easily that $\theta_F = \{(1, 1), (1, a), (a, 1), (a, a), (b, b)\}$ is a congruence relation, so F is a normal hyper filter.

Theorem 5.8 *Let H be a hyper BE-algebra and F be a normal hyper filter of H. Then $(H/F; \bar{\circ}, [1]_F)$ is a hyper BE-algebra and $\{[1]_F\}$ is a hyper filter of H/F.*

Proof. By Theorem 5.2, we only prove that $\{[1]_F\}$ is a hyper filter of H/F. Assume that $A = [1]_F$ and $A \circ B \bigcap \{[1]_F\} \neq \emptyset$ for any $B \in H/F$. The following proves that $B = [1]_F$. Let $B \neq [1]_F$. Then $B = [x]_F$, where $x \in H - F$ and so $[1]_F \circ [x]_F \bigcap \{[1]_F\} \neq \emptyset$, i.e., $[1 \circ x]_F \bigcap \{[1]_F\} \neq \emptyset$. Hence there exists $a \in 1 \circ x$ such that $[a]_F = [1]_F$. This implies that $a \in F$. It follows that $1 \circ x \cap F \neq \emptyset$ and thus $x \in F$, which is a contraction. Therefore $B = [1]_F$. This proof is completed.

Example 5.9 Let $H = \{1, a, b\}$ be a hyper BE-algebra defined by Example 4.2. According to Example 5.7, $F = \{1, a\}$ is a normal hyper filter of H. The following constructs the quotient hyper BE-algebra H/F via F. Since $\theta_F = \{(1,1), (1,a), (a,1), (a,a), (b,b)\}$, then $[1]_F = F = \{1, a\} = [a]_F$, $[b]_F = \{b\}$, $H/F = \{[1]_F, [b]_F\}$ and

$\bar{\circ}$	$[b]_F$	$[1]_F$
$[b]_F$	$[1]_F$	$[1]_F$
$[1]_F$	$[b]_F$	$[1]_F$

Then $(H/F; \bar{\circ}, [1]_F)$ is a hyper BE-algebra.

Definition 5.10 A proper hyper filter F of a hyper BE-algebra H is said to be maximal if it is not a proper subset of any proper hyper filter of H.

Example 5.11 Let $(H; \circ_2, 1)$ be a hyper BE-algebra which is defined in Example 2.4, we can check that $\{1, a\}$ is a maximal hyper filter.

Clearly, a proper hyper filter F of H is maximal if and only if $[F \cup \{a\}) = H$ for each $a \in H - F$.

Definition 5.12 A hyper BE-algebra H is called to be
(1) bounded if H has a bottom element 0;
(2) simple if H has no non-trivial hyper filter.

Example 5.13 (1) Let $H = \{1, a, b\}$ be a hyper BE-algebra as in Example 4.2. Then H is a bounded hyper BE-algebra, where the bottom element $b := 0$.
(2) Let $H = \{1, a, b\}$. Define the hyperoperation " $\circ$ " on H as follows:

$\circ$	1	a	b
1	$\{1\}$	$\{a\}$	$\{b\}$
a	$\{1\}$	$\{1\}$	$\{1, a\}$
b	$\{1\}$	$\{1\}$	$\{1, a\}$

Then $(H; \circ, 1)$ is a simple hyper BE-algebra.

Remark 5.14 A hyper BE-algebra $(H; \circ, 1)$ is simple if and only if $\{1\}$ is a maximal hyper filter of H.

Theorem 5.15 *Let H be a bounded hyper BE-algebra and F be a proper hyper filter of H. Then there is a maximal hyper filter of H containing F.*

Proof. Denote $\Gamma = \{J \in F(L) : F \neq H$ and $F \subseteq J\}$. Then $\Gamma \neq \emptyset$ and $(\Gamma, \subseteq)$ is a poset, where $\subseteq$ is the inclusion relation. Let $\{J_i : i \in I\}$ is a chain in Γ. Then $\bigcup\{J_i : i \in I\}$ is a filter of L by Theorem 3.7 and it is the upper bound of this chain. Since $0 \notin J_i$, we have $0 \notin \bigcup\{J_i : i \in I\}$. Hence $\bigcup\{J_i : i \in I\} \in \Gamma$. By Zorn Lemma, there exists a maximal element M in Γ. Therefore M is indeed a maximal hyper filter.

By Corollary 4.23, the following conclusion holds.

Theorem 5.16 *Let $(H; \circ, 1)$ be a hyper BE-algebra and F be an implicative hyper filter of H. Then F is maximal if and only if $\{x \in H : a \circ x \cap F \neq \emptyset\} = H$ for $a \in H - F$.*

Theorem 5.17 *Let H be a hyper BE-algebra and F be an implicative hyper filter. Then F is maximal if and only if $x \circ y \cap F \neq \emptyset$ and $y \circ x \cap F \neq \emptyset$ for all $x, y \in H - F$.*

Proof. Assume that F is maximal. Then by Corollary 4.26, $[F \cup \{x\}) = \{x \in H : a \circ x \cap F \neq \emptyset\} = H$ for $a \in H - F$. We prove that $x \circ y \cap F \neq \emptyset$ and $y \circ x \cap F \neq \emptyset$ for all $x, y \in H - F$. Or else, without loss of any generality, there exist $x, y \in H - F$ such that $x \circ y \cap F = \emptyset$. Hence $y \notin [F \cup \{x\})$, which implies $y \notin H$. This is a contradiction. Conversely, let $x \in H - F$. Then $x \circ y \cap F \neq \emptyset$. Hence $y \in [F \cup \{x\})$. This shows that $H - F \subseteq [F \cup \{x\})$. Again, $F \subseteq [F \cup \{x\})$. It follows that $H = [F \cup \{x\})$, which implies that F is maximal.

Lemma 5.18 *Let H be a hyper BE-algebra and F be a normal hyper filter of H. If J is a hyper filter containing F, then J/F is also a hyper filter of H/F.*

Proof. Assume that $[x]_F \in J/F$ and $[x]_F \circ [y]_F \bigcap J/F \neq \emptyset$ for any $[x]_F, [y]_F \in H/F$. Then $x \in J$ and there exists $t \in x \circ y$ such that $t \in J$. Hence $x \circ y \cap J \neq \emptyset$. Since J is a hyper filter, we have $y \in J$ and thus $[y]_F \in J/F$. Therefore J/F is a hyper filter of H/F.

Theorem 5.19 *Let H be a bounded hyper BE-algebra and F be a non-trivial normal hyper filter of H. Then the following are equivalent:*

(1) *F is maximal;*
(2) *$H/F = \{H - F, F\}$, where $F = [1]_F$ and $H - F = [x]_F = \{y \in H - F : y\theta_F x, x \in H - F\}$;*
(3) *H/F is simple.*

Proof. (1) $\Leftrightarrow$ (2) Obvious by Theorem 5.17.

(2) $\Rightarrow$ (3) Assume that (2) holds and H/F is not simple. Then $\{[1]_F\}$ is not a maximal hyper filter of H/F by Remark 5.16. Hence according to Theorem 5.8 and Theorem 5.17, there exists a maximal hyper filter of H/F containing $\{[1]_F\}$, which contradicts to $H/F = \{H - F, F\}$. This shows that H/F is simple.

(3) $\Rightarrow$ (1) Assume that H/F is simple. Then $\{[1]_F\}$ is a maximal hyper filter of H/F and $\{[1]_F\} \neq H/F$. Hence there exists $[x]_F \in H/F$ such that $[x]_F \neq [1]_F$. This shows that $x \notin F$ and $F \neq H$. Let F be not a maximal hyper filter of H. Then by Theorem 5.8 and Theorem 5.17, there exists a maximal hyper filter J

of H satisfying $F \subset J \subset H$. Since J is a hyper filter of H, then J/F is a hyper filter of H/F by Lemma 5.18 and $\{[1]_F\} \subseteq J/F \subseteq H/F$. The following proves that $\{[1]_F\} \neq J/F$ and $J/F \neq H/F$. Let $\{[1]_F\} = J/F$. Then for all $x \in J$, we have $[x]_F = [1]_F$ and so $x \in F$, which implies that $J \subseteq F$. This is a contraction. Again, let $J/F = H/F$. Since $J \neq H$, then there exists $x \in H$ such that $x \notin J$. Hence $[x]_F \in H/F = J/F$ and so there exists $a \in J$ such that $[x]_J = [a]_J$. This shows that $x\theta_F a$ and thus $a \circ x \cap F \neq \emptyset$. Therefore $a \circ x \cap J \neq \emptyset$. Considering J is a hyper filter of H, we get $x \in J$, which is a contraction. It follows that $\{[1]_F\} \subset J/F \subset H/F$. This contradicts to the simplicity of H/F. The proof is complete.

6. Conclusions

In this paper, we investigate some types of hyper filters on hyper BE-algebras. we first give some results about hyper filters, then use upper sets to get some representations of hyper filter; Next, we introduce two types of hyper filters: (positive) implicative hyper filters, and focus on discussing the relations of various hyper filters. Also, we give some characterizations of (positive) implicative hyper filters. At last, we consider quotient hyper BE-algebras and get some results.

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