BI-HYPERIDEALS AND QUASI-HYPERIDEALS IN ORDERED SEMIHYPERGROUPS

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Abstract. In this paper, we introduce concepts of bi-hyperideals and quasi-hyperideals of an ordered semihypergroup and present several examples of them. Some properties and relationships between bi-hyperideals and quasi-hyperideals are investigated. In particular, we introduce the concept of intra-regular ordered semihypergroups and give their characterizations in terms of bi-hyperideals and quasi-hyperideals.

Keywords: algebraic hyperstructure, ordered semihypergroup, subsemihypergroup, hyperideal, bi-hyperideal, quasi-hyperideal, quasi-simple, intra-regular.

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1. Introduction and basic definitions

Good and Hughes [12] introduced the notion of bi-ideals. The notion of quasi-ideals was first introduced by Steinfeld [29] for rings and semigroups. Then, many authors studied these concepts, for example see [15], [17], [18], [26]. Ordered semigroups have been studied by several authors, for example, Alimov [1] and Clifford [6], also see [19], [20], [21], [27], [28]. The concept of algebraic hyperstructures was introduced by Marty [22]. Semihypergroups are studied by many authors, for example, Anvariyeh et al. [2], [3], Bonansinga and Corsini [4], Davvaz [7, 8], De Salvo et al. [10], Freni [11], Heidari et al. [14], Hila et al. [15], [16], Leoreanu [25], Yaqoob et al. [30], and many others. The concept of ordering hypergroups

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investigated by Chvalina [5] as a special class of hypergroups and studied by him and many others. In [13], Heidari and Davvaz studied a semihypergroup $(S, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition.

Let S be a nonempty set. A mapping $\circ : S \times S \to \mathcal{P}^*(S)$, where $\mathcal{P}^*(S)$ denotes the family of all nonempty subsets of S, is called a *hyperoperation* on S. The couple $(S, \circ)$ is called a *hypergroupoid*. In the above definition, if A and B are two nonempty subsets of S and $x \in S$, then we denote

$$A \circ B = \bigcup_{a \in A, b \in B} a \circ b, \ x \circ A = \{x\} \circ A \ \text{ and } \ A \circ x = A \circ \{x\}.$$

A hypergroupoid $(S, \circ)$ is called a *semihypergroup* if for every x, y, z in S,

$$x \circ (y \circ z) = (x \circ y) \circ z.$$

That is,

$$\bigcup_{u \in y \circ z} x \circ u = \bigcup_{v \in x \circ y} v \circ z.$$

In [13], Heidari and Davvaz studied a semihypergroup $(S, \circ)$ besides a binary relation $\leq$, where $\leq$ is a partial order relation such that satisfies the monotone condition. Indeed, an *ordered semihypergroup* $(S, \circ, \leq)$ is a semihypergroup $(S, \circ)$ together with a partial order $\leq$ that is *compatible* with the hyperoperation, meaning that for any x, y, z in S,

$$x \leq y \Rightarrow z \circ x \leq z \circ y \text{ and } x \circ z \leq y \circ z.$$

Here, $z \circ x \leq z \circ y$ means for any $a \in z \circ x$ there exists $b \in z \circ y$ such that $a \leq b$. The case $x \circ z \leq y \circ z$ is defined similarly.

A nonempty subset A of an ordered semihypergroup $(S, \circ, \leq)$ is called a *subsemihypergroup* of S if $A \circ A \subseteq A$.

Example 1. Suppose that $S = [0, 1]$, the unit real interval numbers. We define the following hyperoperation on S:

$$\circ : S \times S \to \mathcal{P}^*(S) \text{ by } x \circ y = [0, xy] \text{ for all } x, y \text{ in } S$$

and we consider usual inequality relation $\leq$ of real numbers as a partial order relation on S. Then, it is easy to see that $(S, \circ, \leq)$ is an ordered semihypergroup. If $A = [0, a]$ for some a in S, then A is a subsemihypergroup of S.

Definition 1.1. A nonempty subset A of an ordered semihypergroup $(S, \circ, \leq)$ is called a *left* (respectively, *right*) *hyperideal* of S if it satisfies the following conditions:

(i) $S \circ A \subseteq A$ (respectively, $A \circ S \subseteq A$);

(ii) if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

If A is both a left and a right hyperideal of S, then it is called a *two-sided hyperideal* of S, or simply a *hyperideal* of S.

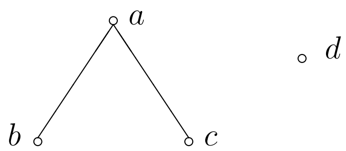
Example 2. We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	a	$\{a, b\}$	$\{a, c\}$	a
b	a	$\{a, b\}$	$\{a, c\}$	a
c	a	$\{a, b\}$	$\{a, c\}$	a
d	a	$\{a, b\}$	$\{a, c\}$	a

$$\leq = \{(a, a), (b, b), (c, c), (d, d), (b, a), (c, a)\}.$$

The detailed proof that $(S, \circ, \leq)$ is an ordered semihypergroup is straightforward. The covering relation and the figure of S are given by:

$$\prec = \{(b, a), (c, a)\}$$



Now, it is easy to see that $\{a, b, c\}$ is a hyperideal of S.

Let A be a nonempty subset of an ordered semihypergroup $(S, \circ, \leq)$. Define

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

Note that condition (ii) in Definition 1.1 is equivalent to $A = (A]$. If A and B are nonempty subsets of S, then

(1) $A \subseteq (A]$;

(2) $(A \cup B] = (A] \cup (B]$;

(3) $((A] \circ (B]] = (A \circ B]$;

(4) $(A] \circ (B] \subseteq (A \circ B]$.

Let a be an element of an ordered semihypergroup $(S, \circ, \leq)$. We define

$L(a)$: the left hyperideal of S generated by a;

$R(a)$: the right hyperideal of S generated by a;

$I(a)$: the hyperideal of S generated by a.

Lemma 1.2. *Let a be an element of an ordered semihypergroup $(S, \circ, \leq)$.*

(1) $L(a) = (a \cup S \circ a]$.

(2) $R(a) = (a \cup a \circ S]$.

(3) $I(a) = (a \cup S \circ a \cup a \circ S \cup S \circ a \circ S]$.

Proof. We will prove (1). The statements (2) and (3) are proved similarly. Clearly, $(a \cup S \circ a] \neq \emptyset$. We have

$$S \circ (a \cup S \circ a] \subseteq (S \circ (a \cup S \circ a)] = (S \circ a \cup S \circ (S \circ a)] \subseteq (S \circ a] \subseteq (a \cup S \circ a].$$

Then, $(a \cup S \circ a]$ is a left hyperideal of S containing a; hence $L(a) \subseteq (a \cup S \circ a]$. The reverse inclusion is clear. ∎

2. Bi-hyperideals

In this section, we introduce the class of bi-hyperideals of an ordered semihypergroup and we present some examples of them. In particular, we show that an ordered semihypergroup is left and right simple if and only if it does not contains proper bi-hyperideals.

Definition 2.1. A subsemihypergroup A of an ordered semihypergroup $(S, \circ, \leq)$ is called a *bi-hyperideal* of S if it satisfies the following conditions:

(i) $A \circ S \circ A \subseteq A$;

(ii) if $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

Note that every left and right hyperideals are bi-hyperideals.

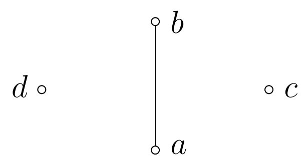
Example 3. We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d
a	a	a	a	a
b	a	a	a	a
c	a	a	$\{a, b\}$	a
d	a	a	$\{a, b\}$	$\{a, b\}$

$$\leq \; = \{(a, a), (b, b), (c, c), (d, d), (a, b)\}.$$

The detailed proof that $(S, \circ, \leq)$ is an ordered semihypergroup is straightforward. The covering relation and the figure of S are given by:

$$\prec \; = \{(a, b)\}$$



It is easy to see that $\{a, b\}$ is a bi-hyperideal of S.

Definition 2.2. An ordered semihypergroup $(S, \circ, \leq)$ is said to be *left* (respectively, *right*) *simple* if it does not contain proper left (respectively, right) hyperideals.

Lemma 2.3. *An ordered semihypergroup* $(S, \circ, \leq)$ *is left simple if and only if* $(S \circ x] = S$ *for all* x *in* S.

Proof. Assume that $(S \circ x] = S$ for all x in S. Let A be a left hyperideal of S, and let $a \in A$. Then, $(S \circ a] = S$. If $y \in S$, then $y \in (S \circ a]$; hence $y \in (w \circ a]$ for some w in S. Since $(w \circ a] \subseteq A$, we have $y \in A$, and so $A = S$. The opposite direction follows by $(S \circ x]$ is a left hyperideal of S for all x in S. ∎

Similarly, we have the following lemma:

Lemma 2.4. *An ordered semihypergroup $(S, \circ, \leq)$ is right simple if and only if $(x \circ S] = S$ for all x in S.*

Definition 2.5. An ordered semihypergroup $(S, \circ, \leq)$ is said to be *regular* if for any $a \in S$, $a \in (a \circ S \circ a]$.

Lemma 2.6. *If an ordered semihypergroup $(S, \circ, \leq)$ is left and right simple, then it is regular.*

Proof. Assume that S is left and right simple. If $a \in S$, then by Lemma 2.3 and 2.4 we have $S = (S \circ a]$ and $S = (a \circ S]$; hence

$$a \in S = (a \circ S] = (a \circ (S \circ a]] \subseteq (a \circ S \circ a].$$ ∎

Now, we prove the main result of this section:

Theorem 2.7. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then, $(S, \circ, \leq)$ is left and right simple if and only if it does not contain proper bi-hyperideals.*

Proof. Assume that $(S, \circ, \leq)$ is left and right simple. Let A be a bi-hyperideal of S. We will show that $S \subseteq A$. Let $x \in S$ and $y \in A$. Since S is left simple, we have $S = (y \cup S \circ y]$. Then, $x \leq y$ or $x \in (w \circ y]$ for some w in S. If $x \leq y$, then $x \in A$. Assume that $x \in (w \circ y]$ for some w in S. Since S is right simple, we have $S = (y \cup y \circ S]$. Since $w \in S$, so $w \leq y$ or $w \in (y \circ u]$ for some u in S. By Lemma 2.6, $(S, \circ, \leq)$ is regular. Then, there exists $b \in S$ such that $y \in (y \circ b \circ y]$. Now, if $w \leq y$, then

$$(w \circ y] \subseteq (y \circ y] \subseteq (y \circ y \circ b \circ y] \subseteq (A \circ S \circ A] \subseteq A;$$

hence $x \in A$. If $w \in (y \circ u]$, then

$$(w \circ y] \subseteq (y \circ u \circ y)] \subseteq (A \circ S \circ A)] \subseteq A;$$

hence $x \in A$. Thus, $S \subseteq A$.

Conversely, assume that $(S, \circ, \leq)$ does not contain proper bi-hyperideals. If A is a left hyperideal of S, then A is a bi-hyperideal of S. By assumption, $S = A$. Similarly, if A is a right hyperideal of S, then A is a bi-hyperideal of S, so $S = A$. ∎

3. Quasi-hyperideals

In this section, we introduce the class of quasi-hyperideals of an ordered semihypergroup and we give some of their properties and relationships between them and bi-hyperideals. We try to use sets instead of elements in the proof of our results similar to [17, 25].

Definition 3.1. A nonempty subset Q of an ordered semihypergroup $(S, \circ, \leq)$ is called a *quasi-hyperideal* of S if it satisfies the following conditions:

(i) $(S \circ Q) \cap (Q \circ S) \subseteq Q$;

(ii) if $x \in Q$ and $y \in S$ such that $y \leq x$, then $y \in Q$.

Note that every quasi-hyperideal is a subsemihypergroup. Indeed, if Q is a quasi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$, then $Q \circ Q \subseteq S \circ Q$ and $Q \circ Q \subseteq Q \circ S$; hence

$$Q \circ Q \subseteq S \circ Q \cap Q \circ S \subseteq Q.$$

Thus, Q is a subsemihypergroup of S.

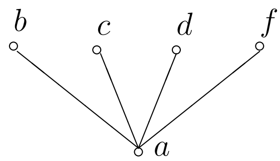
Example 4. We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	a	b	c	d	f
a	a	a	a	a	a
b	a	$\{a, b\}$	a	$\{a, d\}$	a
c	a	$\{a, f\}$	$\{a, c\}$	$\{a, c\}$	$\{a, f\}$
d	a	$\{a, b\}$	$\{a, d\}$	$\{a, d\}$	$\{a, b\}$
f	a	$\{a, f\}$	a	$\{a, c\}$	a

$$\leq = \{(a, a), (b, b), (c, c), (d, d), (f, f), (a, b), (a, c), (a, d), (a, f)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(a, b), (a, c), (a, d), (a, f)\}$$



It is easy to see that the quasi-hyperideals of S are $\{a\}$, $\{a, b\}$, $\{a, d\}$, $\{a, f\}$, $\{a, b, d\}$, $\{a, c, d\}$, $\{a, b, f\}$, $\{a, c, f\}$.

Theorem 3.2. *Let Q be a quasi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. If T is a subsemihypergroup of S, then $Q \cap T = \emptyset$ or $Q \cap T$ is a quasi-hyperideal of T.*

Proof. Let $Q_1 = Q \cap T$. Assume that $Q_1 \neq \emptyset$. Since $Q_1 \subseteq Q$, it follows that

$$(T \circ Q_1) \cap (Q_1 \circ T) \subseteq (S \circ Q) \cap (Q \circ S) \subseteq Q.$$

Since $Q_1 \subseteq T$ and T is a subsemihypergroup of S, we have

$$(T \circ Q_1) \cap (Q_1 \circ T) \subseteq T.$$

Then,

$$(T \circ Q_1) \cap (Q_1 \circ T) \subseteq Q_1.$$

If $x \in Q_1$ and $y \in T$ such that $y \leq x$, then since $x \in Q$ we have $y \in Q$; hence $y \in Q_1$. Therefore, Q_1 is a quasi-hyperideal of T. ∎

Theorem 3.3. *Let $\{Q_i \mid i \in I\}$ be an indexed family of quasi-hyperideals of an ordered semihypergroup $(S, \circ, \leq)$. If $\bigcap_{i \in I} Q_i \neq \emptyset$, then it is a quasi-hyperideal of S.*

Proof. Let $Q = \bigcap_{i \in I} Q_i$. Since, for each $i \in I$,

$$(S \circ Q) \cap (Q \circ S) \subseteq (S \circ Q_i) \cap (Q_i \circ S) \subseteq Q_i,$$

we have

$$(S \circ Q) \cap (Q \circ S) \subseteq Q.$$

If $x \in Q$ and $y \in S$ such that $y \leq x$, then for each $i \in I$ we have $y \in Q_i$; hence $y \in Q$. Therefore, Q is a quasi-hyperideal of S. ∎

Let A be a nonempty subset of an ordered semihypergroup $(S, \circ, \leq)$. Then, the intersection of all quasi-hyperideals of S containing A, denoted by $(A)_q$, is a quasi-hyperideal of S containing A. This is called the *quasi-hyperideal* of S generated by A.

Theorem 3.4. *The intersection of a left and a right hyperideals of an ordered semihypergroup $(S, \circ, \leq)$ is a quasi-hyperideal of S.*

Proof. Let L and R be a left hyperideal and a right hyperideal of S. Let $Q = L \cap R$. We choose $l \in L$ and $r \in R$. Since $l \circ r \subseteq L \cap R$, so $Q \neq \emptyset$. Since

$$S \circ Q \subseteq L \text{ and } Q \circ S \subseteq R,$$

it follows that

$$S \circ Q \cap Q \circ S \subseteq L \cap R = Q.$$

If $x \in Q$ and $y \in S$ such that $y \leq x$, then by $x \in L \cap R$ we have $y \in L \cap R$. Hence, Q is a quasi-hyperideal of S. ∎

Lemma 3.5. *If Q is a quasi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$, then*

$$Q = (Q \cup S \circ Q] \cap (Q \cup Q \circ S].$$

Proof. Let Q be a quasi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$. Clearly,

$$Q \subseteq (Q \cup S \circ Q] \cap (Q \cup Q \circ S].$$

Let $x \in (Q \cup S \circ Q] \cap (Q \cup Q \circ S]$. Then,

$$x \in Q \text{ or } x \in (S \circ Q \cap Q \circ S].$$

By

$$S \circ Q \cap Q \circ S \subseteq Q$$

it follows that $x \in Q$. Hence, $Q = (Q \cup S \circ Q] \cap (Q \cup Q \circ S]$. ■

Theorem 3.6. *If Q is a quasi-hyperideal of an ordered semihypergroup $(S, \circ, \leq)$, then there exist a left hyperideal L and a right hyperideal R of S such that $Q = L \cap R$.*

Proof. Assume that Q is a quasi-hyperideal of S. Let

$$L = (Q \cup Q \circ S] \text{ and } R = (Q \cup S \circ Q].$$

We have $L \neq \emptyset$. Let $x \in L \circ S$, then $x \in l \circ s$ for some l in L and s in S. Let $l \leq p$ for some p in $Q \cup Q \circ S$. Then, $l \circ s \leq p \circ s$. There are two cases to consider:

Case 1: $p \in Q$. By

$$p \circ s \subseteq Q \circ S \subseteq Q \cup Q \circ S,$$

it follows $x \in L$.

Case 2: $p \in Q \circ S$. Let $p \in q \circ s_1$ for some q in Q and s_1 in S. Since

$$p \circ s \subseteq q \circ s_1 \circ s \subseteq Q \circ S \circ S \subseteq Q \cup Q \circ S \circ S,$$

we have $x \in L$. Then, $L \circ S \subseteq L$. Let $x \in L$ and $y \in S$ be such that $y \leq x$. Then, $x \leq z$ for some z in $Q \cup Q \circ S$. Since $y \leq z$, $y \in L$. Hence, L is a left hyperideal of S. Similarly, R is a right hyperideal of S.

By Lemma 3.5, we obtain

$$L \cap R = (Q \cup Q \circ S] \cap (Q \cup S \circ Q] = Q.$$

This completes the proof. ■

Theorem 3.7. *Let A be a nonempty subset of an ordered semihypergroup $(S, \circ, \leq)$. Then, the intersection of all quasi-hyperideals of S containing A is of the form*

$$(A)_q = (A \cup S \circ A] \cap (A \cup A \circ S].$$

Proof. Let Q be the intersection of all quasi-hyperideals $Q_i, i \in I$ of S containing A. By Theorem 3.3, $(A)_q$ is a quasi-hyperideal of S containing A. We have

$$(A \cup S \circ A] \text{ and } (A \cup A \circ S]$$

are left and right hyperideals of S, respectively. By Theorem 3.4,

$$(A \cup S \circ A] \cap (A \cup A \circ S]$$

is a quasi-hyperideal of S containing A. Thus,

$$(A)_q \subseteq (A \cup S \circ A] \cap (A \cup A \circ S].$$

For each $i \in I$, since $A \subseteq Q_i$ and Lemma 3.5, we have

$$(A \cup S \circ A] \cap (A \cup A \circ S] \subseteq (Q_i \cup S \circ Q_i] \cap (Q_i \cup Q_i \circ S] = Q_i.$$

Then,

$$(A)_q \supseteq (A \cup S \circ A] \cap (A \cup A \circ S].$$

Hence,

$$(A)_q = (A \cup S \circ A] \cap (A \cup A \circ S].$$ ■

Definition 3.8. An ordered semihypergroup $(S, \circ, \leq)$ is said to be *quasi-simple* if it is the unique quasi-hyperideal.

Definition 3.9. An ordered semihypergroup $(S, \circ, \leq)$ is called an ordered hypergroup if $(S, \circ)$ is a hypergroup. In fact, for all a in S, we have $a \circ S = S \circ a = S$.

Corollary 3.10. *Every ordered hypergroup is quasi-simple.*

Theorem 3.11. *An ordered semihypergroup $(S, \circ, \leq)$ is quasi-simple if and only if $S = (x \circ S] \cap (S \circ x]$ for all x in S.*

Proof. Assume that S is quasi-simple and let $x \in S$. We will show that

$$R = (x \circ S]$$

is a right hyperideal of S. Let $y \in R \circ S$. Then, $y \in r \circ s$ for some r in R and s in s. Since $r \in R$, so $r \in x \circ s_1$ for some s_1 in S. Thus,

$$y \in r \circ s \subseteq x \circ s_1 \circ s \subseteq x \circ S \subseteq R.$$

Let $y \in R$ and $z \in S$ be such that $z \leq y$. Since $y \in R$, so $y \leq w$ for some w in $x \circ S$. Since $z \leq w$, so $z \in x \circ S \subseteq R$. Therefore, R is a right hyperideal of S. Similarly, $(S \circ x]$ is a left hyperideal of S. By Theorem 3.3,

$$S = (x \circ S] \cap (S \circ x].$$

Conversely, assume that $S = (x \circ S] \cap (S \circ x]$ for all x in S. Let Q be a quasi-hyperideal of S . If $q \in Q$, then by assumption and Lemma 3.5 we get

$$S = (q \circ S] \cap (S \circ q] \subseteq (Q \circ S] \cap (S \circ Q] = Q$$

Thus, $S = Q$. This completes the proof. ■

Definition 3.12. A quasi-hyperideal Q of an ordered semihypergroup $(S, \circ, \leq)$ is said to be *minimal* if it contains no proper quasi-hyperideal.

Theorem 3.13. *Every quasi-simple hyperideal of an ordered semihypergroup $(H, f, \leq)$ is minimal.*

Proof. Let Q be a quasi-simple hyperideal of H. Let Q' be a quasi-hyperideal of H such that $Q' \subseteq Q$. Since Q' is a quasi-hyperideal of Q, $Q' = Q$. Then, Q is a minimal quasi-hyperideal of H. ∎

The notion of intra-regular ordered semihypergroups in introduced by Kehayupulu et al. in [17] and Lee and Lee [25] gave some characterizations of the intra-regular ordered semihypergroups. Note that Definition 3.14 is a generalization of the concept of intra-regular ordered semigroups, since every ordered semigroup is an ordered semihypergroup.

Definition 3.14. An ordered semihypergroup $(S, \circ, \leq)$ is called *intra-regular* if for all s in S there exist x, y in S such that $s \leq x \circ s^2 \circ y$, where we means $s^2 = s \circ s$.

Example 5. Suppose that S is a nonempty set and $\leq$ be a partial order relation on S. We define the following hyperoperation on S:

$$\circ : S \times S \to \mathcal{P}^*(S), \text{ by } x \circ y = \{x, y\} \text{ for all } x, y \text{ in } S.$$

Then, $(S, \circ, \leq)$ is an ordered semihypergroup. Since for every s in S, $s \leq s \circ s^2 \circ s$, we conclude that S is intra-regular.

Example 6. The order semihypergroup $(S, \circ, \leq)$ defined in Example 3 is not intra-regular, since if we consider the element f, then $f^2 = a$ and for all x, y in S, we have $x \circ f^2 \circ y = x \circ a \circ y = a$ and $f \leq a$ is false.

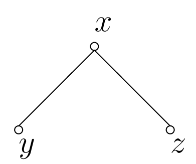
Example 7. We have $(S, \circ, \leq)$ is an ordered semihypergroup where the hyperoperation and the order relation are defined by:

$\circ$	x	y	z
x	x	$\{x, y\}$	$\{x, z\}$
y	x	$\{x, y\}$	$\{x, y\}$
z	x	$\{x, y\}$	z

$$\leq = \{(x, x), (y, y), (z, z), (y, x), (z, x)\}.$$

The covering relation and the figure of S are given by:

$$\prec = \{(y, x), (z, x)\}$$



Now, we have

$$x \leq x \circ x^2 \circ x = x \circ x \circ x = x,$$
$$y \leq x \circ y^2 \circ x = x \circ \{x, y\} \circ x = \{x, y\} \circ x = x,$$
$$z \leq z \circ z^2 \circ z = z \circ z \circ z = z.$$

Therefore, $(S, \circ, \leq)$ is intra-regular.

Let a be an element of an ordered semihypergroup $(S, \circ, \leq)$. We define

$$B(a) \ : \ \text{the bi-hyperideal of } S \text{ generated by } a;$$
$$Q(a) \ : \ \text{the quasi-hyperideal of } S \text{ generated by } a.$$

Lemma 3.15. *Let a be an element of an ordered semihypergroup $(S, \circ, \leq)$.*

(1) $B(a) = (a \cup a^2 \cup a \circ S \circ a]$.

(2) $Q(a) = (a \cup ((a \circ S] \cap (S \circ a])]$.

Proof. (1) Clearly, $(a \cup a^2 \cup a \circ S \circ a] \neq \emptyset$. We have

$$\begin{aligned}(a \cup a^2 \cup a \circ S \circ a] \circ S \circ (a \cup a^2 \cup a \circ S \circ a] \\ &\subseteq ((a \cup a^2 \cup a \circ S \circ a) \circ S \circ (a \cup a^2 \cup a \circ S \circ a)] \\ &\subseteq (a^2 \cup a \circ S \circ a] \\ &\subseteq (a \cup a^2 \cup a \circ S \circ a].\end{aligned}$$

Then, $(a \cup a^2 \cup a \circ S \circ a]$ is a subsemihypergroup of S. Now,

$$\begin{aligned}(a \cup a^2 \cup a \circ S \circ a](a \cup a^2 \cup a \circ S \circ a] \\ &\subseteq ((a \cup a^2 \cup a \circ S \circ a) \circ (a \cup a^2 \cup a \circ S \circ a)] \\ &\subseteq (a \circ S \circ a] \\ &\subseteq (a \cup a^2 \cup a \circ S \circ a].\end{aligned}$$

Hence, $(a \cup a^2 \cup a \circ S \circ a]$ is a bi-hyperideal of S, and thus $B(a) \subseteq (a \cup a^2 \cup a \circ S \circ a]$. The reverse inclusion is clear.

(2) Clearly, $(a \cup ((a \circ S] \cap (S \circ a])] \neq \emptyset$. We have

$$\begin{aligned}(a \cup ((a \circ S] \cap (S \circ a])] \circ S &\subseteq (a \cup (a \circ S]] \circ S \\ &\subseteq (a \circ S \cup (a \circ S] \circ S] \\ &\subseteq (a \circ S \cup (a \circ S]] \\ &= ((a \circ S]] \\ &= (a \circ S].\end{aligned}$$

Similarly,

$$\begin{aligned}S \circ (a \cup ((a \circ S] \cap (S \circ a])] &\subseteq S \circ (a \cup (S \circ a]] \\ &\subseteq (S \circ a \cup S \circ (S \circ a]] \\ &\subseteq (S \circ a \cup (S \circ a]] \\ &= ((S \circ a]] \\ &= (S \circ a].\end{aligned}$$

Hence,

$$\begin{aligned}(a \cup ((a \circ S] \cap (S \circ a])] \circ S \cap S \circ (a \cup ((a \circ S] \cap (S \circ a])] &\subseteq (a \circ S] \cap (S \circ a] \\ &\subseteq ((a \circ S] \cap (S \circ a]] \\ &\subseteq (a \cup ((a \circ S] \cap (S \circ a])].\end{aligned}$$

Therefore, $(a \cup ((a \circ S] \cap (S \circ a])]$ is a quasi-hyperideal of S, and thus

$$Q(a) \subseteq (a \cup ((a \circ S] \cap (S \circ a])].$$

The reverse inclusion is clear. ∎

Theorem 3.16. *Let $(S, \circ, \leq)$ be an ordered semihypergroup. Then, the following statements are true:*

(1) *S is intra-regular if and only if for a bi-hyperideal B and a quasi-hyperideal Q of S, we have $B \cap Q \subseteq (S \circ B \circ Q \circ S]$;*

(2) *S is intra-regular if and only if for a bi-hyperideal B and a quasi-hyperideal Q of S, we have $B \cap Q \subseteq (S \circ Q \circ B \circ S]$.*

Proof. (1) Assume that S is intra-regular. Let $a \in B \cap Q$. Since S is intra-regular, there exist x, y in S such that $a \leq x \circ a^2 \circ y$. We have

$$\begin{aligned} a \leq x \circ a^2 \circ y &\leq x \circ a \circ (x \circ a^2 \circ y) \circ y \\ &= x \circ (a \circ x \circ a) \circ a \circ y^2 \\ &\subseteq S \circ (B \circ S \circ B) \circ Q \circ S \\ &\subseteq S \circ B \circ Q \circ S. \end{aligned}$$

Then, $B \cap Q \subseteq (S \circ B \circ Q \circ S]$.

Conversely, let $a \in S$. We consider a bi-hyperideal $B(a)$ and a quasi-hyperideal $Q(a)$. Then,

$$\begin{aligned} a \in B(a) \cap Q(a) &\subseteq (S \circ B(a) \circ Q(a) \circ S] \\ &= (S \circ (a \cup a^2 \cup a \circ S \circ a] \circ (a \cup ((a \circ S] \cap (S \circ a])] \circ S] \\ &\subseteq ((S \circ a \cup S \circ a^2 \cup S \circ a] \circ (a \cup (a \circ S]] \circ S] \\ &\subseteq ((S \circ a] \circ (a \cup (a \circ S]] \circ S] \\ &\subseteq ((S \circ a] \circ (a \circ S \cup (a \circ S^2]] \\ &\subseteq ((S \circ a^2 \circ S] \cup (S \circ a^2 \circ S^2]] \\ &\subseteq ((S \circ a^2 \circ S]] \\ &= (S \circ a^2 \circ S]. \end{aligned}$$

Thus, S is intra-regular.

(2) Assume that S is intra-regular. Let $a \in B \cap Q$. Since S is intra-regular, there exist x, y in S such that $a \leq x \circ a^2 \circ y$. By

$$\begin{aligned} a \leq x \circ a^2 \circ y &\leq x \circ (x \circ a^2 \circ y) \circ a \circ y \\ &= x^2 \circ a \circ (a \circ y \circ a) \circ y \\ &\subseteq S \circ Q \circ (B \circ S \circ B) \circ S \\ &\subseteq S \circ Q \circ B \circ S, \end{aligned}$$

it follows that $B \cap Q \subseteq (S \circ Q \circ B \circ S]$.

Conversely, let $a \in S$. We consider a bi-hyperideal $B(a)$ and a quasi-hyperideal $Q(a)$ of S; then

$$\begin{aligned} a \in B(a) \cap Q(a) &\subseteq (S \circ Q(a) \circ B(a) \circ S] \\ &= (S \circ (a \cup ((a \circ S] \cap (S \circ a])] \circ (a \cup a^2 \cup a \circ S \circ a] \circ S] \\ &\subseteq (S \circ (a \cup (S \circ a]] \circ (a \circ S \cup a^2 \circ S \cup a \circ S \circ a \circ S]] \\ &\subseteq ((Sa \cup (S^2 \circ a]] \circ (a \circ S]] \\ &\subseteq ((S \circ a](a \circ S]] \\ &\subseteq ((S \circ a^2 \circ S]] \\ &= (S \circ a^2 \circ S]. \end{aligned}$$

Thus, S is intra-regular. ■

Theorem 3.17. *The following statements hold for an ordered semihypergroup $(S, \cdot, \leq)$:*

(1) *S is intra-regular if and only if for a left hyperideal L and a bi-hyperideal B of S, we have $L \cap B \subseteq (L \circ B \circ S]$;*

(2) *S is intra-regular if and only if for a right hyperideal R and a bi-hyperideal B of S, we have $B \cap R \subseteq (S \circ B \circ R]$.*

Proof. (1) Assume that S is intra-regular. Let L be a left hyperideal of S, and B a bi-hyperideal of S. If $a \in L \cap B$, then by S is intra-regular there exists x, y in S such that $a \leq x \circ a^2 \circ y$. We have

$$\begin{aligned} x \circ a^2 \circ y &\leq x \circ (x \circ a^2 \circ y) \circ a \circ y \\ &\subseteq x^2 \circ a \circ (a \circ y \circ a) \circ y \\ &\subseteq S \circ L \circ (B \circ S \circ B) \circ S \\ &\subseteq L \circ B \circ S. \end{aligned}$$

Then, $a \in (L \circ B \circ S]$. This proves that $L \cap B \subseteq (L \circ B \circ S]$.

Conversely, let $a \in S$. We consider a bi-hyperideal $B(a)$ and a left hyperideal $L(a)$ of S; then

$$\begin{aligned} L(a) \cap B(a) &\subseteq (L(a) \circ B(a) \circ S] \\ &\subseteq ((a \cup S \circ a] \circ (a \cup a^2 \cup a \circ S \circ a] \circ S] \\ &\subseteq ((a \cup S \circ a] \circ (a \circ S \cup a^2 \circ S \cup a \circ S \circ a \circ S]] \\ &\subseteq ((a \cup S \circ a] \circ (a \circ S]] \\ &\subseteq (a^2 \circ S \cup S \circ a^2 \circ S]. \end{aligned}$$

Thus, $a \leq t$ for some t in $a^2 \circ S \cup S \circ a^2 \circ S$. If $t \in a^2 \circ S$, then $a \leq a^2 \circ x$ for some x in S. Thus, we have

$$a \leq a^2 \circ x \leq a \circ (a^2 \circ x) \circ x \subseteq a \circ a^2 \circ x^2 \subseteq S \circ a^2 \circ S.$$

If $t \in S \circ a^2 \circ S$, it is obvious. Hence, S is intra-regular.

(2) Let R be a right hyperideal of S, and let B a bi-hyperideal of S. Let $a \in B \cap R$. Since S is intra-regular, there exist x, y in S such that $a \leq x \circ a^2 \circ y$, and so

$$\begin{aligned} a \leq x \circ a^2 \circ y &\leq x \circ a \circ (x \circ a^2 \circ y) \circ y \\ &= x \circ (a \circ x \circ a) \circ a \circ y^2 \\ &\subseteq S \circ (B \circ S \circ B) \circ R \circ S \\ &\subseteq S \circ B \circ R. \end{aligned}$$

Thus, $B \cap R \subseteq (S \circ B \circ R]$.

Conversely, to show that S is intra-regular, let $a \in S$. Consider a bi-hyperideal $B(a)$ and a right hyperideal $R(a)$ of S we have

$$\begin{aligned} B(a) \cap R(a) &\subseteq (S \circ B(a) \circ R(a)] \\ &= (S \circ (a \cup a^2 \cup a \circ S \circ a] \circ (a \cup a \circ S]] \\ &\subseteq ((S \circ a \cup S \circ a^2 \cup S \circ a \circ S \circ a] \circ (a \cup a \circ S]] \\ &\subseteq ((S \circ a] \circ (a \cup a \circ S]] \\ &\subseteq ((S \circ a^2 \cup S \circ a^2 \circ S]] \\ &= (S \circ a^2 \cup S \circ a^2 \circ S]. \end{aligned}$$

Then, $a \leq t$ for some t in $S \circ a^2 \cup S \circ a^2 \circ S$. If $t \in S \circ a^2$, then $a \leq x \circ a^2$ for some x in S; hence

$$a \leq x \circ a^2 \leq x \circ (x \circ a^2) \circ a = x^2 \circ a^2 \circ a \subseteq S \circ a^2 \circ S.$$

If $t \in S \circ a^2 \circ S$, it is obvious. Hence, S is intra-regular. ∎

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