SOME RESULTS ON HYPERCYCLICITY OF TUPLE OF OPERATORS

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Abstract. In this paper we extend some results of hypercyclicity from a single operator to a tuple of commuting operators. In particular we extend theorem of Nathan S. Feldman to a tuple of commuting operators.

Keywords: tuple of operators, hypercyclic vector, hypercyclic operator for tuple, topologically transitive, $d-$dense orbit.

1. Introduction

For a infinite-dimensional separable complex Banach space $X, B(X)$ will denote the algebra of all bounded linear operators on X. For $x \in X$, the orbit of x under T is the set $Orb(T, x) = \{T^n x : n \in \mathbb{N}\}$. A vector x is called a hypercyclic vector for T if $Orb(T, x)$ is dense in X and the operator T is said to be hypercyclic if there is some vector $x \in X$ is hypercyclic see [2]. Let T be a bounded linear operator and U a non-empty open subset of X. Suppose that $x \in X$, and $d > 0$. The orbit $Orb(T, x)$ is called $d-$dense in U if for any $y \in U, B(y, d) \cap Orb(T, x) \neq \emptyset$. In 2002 Nathan S. Feldman studied the perturbations of hypercyclic vectors and show that if T has a $d-$dense orbit, then T is hypercyclic see [4, Theorem 2.1] and see also [6, Theorem 8, p. 344]. In 2007 Enhui Shi , Yuwu Yao, Lizhen Zhou, Youcheng Zhou give a local version of Theorem of Nathan S. Feldman see [8, Theorem 3.7]. Recently, in 2013 B. Yousefi and K. Jahedi extends Theorem 3.7 in [8] for a tuples see [10, Theorem 2.2]. In 2010, Vladimir Müller proved for $T \in B(X)$ be mixing, $\lambda \in \mathbb{C}, |\lambda| = 1$. Then λT is mixing see [7, Proposition 6].

By an n-tuple of operators we mean a finite sequence of length n of commuting continuous linear operators on a Banach space X.

Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional separable Banach space X. If $d > 0$, we say a n-tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ of bounded linear operators acting on a separable infinite dimensional Banach space X has a $d-$dense orbit in a nonempty open subset U of X, if there exists $x \in X$ such that for any $y \in U, B(y,d) \cap Orb(T,x) \neq \emptyset$.

In this paper, we extend theorem of Nathan S. Feldman see [4, Theorem 2.1] and see also [6, Theorem 8, p. 344] from a single operator to a tuple of commuting operators. Also, we show that if $\mathcal{T} = (T_1, T_2, ..., T_n)$ is topologically mixing, with $|\lambda| = 1$ for all $\lambda \in \mathbb{C}$, then there exist $M(i) \in \mathbb{N}$ such that

$$||\lambda^{m(i)}(T_1^{m(1)}T_2^{m(2)}...T_n^{m(n)})u - y|| \leq \varepsilon, \ \forall\, m(i) \geq M(i), \text{ for } i = 1, 2, ..., n.$$

Definition 1.1. [5], [9] Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-tuple of operators acting on an infinite dimensional separable Banach space X. We will let

$$\mathcal{F} = \{T_1^{k_1}T_2^{k_2}...T_n^{k_n} : \ k_i \in \mathbb{Z}_+, \quad i = 1, ..., n\}$$

be the semigroup generated by $\mathcal{T}$. For $x \in X$, the orbit of x under the tuple $\mathcal{T}$ is the set

$$Orb(\mathcal{T}, x) = \{Sx : S \in \mathcal{F}\}.$$

A vector x is called a hypercyclic vector for $\mathcal{T}$ if $Orb(\mathcal{T}, x)$ is dense in X and in this case the tuple $\mathcal{T}$ is called hypercyclic.

Definition 1.2. [11] An n-tuple $\mathcal{T} = (T_1, T_2, ..., T_n)$ is called topologically mixing if for any given open sets U and V, there exist positive integers $M(1), M(2), ..., M(n)$ such that

$$T_1^{m(1)}...T_n^{m(n)}(U) \cap V \neq \emptyset, \ \forall m(i) \geq M(i), \ i \ = \ 1, ..., n.$$

Definition 1.3. [9] We say that the pair $\mathcal{T} = (T_1, T_2)$ is topologically transitive if for every nonempty open subsets U and V of X there exists $S \in \mathcal{F}$ such that $S(U) \cap V \neq \emptyset$.

Definition 1.4. [3, Definition 3.68] Let A be a subset of a metric space (X, d). A subset A_ε of A is an ε-net for A if for every point x of A there exists a point y in A_ε such that $d(x, y) < \varepsilon$. A subset A of X is totally bounded in (X, d) if for every real number $\varepsilon > 0$ there exists a finite ε-net for A.

We have the equivalent the hypercyclic and topologically transitive for $\mathcal{T}$ (see [9, Lemma 2.2] and see also [5, Proposition 2.3]).

2. Main results

For simplicity, we state and prove our results for a pair that is a tuple with $n = 2$, and the general case follows by a similar method. Note that if T_1, T_2 are commutative bounded linear operators on a Banach space X, and $\{m_j\}, \{n_j\}$ are two sequences of natural numbers, then we say $\{T_1^{m_j}T_2^{n_j} : j \in \mathbb{N}\}$ is hypercyclic if there exists $x \in X$ such that $\{T_1^{m_j}T_2^{n_j}x : j \in \mathbb{N}\}$ is dense in X.

The following theorem extends the theorem of Nathan S. Feldman (see [4, Theorem 2.1] and also [6, Theorem 8, p. 344]) from a single operator to a tuple of commuting operators.

Theorem 2.1. *Let $\mathcal{T} = (T_1, T_2, ..., T_n)$ be an n-uplet of commuting continuous linear operators acting on an infinite dimensional separable Banach space X, $d > 0$, and let $x \in X$ satisfying that for each $y \in X$ there is the sequences $\{m_j^i\}_j$, for $i = 1, ..., n$, of natural numbers with $||T_1^{m_j^1}T_2^{m_j^2}...T_n^{m_j^n}x - y|| < d$. Then, $\mathcal{T} = (T_1, T_2, ..., T_n)$ is hypercyclic.*

Proof. For simplicity, we take $n = 2$. Let U, V be non-empty open subsets of X. We show that

$$T_1^{m_j}T_2^{n_j}U \cap V \neq \emptyset$$

for some $m_j, n_j \in \mathbb{N}$ for each $j = 0, 1, ...$ ∎

Choose $u \in U, v \in V$, and $\varepsilon > 0$ such that $\{y \in X : ||y - u|| < \varepsilon\} \subset U$ and $\{y \in X : ||y - v|| < \varepsilon\} \subset V$.

Let $x' = \frac{\varepsilon x}{3d}$. We show first that the set $Orb(\mathcal{T}, x') = \{T_1^{m_j}T_2^{n_j}x', j \in \mathbb{N}\}$ intersects each open ball with radius $\frac{\varepsilon}{3}$. If $y \in X$, then there is an $m_j, n_j \in \mathbb{N}$ such that

$$\left\|T_1^{m_j}T_2^{n_j}x - \frac{3dy}{\varepsilon}\right\| < d.$$

Therefore,

$$||T_1^{m_j}T_2^{n_j}x' - y|| < \frac{\varepsilon}{3d}\left\|T_1^{m_j}T_2^{n_j}x - \frac{3dy}{\varepsilon}\right\| < \frac{\varepsilon}{3}.$$

Next, we show that $Orb(\mathcal{T}, x')$ intersects each open ball with radius ε in infinite set. Suppose on the contrary that there is a $y \in X$ such that the set

$$\{m_j, n_j, j \in \mathbb{N} : ||T_1^{m_j}T_2^{n_j}x' - y|| < \varepsilon\}$$

is finite. Since the ball $B(y, \frac{2\varepsilon}{3})$ cannot be covered by a finite number of balls of radii $\frac{\varepsilon}{3}$, hence there is a

$$y_1 \in B\left(y, \frac{2\varepsilon}{3}\right)$$

such that

$$dist\{y_1, Orb(\mathcal{T}, x')\} \geq \frac{\varepsilon}{3}.$$

Thus

$$Orb(\mathcal{T}, x') \cap B\left(y_1, \frac{\varepsilon}{3}\right) = \emptyset,$$

a contradiction. Hence

$$B(v, \varepsilon) \cap \{T_1^{m_j}T_2^{n_j}x' : j \in \mathbb{N}\}$$

has infinite elements. In particular, there exist $m_j, n_j \in \mathbb{N}$ satisfying $m_j^{'} > m_j$ and $n_j^{'} > n_j$ such that

$$T_1^{m_j} T_2^{n_j} x^{'} \in B(u, \varepsilon) \subset U \quad \text{and} \quad T_1^{m_j^{'}} T_2^{n_j^{'}} x^{'} \in B(v, \varepsilon) \subset V.$$

Hence

$$T_1^{m_j^{'} - m_j} T_2^{n_j^{'} - n_j} T_1^{m_j} T_2^{n_j} x^{'} \in V,$$

and so

$$T_1^{m_j^{'} - m_j} T_2^{n_j^{'} - n_j} U \cap V \neq \emptyset.$$

Thus there exist $S \in \mathcal{F}$ such that $SU \cap V \neq \emptyset$. By [9, Lemma 2.2] and see also [5, Proposition 2.3], $\mathcal{T} = (T_1, T_2)$ is hypercyclic. ∎

In the proof of the following proposition, we use a method of the proof of Proposition 6 [7]. Denote by $\mathbb{T}$ the unit circle in the complex plane, $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\}$.

Proposition 2.1. *Let $\mathcal{T} = (T_1, T_2, ...T_n)$ be a n-uplet of commuting continuous linear operators acting on an infinite dimensional separable Banach space X, Let $\mathcal{T}$ is topologically mixing, with $|\lambda| = 1$ for all $\lambda \in \mathbb{C}$. Then there exist $M(i) \in \mathbb{N}$ such that*

$$||\lambda^{m(i)} (T_1^{m(1)} T_2^{m(2)} ... T_n^{m(n)}) u - y|| \leq \varepsilon, \quad \forall\, m(i) \geq M(i), \quad \textit{for } i = 1, 2, ..., n.$$

Proof. For simplicity, we take $n = 2$. Let $U, V \subset X$ be non-empty open subsets of X. Choose $y \in V, y \neq 0$. Let $\varepsilon > 0$ satisfy $B(y, \varepsilon) \subset V$, where $B(y, \varepsilon)$ denotes the open ball centered at y with radius ε.

Let $\mu_1, ..., \mu_k \in \mathbb{T}$ be a finite $\frac{\varepsilon}{2||y||}$-net in $\mathbb{T}$. Since $\mathcal{T} = (T_1, T_2)$ is topologically mixing, for each $j = 1, ..., k$, there exists $M_j(i) \in \mathbb{N}$ where $i = 1, 2$ such that

$$T_1^{m(1)} T_2^{m(2)} U \cap B(\mu_j y, \frac{\varepsilon}{2}) \neq \emptyset \text{ for all } m(i) \geq M_j(i).$$

Let $m(i) \geq \max\{M_j(i),\ j = 1, ..., k\}$, for all $i = 1, 2$. Find j such that $|\lambda^{-m(i)} - \mu_j| \leq \frac{\varepsilon}{2||y||}$ for all $i = 1, 2$. Find $u \in U$ such that $||T_1^{m(1)} T_2^{m(2)} u - \mu_j y|| < \frac{\varepsilon}{2}$. Then

$$\begin{aligned} ||\lambda^{m(i)} (T_1^{m(1)} T_2^{m(2)}) u - y|| &= ||(T_1^{m(i)} T_2^{m(2)}) u - \lambda^{-m(i)} y|| \\ &\leq ||T_1^{m(1)} T_2^{m(2)} u - \mu_j y|| + ||\mu_j y - \lambda^{-m(i)} y|| \\ &\leq \frac{\varepsilon}{2} + ||y||\, |\mu_j - \lambda^{-m(i)}| \leq \varepsilon. \end{aligned}$$

Hence

$$||\lambda^{m(i)} (T_1^{m(1)} T_2^{m(2)}) u - y|| \leq \varepsilon.$$

Then, there exist $M(i) = \max\{M_j(i),\ j = 1, ..., k\}$, for all $i = 1, 2$, such that

$$||\lambda^{m(i)} (T_1^{m(1)} T_2^{m(2)}) u - y|| \leq \varepsilon \qquad \forall\, m(i) \geq M(i), \text{ for } \ i = 1, 2.$$ ∎

We will finish with a corollary of the following theorem.

Theorem 2.2. [1, Theorem 1.1]. *Let X be a complex topological vector space, let $T \in \mathcal{L}(X)$ and let $x \in X$. The following are equivalent:*

(i) *x is hypercyclic for T;*

(ii) $\overline{\{e^{iP(n)}T^n x;\, n \in \mathbb{N}^*\}} = X$ *for any polynomial* $P \in \mathbb{R}[t]$;

(iii) $\overline{\{e^{iP(n)}T^n x;\, n \in \mathbb{N}^*\}}^{\circ} \neq \emptyset$ *for some polynomial* $P \in \mathbb{R}[t]$.

Corollary 2.1. *Let $T \in \mathcal{L}(X)$, $k \in \mathbb{N}^*$, and $P \in \mathbb{R}[t]$. Suppose there are vectors $x_1, x_2, ..., x_k \in X$ such that*

$$\overline{\bigcup_{j=1}^{k} \{e^{iP(n)}T^n x_j;\, n \in \mathbb{N}^*\}} = X.$$

Then T is hypercyclic.

Proof. We have

$$X = \overline{\bigcup_{j=1}^{k} \{e^{iP(n)}T^n x_j;\, n \in \mathbb{N}^*\}} = \bigcup_{j=1}^{k} \overline{\{e^{iP(n)}T^n x_j;\, n \in \mathbb{N}^*\}}.$$

By [3, Theorem 3.58, p. 147], there is $1 \leq j_0 \leq k$ such that

$$\overline{\{e^{iP(n)}T^n x_{j_0};\, n \in \mathbb{N}^*\}}^{\circ} \neq \emptyset.$$

By Theorem 1.2, x_{j_0} is hypercyclic for T. ∎

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