EXTENSION OF MITTAG–LEFFLER DENSITY AND PROCESSES

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Abstract. An attempt is made to investigate some properties of Mittag-Leffler density. In this paper, structural representation of the Mittag–Leffler variable and extension of Mittag–Leffler stochastic process have also been discussed.

Keywords: Mittag–Leffler density, moment, Laplace transformation, positive levy random variable, Levy process, infinite divisibility, stochastic process.

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1. Introduction

In 1903, the Swedish mathematician Gosta Mittag–Leffler [11] introduced the function $E_\alpha(z)$, is defined by

$$E_\alpha(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + 1)}, \quad \alpha \in \mathbf{C},\ \Re(\alpha) > 0,\ z \in \mathbf{C}, \tag{1}$$

where $\Gamma(z)$ is the familiar Gamma function. The Mittag–Leffler function reduces immediately to the exponential function $e^z = E_1(z)$ when $\alpha = 1$ for $0 < \alpha < 1$, it interpolates between the pure exponential e^z and a geometric function $\frac{1}{1-z} = \sum_{n=0}^{\infty} z^n, (|z| < 1)$. Its importance has been realized during the last two decades due to its involvement in the problems of applied sciences such as physics, chemistry, biology and engineering. Mittag–Leffler function occurs naturally in the solution of fractional order differential or integral equations.

In 1905, a generalization of $E_\alpha(z)$ was studied by Wiman [11] who defined the function $E_{\alpha,\beta}(z)$ as follows:

$$E_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{z^n}{\Gamma(\alpha n + \beta)}, \quad \alpha, \beta \in \mathbf{C},\ \Re(\alpha) > 0,\ \Re(\beta) > 0,\ z \in \mathbf{C} \tag{2}$$

The function $E_{\alpha,\beta}(z)$ is now known as Wiman function.

In 1971, Prabhakar [8] introduced the function $E^{\gamma}_{\alpha,\beta}(z)$ defined by,

$$E^{\gamma}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_n}{\Gamma(\alpha n + \beta)} \cdot \frac{z^n}{n!}, \quad \alpha, \beta, \gamma \in \mathbf{C}, \Re(\alpha) > 0, \Re(\beta) > 0, \tag{3}$$

where

$$(\gamma)_n = \gamma(\gamma + 1) \dots (\gamma + n - 1) = \frac{\Gamma(\gamma + n)}{\Gamma(\gamma)},$$

whenever $\Gamma(\gamma)$ is defined, $(\gamma)_0 = 1$, $\gamma \neq 0$.

In the sequel to this study, Shukla and Prajapati [10] investigated the function $E^{\gamma,q}_{\alpha,\beta}(z)$ defined for $\alpha, \beta, \gamma \in \mathbf{C}$, $\Re(\alpha) > 0$, $\Re(\beta) > 0$, $q \in (0, 1) \cup \mathbf{N}$ as,

$$E^{\gamma,q}_{\alpha,\beta}(z) = \sum_{n=0}^{\infty} \frac{(\gamma)_{qn}}{\Gamma(\alpha n + \beta)} \cdot \frac{z^n}{n!} \tag{4}$$

The function $E^{\gamma,q}_{\alpha,\beta}(z)$ converges absolutely for all $z \in \mathbf{C}$ if $q < \Re(\alpha) + 1$ (entire function of order $\Re(\alpha)^{-1}$) and for $|z| < 1$ if $q = \Re(\alpha) + 1$.

It is easily seen that (4) is an obvious generalization of (1),(2),(3) and the exponential function e^z as follows:

$$E^{1,1}_{1,1} = e^z, E^{1,1}_{\alpha,1} = E_\alpha(z), E^{1,1}_{\alpha,\beta}(z) = E_{\alpha,\beta}(z), E^{\gamma,1}_{\alpha,\beta}(z) = E^{\gamma}_{\alpha,\beta}(z)$$

Many properties of the function follow from Mittag–Leffler integral representation,

$$E_\alpha(z) = \frac{1}{2\pi i} \int_{\mathbf{C}} \frac{y^{\alpha-1} e^y}{y^\alpha - z} dz,$$

where the path of integration C is a loop which starts and ends at $-\infty$ and encircles the circular disc $|y| \leq z^{\frac{1}{\alpha}}$.

Mittag–Leffler functions and distributions have received the attention from mathematicians, statisticians, and scientists in physical and chemical sciences. Pillai and Sandhya [7] introduced the Mittag–Leffler distribution in terms of Mittag-Leffler functions.

We will use following terminologies in the analysis of the problem undertaken.

Mellin transform

Let a function $f(t)$ be defined for positive t and let it satisfy the conditions

$$\int_0^1 |f(t)|t^{\sigma_1-1}dt < +\infty, \quad \int_1^\infty |f(t)|t^{\sigma_2-1}dt < +\infty$$

for a proper choice of the numbers σ_1 and σ_2. The function

$$F(s) = \int_0^\infty f(t)t^{s-1}dt, \quad s = \sigma + i\tau, \sigma_1 < \sigma < \sigma_2 \tag{5}$$

is the Mellin transform [1] of the function $f(t)$. The inversion formula of the Mellin transform is,

$$f(t) = \frac{1}{2\pi i}\int_{\sigma-i\infty}^{\sigma+i\infty} F(s)t^{-s}ds, \quad t > 0, \quad \sigma_1 < \sigma < \sigma_2,$$

where the integral is taken along the straight line l, $\Re(s) = \sigma$ parallel to the imaginary axis of the s plane and is understood to be the principal value.

Laplace transform

The function $F(s)$ of the complex variable defined by,

$$F(s) = \mathcal{L}\{f(t); s\} = \int_0^\infty e^{-st}f(t)dt, \tag{6}$$

is called the Laplace transform [4] of the function $f(t)$.

The original $f(t)$ can be restored from the Laplace transform $F(s)$ with the help of the inverse Laplace transform,

$$f(t) = \mathcal{L}^{-1}\{F(s); t\} = \int_{c-i\infty}^{c+i\infty} e^{-st}F(s)ds,$$

where, c lies in the right half plane of the absolute convergence of the Laplace integral.

Levy random variable

A positive Levy random variable [5] $u > 0$, with parameter α is such that the Laplace transform of the density of $u > 0$ is given by e^{-t^α}.

$$E[e^{-tu}] = e^{-t^\alpha},$$

where $E(\cdot)$ denotes the expected value of $(\cdot)$ or the statistical expectation of $(\cdot)$. When $\alpha = 1$ the random variable is degenerate with the density function

$$f_1(x) = \begin{cases} 1; & \text{for} \quad x = 1 \\ 0; & \text{elsewhere} \end{cases}$$

Consider an exponential random variable with density function

$$f_1(x) = \begin{cases} e^{-x}; & \text{for} \quad 0 \leq x < \infty \\ 0; & \text{elsewhere} \end{cases}$$

and with the Laplace transform $L_{f_1}(t) = (1+t)^{-1}$.

Levy processes and infinite divisibility [2]

• Brownian motion

A real-valued process $B = \{B_t : t \geq 0\}$, defined on a probability space (Ω, F, P), is said to be a Brownian motion if the following hold:

(i) The paths of B are P-almost surely continuous.

(ii) $P(B_0 = 0) = 1$.

(iii) For $0 \leq s \leq t, B_t - B_s$ is equal in distribution to B_{t-s}.

(iv) For $0 \leq s \leq t, B_t - B_s$ is independent of $\{B_u : u \leq s\}$.

(v) For each $t > 0$, B_t is equal in distribution to a normal random variable with variance t.

• Poisson process

A process valued on the non-negative integers $N = \{N_t : t \geq 0\}$, defined on a probability space (Ω, F, P), is said to be a Poisson process with intensity $\lambda > 0$ if the following hold:

(i) the paths of N are P-almost surely right continuous with left limits.

(ii) $P(N_0 = 0) = 1$.

(iii) For $0 \leq s \leq t, N_t - N_s$ is equal in distribution to N_{t-s}.

(iv) For $0 \leq s \leq t, N_t - N_s$ is independent of $\{N_u : u \leq s\}$.

(v) For each $t > 0$, N_t is equal in distribution to a Poisson random variable with variance λt.

• Levy process

A process $X = \{X_t : t \geq 0\}$, defined on a probability space(Ω, F, P), is said to be a Levy process if it possesses the following properties.

(i) the paths of X are P-almost surely right continuous with left limits.

(ii) $P(X_0 = 0) = 1$.

(iii) For $0 \leq s \leq t, X_t - X_s$ is equal in distribution to X_{t-s}.

(iv) For $0 \leq s \leq t, X_t - X_s$ is independent of $\{X_u : u \leq s\}$.

Conditional expectation [5]

For two random variables x and y having a joint distribution,

$$e(x) = E[E(x|y)]$$

whenever all the expected values exist, where the inside expectation is taken in the conditional space of x given y and the outside expectation is taken in the marginal space of y.

Moments

The r^{th} moment of a variable x about any point $x = A$, usually denoted by ${\mu_r}'$, and defined as,

$${\mu_r}' = \frac{1}{N}\sum_i f_i(x_i - A)^r, \qquad \sum_i f_i = N = \frac{1}{N}\sum_i f_i(d_i)^r,$$

where $d_i = x_i - A$. The r^{th} moment of a variable about the mean $\bar{x}$, usually denoted by μ_r is given by,

$$\mu_r = \frac{1}{N}\sum_i f_i\,(x_i - \bar{x})^r = \frac{1}{N}\sum_i f_i {z_i}^r,$$

where, $z_i = x_i - \bar{x}$.

2. Theorem on generalized Mittag–Leffler and related results

Theorem 1. *If*

$$f(x) = \frac{x^{\alpha\beta-1}}{\delta^\beta} E^{\beta,\gamma}_{\alpha,\alpha\beta}\left(-\frac{x^\alpha}{\delta}\right), \tag{7}$$

then

$$L_f(t) = \frac{1}{(\delta t^\alpha)^\beta}\left(1 + \frac{1}{\delta t^\alpha}\right)^{-\beta,\gamma}, \tag{8}$$

where $|\frac{1}{(\delta t^\alpha)^\beta}\,(1+\delta t^\alpha)^{-\beta,\gamma}| \le k$, *$k$ is an arbitrary constant. α and β are non-negative and nonzero values,* $E^{\beta,\gamma}_{\alpha,\alpha\beta}\left(-\frac{x^\alpha}{\delta}\right)$ *is the generalized Miitag–Leffler function defined by* (4) *and* $(1+\delta t^\alpha)^{-\beta,\gamma}$ *is defined by Prajapati and Shukla* [9].

Proof. Since

$$f(x) = \frac{x^{\alpha\beta-1}}{\delta^\beta} E^{\beta,\gamma}_{\alpha,\alpha\beta}\left(-\frac{x^\alpha}{\delta}\right),$$

by simplification, this can be easily written as

$$= \frac{x^{\alpha\beta-1}}{\delta^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\Gamma(\alpha k+\alpha\beta)} \left(\frac{x^{\alpha}}{\delta}\right)^k$$

$$f(x) = \frac{1}{\delta^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\delta^k \Gamma(\alpha k+\alpha\beta)} x^{\alpha k+\alpha\beta-1}.$$

By taking the Laplace transform, we get

$$\begin{aligned} L_f(t) &= \frac{1}{\delta^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\delta^k \Gamma(\alpha k+\alpha\beta)} \int_0^{\infty} e^{-tx} x^{\alpha k+\alpha\beta-1} dx \\ &= \frac{1}{\delta^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\delta^k \Gamma(\alpha k+\alpha\beta)} \frac{\Gamma(\alpha k+\alpha\beta)}{t^{\alpha k+\alpha\beta}} \\ &= \frac{1}{\delta^{\beta}} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\delta^k} t^{-\alpha k-\alpha\beta} \\ L_f(t) &= \frac{1}{(\delta t^{\alpha})^{\beta}} (1+\delta t^{\alpha})^{-\beta,\gamma}. \end{aligned}$$

■

By writing $f(x)$ in the Mellin–Barnes integral form as

$$\begin{aligned} f(x) &= \frac{x^{\alpha\beta-1}}{\delta^{\beta}\Gamma(\beta)} \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{\Gamma(s)\Gamma(\beta-\gamma s)}{\Gamma(\alpha\beta-\alpha s)} \left(\frac{x^{\alpha}}{\delta}\right)^{-s} ds, \ 0<c<\beta, \\ &= \frac{1}{\Gamma(\beta)} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\right)}{\delta^{\frac{1}{\alpha}}\alpha\Gamma(1-s)} \left(\frac{x}{\delta^{\frac{1}{\alpha}}}\right)^{-s} ds, \ 1-\alpha\beta<c_1<1, \\ &= \frac{1}{\Gamma(\beta)} \frac{1}{2\pi i} \int_{c_1-i\infty}^{c_1+i\infty} \frac{\Gamma\left(1+\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\left(\frac{\gamma}{\alpha}-\frac{\gamma s}{\alpha}\right)\right)}{\delta^{\frac{1}{\alpha}}\Gamma(2-s)} \left(\frac{x}{\delta^{\frac{1}{\alpha}}}\right)^{-s} ds. \end{aligned}$$

If $f(x)$ is a density, then the kernel in the Mellin–Barnes integral as the $(s-1)^{\text{th}}$ moment $E(s^{s-1})$

$$E(s^{s-1}) = \frac{1}{\Gamma(\beta)\delta^{\frac{\gamma}{\alpha}}} \frac{\Gamma\left(1+\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\left(\frac{\gamma}{\alpha}-\frac{\gamma s}{\alpha}\right)\right)}{\delta^{-\frac{\gamma s}{\alpha}}\Gamma(2-s)} \tag{9}$$

$$= \frac{1}{\Gamma(\beta)} \frac{\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)\delta^{\frac{\gamma(s-1)}{\alpha}}}{\alpha\Gamma(1-s)}. \tag{10}$$

If we subsitute $s=1$ in the equation (9) then R.H.S. will be equal to 1. Thus $f(x)$ is a density function, which is known as the Extension of Generalized Mittag–Leffler density. On setting $\gamma = 1$, this reduces to Generalized Mittag–Leffler density as suggested by Mathai [4].

The $\rho-$ moment of the Mittag–Leffler density $f(x)$ (Mathai [4]) is,

$$\begin{aligned} E(x^\rho) &= \frac{1}{\Gamma(\beta)}\frac{1}{\delta^{-\frac{\gamma\rho}{\alpha}}}\frac{\Gamma\left(1-\frac{\rho}{\alpha}\right)\Gamma\left(\beta+\frac{\gamma\rho}{\alpha}\right)}{\Gamma(1-\rho)}; -\alpha < \mathrm{R}(\rho) < \alpha < 1 \\ E(x^{s-1}) &= \frac{1}{\Gamma(\beta)}\frac{1}{\delta^{\frac{\gamma}{\alpha}}}\frac{\Gamma\left(1+\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)}{\Gamma(2-s)\delta^{-\frac{\gamma s}{\alpha}}} \\ &= \frac{1}{\Gamma(\beta)}\frac{1}{\delta^{\frac{\gamma}{\alpha}}}\frac{\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)}{(1-s)\Gamma(1-s)\delta^{-\frac{\gamma s}{\alpha}}} \\ &= \frac{1}{\alpha}\frac{1}{\Gamma(\beta)}\frac{1}{\delta^{\frac{\gamma}{\alpha}}}\frac{\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)}{\Gamma(1-s)\delta^{-\frac{\gamma s}{\alpha}}}, \end{aligned}$$

where, $(1-s)>0, \beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}>0$, which is same as (10).

Case I. ([5]) If $y = ax,\ a > 0$ and if x has a Mittag–Leffler distribution then the density of y can also be represented as a Mittag–Leffler function with the Laplace transform,

$$\begin{aligned} L_x(t) &= (1+t^\alpha)^{-1} \\ \therefore L_y(t) &= (1+(at)^\alpha)^{-1}, \quad a>0,\ |(at)^\alpha|<1. \end{aligned}$$

Case II. Let

$$E(x^h) = \frac{1}{\Gamma(\beta)}\frac{1}{\delta^{-\frac{\gamma h}{\alpha}}}\frac{\Gamma\left(1-\frac{h}{\alpha}\right)\Gamma\left(\beta+\frac{\gamma h}{\alpha}\right)}{\Gamma(1-h)}; \quad -\alpha < R(h) < \alpha < 1$$

This can easily reduce to,

$$E(x^0) = \frac{1}{\Gamma(\beta)}\frac{\Gamma(1)\Gamma(\beta)}{\Gamma(1)} = 1$$

from (5), $f(x)$ is non-negative function for all x,

$$E(x^h) = \int_0^\infty x^h g(x)dx \qquad \text{for } h=0.$$

Hence, $f(x)$ is a density function for a positive random variable x. However this is interesting to see that, it is not possible to show that $\int_0^\infty x^h g(x)dx = 1$ by using the series of Mittag–Leffler function.

3. Structural representation of the Mittag–Leffler variable

Let u be a positive Levy variable with the Laplace transform $e^{-t^\alpha}; 0 < \alpha \le 1$ and let v be a gamma random variable with parameters η and δ or with the Laplace transform $\frac{1}{(\delta t^\alpha)^\beta}\left(1+\frac{1}{\delta t^\alpha}\right)^{-\beta,\gamma}$, where u and v are statistically independently distributed.

Theorem 2. *If u and v are defined as above, then*

$$w \sim uv^{\frac{\gamma}{\alpha}}, \tag{11}$$

where w is a generalized Mittag–Leffler variable with the Laplace transform $\frac{1}{(\delta t^\alpha)^\beta}\left(1+\frac{1}{\delta t^\alpha}\right)^{-\beta,\gamma}$, provided $\frac{1}{(\delta t^\alpha)^\beta}\left(1+\frac{1}{\delta t^\alpha}\right)^{-\beta,\gamma}$ is convergent and β is non-negative value, and $\sim$ means 'distributed as' or both sides have the same distribution.

Proof. By denoting the Laplace transform of the density of w by $L_w(t)$ and treating it as an expected value,

$$L_w(t) = E\left(e^{-tv^{\frac{\gamma}{\alpha}}u}\right) = E\left(E\left(e^{-\left(tv^{\frac{\gamma}{\alpha}}u\right)}|v\right)\right)$$

By simplification, we obtain

$$= E\left[e^{-\left(tv^{\frac{\gamma}{\alpha}}\right)^\alpha}\right] = E\left[-t^\alpha v^\gamma\right] = \frac{1}{(\delta t^\alpha)^\beta}\left(1+\frac{1}{\delta t^\alpha}\right)^{-\beta,\gamma}$$

From the structural representation (11) and the Mellin transform (5), we have,

$$E(w^{s-1}) = E(u^{s-1})E\left(v^{\frac{\gamma}{\alpha}}\right)^{s-1} \tag{12}$$

Since, u and v are statistically independent, the left hand side of (12) is written as,

$$E(w^{s-1}) = \frac{\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)\delta^{\frac{\gamma(s-1)}{\alpha}}}{\alpha\Gamma(\beta)\Gamma(1-s)} \tag{13}$$

One can easily compute $E\left(v^{\frac{\gamma}{\alpha}}\right)^{s-1}$ by using Gamma density, this gives

$$\begin{aligned}
E(v^{\frac{\gamma}{\alpha}})^{s-1} &= \frac{1}{\Gamma(\beta)}\,\frac{1}{\delta^\beta}\int_0^\infty \left(v^{\frac{\gamma}{\alpha}}\right)^{s-1} v^{\beta-1}e^{-\frac{v}{\delta}}dv \\
&= \frac{1}{\Gamma(\beta)}\frac{1}{\delta^\beta}\int_0^\infty v^{\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)-1}e^{-\frac{v}{\delta}}dv \\
&= \frac{1}{\Gamma(\beta)}\,\frac{1}{\delta^\beta}\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)\delta^{\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}} \\
&= \frac{1}{\Gamma(\beta)}\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)\delta^{\frac{\gamma(s-1)}{\alpha}}
\end{aligned}$$

Finally, we arrive at

$$E(v^{\frac{\gamma}{\alpha}})^{s-1} = \frac{1}{\Gamma(\beta)}\Gamma\left(\beta-\frac{\gamma}{\alpha}+\frac{\gamma s}{\alpha}\right)\delta^{\frac{\gamma(s-1)}{\alpha}} \tag{14}$$

For $R(s) > 1-\alpha\beta,\ 0<\alpha\le 1,\ \beta>0$ (13) and (14), we have $(s-1)^{th}$ moment of Levy variable.

$$E[u^{s-1}] = \frac{\Gamma\left(\frac{1}{\alpha}-\frac{s}{\alpha}\right)}{\alpha\Gamma(1-s)} = \frac{\Gamma\left(1+\frac{1}{\alpha}-\frac{s}{\alpha}\right)}{\Gamma(2-s)};\ 0<\alpha\le 1.$$

This leads us to the proof of the theorem. ∎

4. Extension of Mittag–Leffler stochastic process

The stochastic process $\{x(t), t > 0\}$ having stationary independent increment with $x(0) = 0$ & $x(1)$ having the Laplace transform,

$$L_{x(1)}(\xi) = (1 + \xi^{\alpha})^{-1}, 0 < \alpha \leq 1, \xi > 0$$

which is the Laplace transform of a Mittag–Leffler random variable, is called the Mittag–Leffler stochastic process. Then the Laplace transform of $x(t)$, denoted by $L_{x(t)}(\xi)$, is given by,

$$L_{x(1)}(\xi) = \frac{1}{(\xi^{\alpha})^{\beta}} \left(1 + \frac{1}{\xi^{\alpha}}\right)^{-\beta,\gamma} \tag{15}$$

The density corresponding to the Laplace transform of (15) or the density of $x(t)$ is then available as the following:

$$\begin{aligned} f_{x(t)}(x) &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\Gamma(\alpha k + \alpha\beta)} x^{\alpha k + \alpha\beta - 1} \\ &= x^{\alpha\beta - 1} \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{(\beta)_{\gamma k}}{\Gamma(\alpha k + \alpha\beta)} x^{\alpha k} \\ &= x^{\alpha\beta - 1} E_{\alpha,\alpha\beta}^{\beta,\gamma}(-x^{\alpha}),\ 0 < \alpha \leq 1,\ x \geq 0, t > 0. \end{aligned} \tag{16}$$

The distribution function of $x(t)$ is given by,

$$\begin{aligned} F_{x(t)}(x) &= \int_0^x f_{x(t)}(y) dy \\ &= \sum_{k=0}^{\infty} (-1)^k \frac{(\beta)_{\gamma k}}{k!} \frac{x^{\alpha k + \alpha\beta}}{(\alpha k + \alpha\beta)\Gamma(\alpha k + \alpha\beta)} \\ &= \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(\beta + \gamma k)}{\Gamma(\beta)} \frac{x^{\alpha k + \alpha\beta}}{\Gamma(\alpha k + \alpha\beta + 1)};\ 0 < \alpha \leq 1. \end{aligned} \tag{17}$$

Special case of distribution function

On replacing $\alpha\beta$ by αt, β by t & γ by 1 in equation (16), it reduces to

$$F_x(t) = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} \frac{\Gamma(t + k)}{\Gamma(t)} \frac{x^{\alpha k + \alpha t}}{\Gamma(\alpha k + \alpha t + 1)}; 0 < \alpha \leq 1, t > 0 \tag{18}$$

This result was given by Pillai [6].

References

[1] Krasnov, M., Kiselev, A., Makarenko, G., *Problems and Exercises in Integral Equations*, MIR Publishers, Moscow, 1971.

[2] Kyprianou, A.E., *Lecture notes on Levy processes and continuous-state branching processes.* Part I.

[3] Levy, P., *Calcul des probabilités*, Gauthier-Villars, Paris, 1925.

[4] Mathai, A.M., Haubold, H.J., *Special Functions for Applied Scientists*, Springer, New York, 2010.

[5] Mathai, A.M., Saxena, R.K., Haubold, H.J., *The H-Function* (*Theory and Applications*), Springer, New York, 2010.

[6] Pillai, R.N., *On Mittag-Leffler and related distributions*, Ann. Inst. Statist. Math., 42 (1) (1990), 157-161.

[7] Pillai, R.N., Sandhya, E., *Distributions with complete monotone derivative and geometric infinite divisibility*, Adv. Appl. Prob., 22 (1990), 751-754.

[8] Prabhakar, T.R., *A singular integral equation with a generalized Mittag-Leffler function in the Kernel*, Yokohama Math. J., 19 (1971), 7-15.

[9] Prajapati, J.C., Shukla, A.K., *Decomposition of generalized Mittag-Leffler function and its properties*, Advances in Pure Mathematics, 2 (2012), 8-14.

[10] Shukla, A.K., Prajapati, J.C., *On a generalization of Mittag-Leffler function and its properties*, J. Math. Anal. Appl., 336 (2007), 797-811.

[11] Wiman, A., *Uber de fundamental satz in der theorie der funktionen*, Acta Math., 29 (1905), 191-201.

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