ON RADIAL DISTRIBUTION OF JULIA SETS OF SOLUTIONS TO COMPLEX DIFFERENTIAL EQUATIONS WITH MEROMORPHIC COEFFICIENTS

Guowei Zhang

School of Mathematics and Statistics
Anyang Normal University
Anyang, Henan, 455000
China
e-mail: herrzgw@foxmail.com

Abstract. In this paper, we mainly investigate the radial distribution of Julia set of solutions to some second order complex linear differential equations with meromorphic coefficients, and find out the lower bound of it. The radial distributions of Julia sets of derivatives of these solutions are also obtained.

Keywords: radial distribution, Julia set, complex differential equation, derivative, deficient value.

2000 MR Subject Classification: 30D05, 37F10, 37F50.

1. Introduction and main results

In this article, we assume the reader is familiar with standard notations and basic results of Nevanlinna's value distribution theory; see [6], [7], [10], [18], [19]. Let f be a meromorphic function in the whole complex plane. We use $\sigma(f)$ and $\mu(f)$ to denote the order and lower order of f respectively; see [19, p.10] for the definitions. Some basic knowledge of complex dynamics of meromorphic functions is also needed; see [3], [21]. We define $f^n, n \in \mathbb{N}$ denote the nth iterate of f. The Fatou set $F(f)$ of transcendental meromorphic function f is the subset of the plane $\mathbb{C}$ where the iterates f^n of f form a normal family. The complement of $F(f)$ in $\mathbb{C}$ is called the Julia set $J(f)$ of f. It is well known that $F(f)$ is open and completely invariant under f, $J(f)$ is closed and non-empty.

We denote $\Omega(\alpha,\beta) = \{z \in \mathbb{C} | \arg z \in (\alpha,\beta)\}$, where $0 < \alpha < \beta < 2\pi$. Given $\theta \in [0, 2\pi)$, if $\Omega(\theta-\varepsilon, \theta+\varepsilon) \cap J(f)$ is unbounded for any $\varepsilon > 0$, then we call the ray $\arg z = \theta$ the radial distribution of $J(f)$. Define $\Delta(f) = \{\theta \in [0, 2\pi) | \arg z = \theta$ is the radial distribution of $J(f)\}$. Obviously, $\Delta(f)$ is closed and so measurable. We use the $mes\Delta(f)$ to denote the linear measure of $\Delta(f)$. Many important results of radial distribution of transcendental meromorphic functions have been obtained, for example [1], [12], [13], [14], [15], [16], [22]. Recently, Huang and Wang [8], [9] considered the radial distribution of Julia sets of entire solutions of some special linear complex differential equations and obtained the lower bound of them.

In the present paper, we devote to investigate the radial distributions of Julia sets of solutions of some kind of second order complex differential equations which was studied in [17]. The coefficients of this equation are meromorphic and have some special properties. In order to introduce these properties, we give the definition of the so called $\mathscr{E}\mathscr{F}$ Class firstly. In what follows we use the notations $\Omega(\alpha,\beta,r)=\{z : \arg z \in \Omega(\alpha,\beta), |z| < r\}$, $\Omega(r,\alpha,\beta)=\{z : \arg z \in \Omega(\alpha,\beta), |z| \geq r\}$ and denote by $\overline{\Omega}(r,\alpha,\beta)$ the closure of $\Omega(r,\alpha,\beta)$.

Definition. Let f be a meromorphic function in the finite complex plane $\mathbb{C}$ of order $0 < \sigma(f) < \infty$. A ray $\arg z = \theta$ starting from the origin is called a zero-pole accumulation ray of $f(z)$, if for any given real number $\varepsilon > 0$, the following equality holds

$$\limsup_{r\to\infty} \frac{\log n\{\overline{\Omega}(\theta-\varepsilon,\theta+\varepsilon), f=0\} + \log n\{\overline{\Omega}(\theta-\varepsilon,\theta+\varepsilon), f=\infty\}}{\log r} = \sigma(f). \tag{1.1}$$

Edrei and Fuchs [4], [5] proved that the number of deficient values of a meromorphic function is finite when its zeros and poles are located near some curves. The weaker form of their result can be stated as follows.

Theorem A. [20, Theorem 3.10] *Let f be a meromorphic function in the complex plane $\mathbb{C}$ of order $0 < \sigma(f) < \infty$. Assume that $f(z)$ has q zero-poles accumulation rays and p deficient values other than 0 and ∞, then $p \leq q$.*

We shall say that a meromorphic function $f \in \mathscr{E}\mathscr{F}$, called it Edrei-Fuchs Class if $f(z)$ satisfies the conditions of Theorem A with $p = q \geq 1$, that is, $f(z)$ is of finite and positive order, and has p zero-pole accumulation rays and p non-zero finite deficient values. In [17], assume that one of coefficients of the second order linear complex differential equations belongs to Edrei-Fuchs Class, the authors proved every nontrivial solutions of this equation is of infinite order. Actually, they obtained

Theorem B. [17] *Let $A(z) \in \mathscr{E}\mathscr{F}$ be a meromorphic function and let $B(z)$ be a transcendental meromorphic function having a deficient value ∞. Then every nontrivial solution f of equation*

$$f'' + A(z)f' + B(z)f = 0 \tag{1.2}$$

is of infinite order.

Our main aim of this article is to estimate the lower bound of radial distribution of Julia set to solutions of equation (1.2).

Theorem 1.1 *Let $A(z) \in \mathscr{E}\mathscr{F}$ be a meromorphic function and let $B(z)$ be a transcendental meromorphic function with finite order having a deficient value ∞, let f be a nontrivial solution of equation* (1.2) *and $J(f)$ has an unbounded component, then* $\operatorname{mes}\Delta(f) \geq M_2 > 0$, *where M_2 is a constant defined in Remark* 1*, Section* 2.

Furthermore, we study the radial distribution of Julia set of the derivatives of the nontrivial solutions of equations (1.2). Indeed, we obtain the following results.

Theorem 1.2 *Let $A(z) \in \mathscr{E}\mathscr{F}$ be a meromorphic function and let $B(z)$ be a transcendental meromorphic function with finite order having a deficient value ∞, let f be a nontrivial solution of equation* (1.2)*. Moreover, if $J(f)$ and $J(f^{(k)})$ both have an unbounded component, then* $\operatorname{mes}(\Delta(f) \cap \Delta(f^{(k)})) \geq M_2$, *where k is a positive integer.*

By Theorem 1.2, we immediately have

Corollary 1.1 *Under the hypothesis of Theorem* 1.2 *we have $mes\Delta(f^{(k)}) \geq M_2$, where k is a positive integer.*

2. Preliminary lemmas

In this article, for a measurable set $E \subset (0, \infty)$, we define the logarithmic measure of E by $m_l(E) = \int_E \frac{dt}{t}$. We also define the upper and lower logarithmic densities of $E \subset [1, \infty)$ respectively, by

$$\overline{\log \operatorname{dens}} E = \limsup_{r\to\infty} \frac{m_l(E \cap [0, r])}{\log r},$$
$$\underline{\log \operatorname{dens}} E = \liminf_{r\to\infty} \frac{m_l(E \cap [0, r])}{\log r}. \tag{2.1}$$

In the following, we recall the Nevanlinna characteristic in an angle; see [6]. Let $g(z)$ be meromorphic on the angle $\overline{\Omega}(\alpha, \beta)$, where $\beta - \alpha \in (0, 2\pi]$. Following [6], we define

$$A_{\alpha,\beta}(r, g) = \frac{w}{\pi} \int_1^r \left(\frac{1}{t^w} - \frac{t^w}{r^{2w}} \right) \{\log^+ |g(te^{i\alpha})| + \log^+ |g(te^{i\beta})|\} \frac{dt}{t};$$
$$B_{\alpha,\beta}(r, g) = \frac{2w}{\pi r^w} \int_\alpha^\beta \log^+ |g(re^{i\theta})| \sin w(\theta - \alpha) d\theta;$$
$$C_{\alpha,\beta}(r, g) = 2 \sum_{1<|b_n|<r} \left(\frac{1}{|b_n|^w} - \frac{|b_n|^w}{r^{2w}} \right) \sin w(\beta_n - \alpha),$$

where $w = \pi/(\beta - \alpha)$, and $b_n = |b_n|e^{i\beta_n}$ are poles of $g(z)$ in $\overline{\Omega}(\alpha, \beta)$ appearing according to their multiplicities. The Nevanlinna angular characteristic is defined as

$$S_{\alpha,\beta}(r, g) = A_{\alpha,\beta}(r, g) + B_{\alpha,\beta}(r, g) + C_{\alpha,\beta}(r, g).$$

In particular, we denote the order of $S_{\alpha,\beta}(r,g)$ by

$$\sigma_{\alpha,\beta}(g) = \limsup_{r\to\infty} \frac{\log S_{\alpha,\beta}(r,g)}{\log r}.$$

We call W is a hyperbolic domain if $\overline{\mathbb{C}}\backslash W$ contains at least three points, where $\overline{\mathbb{C}}$ is the extended complex plane. For an $a \in \mathbb{C}\backslash W$, define $C_W(a) = \inf\{\lambda_W(z)|z-a| : \forall z \in W\}$, where $\lambda_W(z)$ is the hyperbolic density on W. It is well known that if every component of W is simply connected, then $C_W(a) \geq 1/2$; see [22]. For a finite number $a \in J(f)$, if there is a component U in $F(f)$ such that $C_U(a) > 0$, then we call $C_{F(f)}(a) > 0$, where $f(z)$ is a transcendental meromorphic function in $\mathbb{C}$.

Lemma 2.1. ([22, Lemma 2.2]) *Let $f(z)$ be analytic in $\Omega(r_0,\theta_1,\theta_2)$, U be a hyperbolic domain, and $f : \Omega(r_0,\theta_1,\theta_2) \to U$. If there exists a point $a \in \partial U\backslash\{\infty\}$ such that $C_U(a) > 0$, then there exists a constant $d > 0$ such that, for sufficiently small $\varepsilon > 0$, we have*

$$|f(z)| = O(|z|^d), \quad z \to \infty, \ z \in \Omega(r_0,\theta_1+\varepsilon,\theta_2-\varepsilon).$$

The next lemma shows some estimates for the logarithmic derivative of functions being analytic in an angle. Before this, we recall the definition of an R-set; for reference, see [10]. Set $B(z_n,r_n) = \{z : |z-z_n| < r_n\}$. If $\sum_{n=1}^{\infty} r_n < \infty$ and $z_n \to \infty$, then $\bigcup_{n=1}^{\infty} B(z_n,r_n)$ is called an R-set. Clearly, the set $\left\{|z| : z \in \bigcup_{n=1}^{\infty} B(z_n,r_n)\right\}$ is of finite linear measure.

Lemma 2.2. ([9, Lemma 2.2]) *Let $z = re^{i\psi}, r_0+1 < r$ and $\alpha \leq \psi \leq \beta$, where $0 < \beta - \alpha \leq 2\pi$. Suppose that n (≥ 2) is an integer, and that $g(z)$ is analytic in $\Omega(r_0,\alpha,\beta)$ with $\sigma_{\alpha,\beta}(g) < \infty$. Choose $\alpha < \alpha_1 < \beta_1 < \beta$. Then, for every $\varepsilon_j \in (0,(\beta_j-\alpha_j)/2)$ $(j = 1,2,\ldots,n-1)$ outside a set of linear measure zero with*

$$\alpha_j = \alpha + \sum_{s=1}^{j-1} \varepsilon_s, \quad \beta_j = \beta - \sum_{s=1}^{j-1} \varepsilon_s, \quad j = 2,3,\ldots,n-1.$$

there exists $K > 0$ and $M > 0$ only depending on g, $\varepsilon_1,\ldots,\varepsilon_{n-1}$ and $\Omega(\alpha_{n-1},\beta_{n-1})$, and not depending on z, such that

$$\left|\frac{g'(z)}{g(z)}\right| \leq Kr^M(\sin k(\psi-\alpha))^{-2}$$

and

$$\left|\frac{g^{(n)}(z)}{g(z)}\right| \leq Kr^M\left(\sin k(\psi-\alpha)\prod_{j=1}^{n-1}\sin k_{\varepsilon_j}(\psi-\alpha_j)\right)^{-2}$$

for all $z \in \Omega(\alpha_{n-1},\beta_{n-1})$ outside an R-set H, where $k = \pi/(\beta-\alpha)$ and $k_{\varepsilon_j} = \pi/(\beta_j-\alpha_j)$ $(j = 1,2,\ldots,n-1)$.

Lemma 2.3. ([11]) *Let* $T(r) > 1$ *be a nonconstant increasing function in* $(0, +\infty)$ *of finite order* σ, *i.e.,*

$$\limsup_{r\to\infty} \frac{\log T(r)}{\log r} = \sigma < \infty. \tag{2.2}$$

For any η *such that* $0 \le \eta < \sigma$, *if* $\sigma > 0$, *and* $\eta = 0$, *if* $\sigma = 0$, *define*

$$E(\eta) = \{r \ge 1 : r^{\eta} < T(r)\}. \tag{2.3}$$

Then $\overline{\log\mathrm{dens}}\, E(\eta) > 0$.

Lemma 2.4. ([17]) *Let* $A(z)$ *be a meromorphic function of order* $0 < \sigma(A) < +\infty$ *having* p *finite deficient values* $a_1, a_2, \ldots, a_p, (p \ge 1)$ *and let* $B(z)$ *be a meromorphic function with finite order having a deficient value* ∞. *Suppose that* $\beta > 1$ *and* $0 < \eta < \sigma(A)$ *are two constants. Then there exists a sequence* $\{t_n\}$ *such that*

$$\lim_{n\to\infty} \frac{t_n^{\eta}}{T(t_n, A)} = 0. \tag{2.4}$$

Moreover, for every sufficiently large n, *there is a set* $F_n \subset [t_n, (\beta+1)t_n]$ *with* $mes(F_n) \le (\beta-1)t_n/4$ *such that, for all* $R \in [t_n, \beta t_n]\backslash F_n$, *the arguments* θ *sets* $E_{\nu}(R)$, $(\nu = 1, 2, \ldots, p)$ *and* $E_{\infty}(R)$ *satisfying the following inequalities*

$$\mathrm{mes}(E_{\nu}(R)) := \mathrm{mes}\left(\left\{\theta \in [0, 2\pi) | \log \frac{1}{|A(Re^{i\theta}) - a_{\nu}|} \ge \frac{\delta_0}{4} T(R, A)\right\}\right) \ge M_1 > 0; \tag{2.5}$$

and

$$\mathrm{mes}(E_{\infty}(R)) := \mathrm{mes}\left(\left\{\theta \in [0, 2\pi) | \log |B(Re^{i\theta})| \ge \frac{\delta_1}{4} T(R, B)\right\}\right) \ge M_2 > 0, \tag{2.6}$$

where M_1, M_2 *are two positive constants depending only on* $\sigma(A), \sigma(B)$, $\delta_0 = \min\{\delta(a_{\nu}, A),\ \nu = 1, 2, \ldots, p\}$, $\delta_1 = \delta(\infty, B), \beta$ *and* η.

Remark 1. From the proof of Lemma 2.4 in [17], we know that

$$M_1 := \frac{\delta_0}{4} \left\{ \frac{(2\beta+4)^{H_1}}{2\pi} \left[(2\beta+1) + \frac{2\log \frac{16e(2\beta+1)}{\beta-1}}{\log \frac{\beta+2}{\beta+1}} \right] \right\}^{-1}; \tag{2.7}$$

$$M_2 := \frac{\delta_1}{4} \left\{ \frac{(2\beta+4)^{H_2}}{2\pi} \left[(2\beta+1) + \frac{2\log \frac{16e(2\beta+1)}{\beta-1}}{\log \frac{\beta+2}{\beta+1}} \right] \right\}^{-1} \tag{2.8}$$

where $H_1 = \frac{\sigma(A)}{h_1} + 1$ and h_1 is defined by $h_1 := \overline{\log\mathrm{dens}}\, E(\eta_1) := \overline{\log\mathrm{dens}}\, \{r \ge 1 : r^{\eta_1} < T(r, A)\} > 0$, $\eta < \eta_1 < \sigma(A)$; $H_2 = \frac{\sigma(B)}{h_2} + 1$ and h_2 is defined by $h_2 := \overline{\log\mathrm{dens}}\, E(\eta_2) := \overline{\log\mathrm{dens}}\, \{r \ge 1 : r^{\eta_2} < T(r, B)\} > 0$, $\eta < \eta_2 < \sigma(B)$ according to Lemma 2.3.

Lemma 2.5. ([17, Lemma 2.6]) *Let $A(z) \in \mathscr{E}\mathscr{F}$, then for any given sufficiently small $\varepsilon > 0$ and $\beta > 1$, when n is sufficiently large, there exists a sequence of angular regions $\overline{\Omega}(\theta_{k_\nu} + 2\varepsilon, \theta_{k_\nu+1} - 2\varepsilon, t_n, \beta t_n) := \{z : t_n \le |z| \le \beta t_n, \theta_{k_\nu} + 2\varepsilon \le \arg z \le \theta_{k_\nu+1} - 2\varepsilon\}$, $n = 1, 2, \ldots, \nu = 1, 2, \ldots, p$ such that for every $1 \le \nu \le p$, the following inequality*

$$\log \frac{1}{|A(z) - a_\nu|} > \log \frac{4}{d} \tag{2.9}$$

holds for $z \in \overline{\Omega}(\theta_{k_\nu} + 2\varepsilon, \theta_{k_\nu+1} - 2\varepsilon, t_n, \beta t_n) \backslash \bigcup_{\nu=1}^{p} (\gamma_\nu)_n$, where $\bigcup_{\nu=1}^{p} (\gamma_\nu)_n$ are some disks with the sum of total radius not exceeding $p\varepsilon t_n/8$ and $t_n, \beta t_n$ are defined by Lemma 2.4 and $d = \min_{1 \le \nu \ne \nu' \le p} \{|a_\nu - a_{\nu'}|\}$ and a_ν are deficient values of $A(z)$.

3. Proof of theorems

Proof of Theorem 1.1. By Theorem B, we have already known that every nontrivial meromorphic solution f of (1.2) is of infinite order. We shall obtain the assertion by reduction to contradiction. At first, we suppose that $\operatorname{mes} \Delta(f) < M_2$, so $\zeta = M_2 - \operatorname{mes} \Delta(f) > 0$. Since $\Delta(f)$ is closed, obviously $S = [0, 2\pi) \backslash \Delta(f)$ is open, so it consists of at most countably many open intervals. We can choose finitely many open intervals $I_i = (\alpha_i, \beta_i)$, $i = 1, 2, \ldots, m$ satisfying $[\alpha_i, \beta_i] \subset S$ and $\operatorname{mes}\left(S \backslash \bigcup_{i=1}^{m} I_i\right) < \zeta/4$. For the angular domain $\Omega(\alpha_i, \beta_i)$, it is easy to see that $(\alpha_i, \beta_i) \cap \Delta(f) = \emptyset$ and $\Omega(r, \alpha_i, \beta_i) \cap J(f) = \emptyset$ for sufficiently large r. This implies that, for each $i = 1, 2, \ldots, m$, there exist the corresponding r_i and unbounded Fatou component U_i of $F(f)$ such that $\Omega(r_i, \alpha_i, \beta_i) \subset U_i$; see [2]. Because the poles of f are in the set $J(f)$; see [3, Section 2.1], then f does not have poles in $\Omega(r, \alpha_i, \beta_i)$. By the assumption $J(f)$ has an unbounded component, say Γ, then the mapping $f : \Omega(r_i, \alpha_i, \beta_i) \to \mathbb{C} \backslash \Gamma$ is analytic. Since we have chosen Γ such that $\mathbb{C} \backslash \Gamma$ is simply connected, for any $a \in \Gamma \backslash \{\infty\}$, we have $C_{\mathbb{C} \backslash \Gamma}(a) \ge 1/2$. Thus, applying Lemma 2.1 to f in every $\Omega(r_i, \alpha_i, \beta_i)$, there exist a positive constant d_1 such that, for $z \in \bigcup_{i=1}^{m} \Omega(r_i, \alpha_i + \varepsilon, \beta_i - \varepsilon)$,

$$|f(z)| = O(|z|^{d_1}), \qquad as |z| \to \infty, \tag{3.1}$$

where $0 < \varepsilon < \min\{\zeta/(16m), (\beta_i - \alpha_i)/8\}$, $i = 1, 2, \ldots, m$; see [16], [22]. Thus, recalling the definition of $S_{\alpha,\beta}(r, f)$, we immediately have that

$$S_{\alpha_i+\varepsilon, \beta_i-\varepsilon}(r, f) = O(1), \qquad (i = 1, 2, \ldots, m) \tag{3.2}$$

So $\sigma_{\alpha_i+\varepsilon, \beta_i-\varepsilon}(r, f)$ is finite. Therefore, by Lemma 2.2, there exist two constants $M > 0$ and $K > 0$ such that

$$\left| \frac{f^{(s)}(z)}{f(z)} \right| \le K r^M, \qquad (s = 1, 2) \tag{3.3}$$

for all $z \in \bigcup_{i=1}^{m} \Omega(r_i, \alpha_i + 2\varepsilon, \beta_i - 2\varepsilon)$, outside a R-set H.

By Lemma 2.4, there exists a sequence $\{t_n\}$ satisfying (2.4), such that for all $R_n \in [t_n, \beta t_n]\backslash F_n$, where $F_n \subset [t_n, (\beta+1)t_n]$ is a set with $\text{mes}(F_n) \leq (\beta-1)t_n/4$, and for sufficiently large n, we have

$$\text{mes}(E_\infty(R_n)) > M_2 - \zeta/4. \tag{3.4}$$

Therefore, we have

$$\begin{aligned}\text{mes}(E_\infty(R_n) \cap S) &= \text{mes}(E_\infty(R_n)\backslash(\Delta(f) \cap E_\infty(R_n))) \\ &\geq \text{mes}(E_\infty(R_n)) - \text{mes}(\Delta(f)) > \frac{3\zeta}{4} > 0.\end{aligned} \tag{3.5}$$

Then, for each n, we have

$$\begin{aligned}\text{mes}\left(\left(\bigcup_{i=1}^{m} I_i\right) \cap E_\infty(R_n)\right) &= \text{mes}(S \cap E_\infty(R_n)) \\ -\text{mes}\left(\left(S\backslash\bigcup_{i=1}^{m} I_i\right) \cap E_\infty(R_n)\right) &> \frac{3\zeta}{4} - \frac{\zeta}{4} = \frac{\zeta}{2}.\end{aligned} \tag{3.6}$$

Thus, there exists an open interval $I_{i_0} = (\alpha, \beta) \subset \bigcup_{i=1}^{m} I_i \subset S$ such that, for infinitely many n,

$$\text{mes}(E_\infty(R_n) \cap (\alpha, \beta)) > \frac{\zeta}{2m} > 0. \tag{3.7}$$

Without loss of generality, we can assume that (3.7) holds for all n.

Suppose that $A(z)$ has p non-zero finite deficient values, $a_1, a_2, \ldots, a_p$ with deficiency $\delta(a_\nu, A) > 0, 1 \leq \nu \leq p$ and has p zero-pole accumulation rays, $0 \leq \theta_1 < \theta_2 < \ldots < \theta_p < \theta_1 + 2\pi$. From equation (1.2), we have the following inequality

$$|B(z)| \leq \left|\frac{f''(z)}{f(z)}\right| + \left|\frac{f'(z)}{f(z)}\right|(|A(z) - a_\nu| + |a_\nu|). \tag{3.8}$$

Let $\omega = \min_{1\leq k\leq \nu}(\theta_{k+1} - \theta_k)$ and $0 < \varepsilon_0 < \min\{\frac{\zeta}{8pm}, \frac{\omega}{2}, \frac{\beta-1}{2p}\}$. According to Lemma 2.5, we choose $R_n^* \in [t_n, \beta t_n]\backslash F_n$ such that for every $n \geq n_0$

$$\{z : |z| = R_n^*\} \cap \left(\bigcup_{\nu=1}^{p}(\gamma_\nu)_n\right) \cap H = \emptyset \tag{3.9}$$

where $\bigcup_{\nu=1}^{p}(\gamma_\nu)_n$ are some disks with the sum of total radius not exceeding $p\varepsilon_0 t_n/8 < \frac{(\beta-1)t_n}{16}$ and H is the R-set mentioned above. Hence, by Lemma 2.5 the following inequalities

$$\log\frac{1}{|A(R_n^* e^{i\varphi}) - a_\nu|} > \log\frac{4}{d}, \qquad \nu = 1, 2, \ldots, p \tag{3.10}$$

holds for sufficiently large n and $R_n^* e^{i\varphi} \in \bigcup_{\nu=1}^{p} \overline{\Omega}(\theta_{k_\nu} + 2\varepsilon_0, \theta_{k_\nu+1} - 2\varepsilon_0, t_n, \beta t_n)$.

On the other hand, from Lemma 2.4, for the sequence R_n^*, the following inequality

$$\text{(3.11)} \quad \operatorname{mes}(E_\infty(R_n^*)) := \operatorname{mes}\left(\left\{\theta \in [0, 2\pi) | \log |B(R_n^* e^{i\theta})| \geq \frac{\delta_1}{4} T(R_n^*, B)\right\}\right) \geq M_2 > 0$$

also holds for sufficiently large n.

Combining with (3.7), there exists a set $E_\infty(R_n^*) \cap (\alpha, \beta) \cap [\theta_{k_{\nu_0}} + 2\varepsilon_0, \theta_{k_{\nu_0}+1} - 2\varepsilon_0]$, $(1 \leq k_{\nu_0} \leq p)$ such that

$$\text{(3.12)} \qquad \operatorname{mes}(E_\infty(R_n^*) \cap (\alpha, \beta) \cap [\theta_{k_{\nu_0}} + 2\varepsilon_0, \theta_{k_{\nu_0}+1} - 2\varepsilon_0]) \geq \frac{\zeta}{4mp}.$$

Thus, for sufficiently large n, we choose $\varphi_n \in E_\infty(R_n^*) \cap (\alpha, \beta) \cap [\theta_{k_{\nu_0}} + 2\varepsilon_0, \theta_{k_{\nu_0}+1} - 2\varepsilon_0]$ such that (3.3), (3.10) and (3.11) hold. Combining (3.3), (3.8) with (3.10), we get

$$\text{(3.13)} \qquad |B(R_n^* e^{i\varphi_n})| \leq K(R_n^*)^M \left(1 + \frac{d}{4} + |a_\nu|\right).$$

Then, by (3.11) we obtain

$$\text{(3.14)} \qquad \frac{\delta_1}{4} T(R_n^*, B) \leq \log K + M \log R_n^* + \log \left(1 + \frac{d}{4} + |a_\nu|\right).$$

This implies that $B(z)$ is a rational function, which is a contradiction. Thus, we complete the proof. ■

Proof of Theorem 1.2. We know that every nontrivial solution f of equation (1.2) is a transcendental meromorphic function with infinite order. We also obtain the assertion by reduction to contradiction. Assume that

$$\text{(3.15)} \qquad \operatorname{mes}(\Delta(f) \cap \Delta(f^{(k)})) < M_2$$

and so

$$\text{(3.16)} \qquad \xi := M_2 - \operatorname{mes}(\Delta(f) \cap \Delta(f^{(k)})) > 0.$$

We shall show that there must exist an open interval

$$\text{(3.17)} \qquad I = (\alpha, \beta) \subset \Delta(f^{(k)})^c, \quad 0 < \beta - \alpha < M_2$$

such that

$$\text{(3.18)} \qquad \lim_{n \to \infty} \operatorname{mes}(\Delta(f) \cap E_\infty(R_n) \cap I) > 0,$$

where $\Delta(f^{(k)})^c := [0, 2\pi) \backslash \Delta(f^{(k)})$ and $E_\infty(R_n)$ is as defined in (2.6). In order to achieve this goal, we shall prove the following firstly.

$$\text{(3.19)} \qquad \lim_{n \to \infty} \operatorname{mes}(E_\infty(R_n) \backslash \Delta(f)) = 0.$$

Otherwise, suppose that there is a subseries $\{R_{n_k}\}$ such that

$$\lim_{k\to\infty} \mathrm{mes}(E_\infty(R_{n_k})\backslash\Delta(f)) > 0, \tag{3.20}$$

then there exists $\theta_0 \in \Delta(f)^c$ and $\eta > 0$ satisfying

$$\lim_{k\to\infty} \mathrm{mes}((\theta_0-\eta,\theta_0+\eta)\cap(E_\infty(R_{n_k})\backslash\Delta(f))) > 0. \tag{3.21}$$

Since $\arg z = \theta_0$ is not a radial distribution of $J(f)$, there exists $r_0 > 0$ such that

$$\Omega(r_0,\theta_0-\eta,\theta_0+\eta)\cap J(f) = \emptyset. \tag{3.22}$$

This implies that there exists an unbounded component U of Fatou set $F(f)$, such that $\Omega(r_0,\theta_0-\eta,\theta_0+\eta)\subset U$ and $f(\Omega(r_0,\theta_0-\eta,\theta_0+\eta))\subset F(f)$ is analytic. Note that $J(f)$ has an unbounded component, applying Lemma 2.1 to f in $\Omega(r_0,\theta_0-\eta,\theta_0+\eta)$, for any $\zeta > 0, \zeta < \eta$, we have

$$|f(z)| = O(|z|^{d_1}),\quad z\in\Omega(r_0,\theta_0-\eta+\zeta,\theta_0+\eta-\zeta),\ |z|\to\infty, \tag{3.23}$$

where d_1 is a positive constant. Recalling the definition of $S_{\alpha,\beta}(r,f)$, we immediately get that

$$S_{\theta_0-\eta+\zeta,\theta_0+\eta-\zeta}(r,f) = O(1). \tag{3.24}$$

Thus, $\sigma_{\theta_0-\eta+\zeta,\theta_0+\eta-\zeta}(r,f)$ is finite. Therefore, by Lemma 2.2, there exists constants $M > 0$ and $K > 0$ such that (3.3) holds for all $z\in\Omega(r_0,\theta_0-\eta+\zeta,\theta_0+\eta-\zeta)$, outside a R-set H.

Since ζ can be chosen sufficiently small, from (3.21) we have

$$\lim_{k\to\infty} \mathrm{mes}((\theta_0-\eta+\zeta,\theta_0+\eta-\zeta)\cap E_\infty(R_{n_k})) > 0. \tag{3.25}$$

Thus, similarly as (3.9) we can find an infinite series $\{R^*_{n_k}e^{i\theta_{n_k}}\}$ such that, for all sufficiently large k and sufficiently small ε_0, (3.3) (3.10) and (3.11) hold when $\theta_{n_k}\in(\theta_0-\eta+\zeta,\theta_0+\eta-\zeta)\cap E_\infty(R_{n_k})\cap[\theta_{k_{\nu_0}}+2\varepsilon_0,\theta_{k_{\nu_0}+1}-2\varepsilon_0]$. From (3.3), (3.8), (3.10), (3.11), as the same argument of (3.12) and (3.13) in the proof of Theorem 1.1, we can obtain contradiction. These implies (3.19) is valid.

By Theorem 1.1, we know that

$$\mathrm{mes}\Delta(f) \ge M_2. \tag{3.26}$$

From Lemma 2.4, we have, for all sufficiently large n and any positive ε,

$$\mathrm{mes}(E_\infty(R_n)) > M_2-\varepsilon. \tag{3.27}$$

Combining (3.19), (3.26) and (3.27) follows that, for all sufficiently large n,

$$\mathrm{mes}(\Delta(f)\cap E_\infty(R_n)) \ge M_2-\xi/4, \tag{3.28}$$

where ξ is defined in (3.16). Since $\Delta(f^{(k)})$ is closed, clearly $\Delta(f^{(k)})^c$ is open, so it consists of at most countably open intervals. We can choose finitely many open intervals I_j $(j = 1, 2, \ldots, m)$, satisfying

$$I_j \subset \Delta(f^{(k)})^c, \quad \text{mes}\left(\Delta(f^{(k)})^c \backslash \bigcup_{i=1}^{m} I_i\right) < \xi/4. \tag{3.29}$$

Since, for sufficiently large n,

$$\begin{aligned} &\text{mes}\left(\Delta(f) \cap E_\infty(R_n) \cap \left(\bigcup_{i=1}^{m} I_i\right)\right) + \text{mes}(\Delta(f) \cap E_\infty(R_n) \cap \Delta(f^{(k)})) \\ &= \text{mes}\left(\Delta(f) \cap E_\infty(R_n) \cap \left(\Delta(f^{(k)}) \cup \left(\bigcup_{i=1}^{m} I_i\right)\right)\right) \geq M_2 - \xi/2, \end{aligned} \tag{3.30}$$

we have

$$\begin{aligned} &\text{mes}(\Delta(f) \cap E_\infty(R_n) \cap \left(\bigcup_{i=1}^{m} I_i)\right) \\ &\geq M_2 - \xi/2 - \text{mes}(\Delta(f) \cap E_\infty(R_n) \cap \Delta(f^{(k)})) \\ &\geq M_2 - \xi/2 - \text{mes}(\Delta(f) \cap \Delta(f^{(k)})) = \xi/2. \end{aligned} \tag{3.31}$$

Thus, there exists an open interval $I_{i_0} = (\alpha, \beta) \subset \bigcup_{i=1}^{m} I_i \subset \Delta(f^{(k)})^c$ such that, for infinitely many sufficiently large n,

$$\text{mes}(\Delta(f) \cap E_\infty(R_n) \cap I_{i_0}) \geq \frac{\xi}{2m} > 0. \tag{3.32}$$

Then, we prove (3.18) holds.

From (3.18), we know that there are $\widetilde{\theta_0}$ and $\widetilde{\eta} > 0$ such that

$$(\widetilde{\theta_0} - \widetilde{\eta}, \widetilde{\theta_0} + \widetilde{\eta}) \subset I \tag{3.33}$$

and

$$\lim_{n\to\infty} \text{mes}(\Delta(f) \cap E_\infty(R_n) \cap (\widetilde{\theta_0} - \widetilde{\eta}, \widetilde{\theta_0} + \widetilde{\eta})) > 0. \tag{3.34}$$

Then, there exists $\widetilde{r_0}$ such that $\Omega(\widetilde{r_0}, \widetilde{\theta_0} - \widetilde{\eta}, \widetilde{\theta_0} + \widetilde{\eta}) \cap J(f^{(k)}(z)) = \emptyset$. Noting that $J(f^{(k)})$ has an unbounded component, by Lemma 2.1 and the similar argument between (3.22) and (3.23), for any $\widetilde{\zeta} > 0, \widetilde{\zeta} < \widetilde{\eta}$, we have

$$|f^{(k)}(z)| = O(|z|^{d_2}), \quad z \in \Omega(\widetilde{r_0}, \widetilde{\theta_0} - \widetilde{\eta} + \widetilde{\zeta}, \widetilde{\theta_0} + \widetilde{\eta} - \widetilde{\zeta}), \ |z| \to \infty, \tag{3.35}$$

where d_2 is a positive constant.

Fixed $r_N e^{i\theta_N} \in \{r_n e^{i\theta_n}\}$, and take a $r_n e^{i\theta_n} \in \{r_n e^{i\theta_n}\}, n > N$. Take a simple Jordan arc γ in $\Omega(\widetilde{r_0}, \widetilde{\theta_0} - \widetilde{\eta}, \widetilde{\theta_0} + \widetilde{\eta})$ which connecting $r_N e^{i\theta_N}$ to $r_N e^{i\theta_n}$ along

$|z| = r_N$, and connecting $r_N e^{i\theta_n}$ to $r_n e^{i\theta_n}$ along $\arg z = \theta_n$. For any $z \in \gamma$, γ_z denotes a part of γ, which connecting $r_N e^{i\theta_N}$ to z. Let $L(\gamma)$ be the length of γ. Clearly, $L(\gamma) = O(r_n)$, $n \to \infty$. By (3.35), it follows

$$|f^{(k-1)}(z)| \leq \int_{\gamma_z} |f^{(k)}(z)||dz| + c_k \leq O(|z|^{d_2} L(\gamma)) + c_k \leq O(r_n^{d_2+1}), \; n \to \infty. \tag{3.36}$$

Similarly, we have

$$\begin{aligned} |f^{(k-2)}(z)| &\leq \int_{\gamma_z} |f^{(k-1)}(z)||dz| + c_{k-1} \leq O(r_n^{d_2+2}), \quad n \to \infty \\ &\vdots \\ |f(z)| &\leq \int_{\gamma_z} |f'(z)||dz| + c_1 \leq O(r_n^{d_2+k}), \quad n \to \infty. \end{aligned} \tag{3.37}$$

where $c_1, c_2, \ldots, c_k$ are constants, which are independent of n. Therefore,

$$S_{\widetilde{\theta}_0 - \widetilde{\eta} + \widetilde{\zeta}, \widetilde{\theta}_0 + \widetilde{\eta} - \widetilde{\zeta}}(r, f) = O(1). \tag{3.38}$$

By Lemma 2.2, we know (3.3) also holds for all $z \in \Omega(\widetilde{r}_0, \widetilde{\theta}_0 - \widetilde{\eta} + \widetilde{\zeta}, \widetilde{\theta}_0 + \widetilde{\eta} - \widetilde{\zeta})$, outside a R-set H. By the similar argument between (3.3) and (3.14) in the proof of Theorem 1.1, we can deduce contradictions. Therefore, it follows

$$\operatorname{mes}(\Delta(f) \cap \Delta(f^{(k)})) \geq M_2. \tag{3.39}$$

The proof is complete. ∎

Acknowledgement. The author wishes to express his thanks to the referee for his/her valuable suggestions and comments. This work was supported by NSFC (no. 11426035, 11301008) and the Key Project of Natural Science Foundation of Educational Committee of Henan Province under Grant no. 15A110008.

References

[1] Baker, I.N., *Sets of non-normality in iteration theory,* J. Lond. Math. Soc., 40 (1965), 499-502.

[2] Baker, I.N., *The domains of normality of an entire function,* Ann. Acad. Sci. Fenn. Ser. AI Math., 1 (1975), 277-283.

[3] Bergweiler, W., *Iteration of meromorphic functions,* Bull. Amer. Math. Soc. (N.S.), 29 (1993), 151-188.

[4] Edrei, A., Fuchs, W., *On meromorphic functions with regions free of poles and zeros,* Acta. Math., 108 (1962), 113-145.

[5] Edrei, A., Fuchs, W., *Bounds for the number of deficient values of certain classes of meromorphic functions,* Proc. Lond. Math. Soc., 12 (1962), 315-344.

[6] Goldberg, A.A., Ostrovskii, I.V., *Value distribution of meromorphic function,* in: AMS Translations of Mathematical Monographs series, 2008.

[7] Hayman, W., *Meromorphic functions*, Clarendon Press, Oxford, 1964.

[8] Huang, Z., Wang, J., *On the radial distribution of Julia sets of entire solutions of $f^{(n)} + A(z)f = 0$,* J. Math. Anal. Appl., 387 (2012), 1106-1113.

[9] Huang, Z., Wang, J., *On limite directions of Julia sets of entire solutions of linear differential equations,* J. Math. Anal. Appl., 409 (2014), 478-484.

[10] Laine, I., *Nevanlinna theory and complex differential equations*, Walter de Gruyter, Berlin, 1993.

[11] Laine, I., Wu, P.C., *Growth of solutions of second order linear differential equations*, Proc. Am. Math. Soc., 128 (9) (2000), 2693-2703.

[12] Qiao, J., *Julia set of entire functions and their derivatives,* Chin. Sci. Bull., 39 (3) (1994), 186-188.

[13] Qiao, J., *Stable domains in the iteration of entire functions* (in Chinese), Acta. Math. Sin., 37 (1994), 702-708.

[14] Qiao, J., *On limiting directions of Julia set,* Ann. Acad. Sci. Fenn. Math., 26 (2001), 391-399.

[15] Qiu, L., Wu, S.J., *Radial distributions of Julia sets of meromorphic functions*, *J. Aust. Math. Soc.*, 81 (3) (2006), 363-368.

[16] Wang, S., *On radial distributions of Julia sets of meromorphic functions,* Taiwanese J. Math., 11 (5) (2007), 1301-1313.

[17] Wu, P.C., Wu, S.J., Zhu, J., *On the growth of solutions of second order complex differential equation with meromorphic coefficients*, J. Inequal. Appl., 2012, 117.

[18] Yang, L., *Value distribution theory*, Springer-Verlag, Berlin, 1993.

[19] Yang, C.C., Yi, H.X., *Uniqueness theory of meromorphic functions*, Science Press/Kluwer Academic Publishers, Beijing/Dordrecht, 2003.

[20] Zhang, G.H., *Theory of entire and meromorphic functions, deficient and asymptotic values and singular directions*, Translations of Mathematical Monographs, Am. Math. Soc., 122 (1993).

[21] Zheng, J.H., *Dynamics of meromorphic functions*, Tsinghua University Press, Beijing, 2006 (in Chinese).

[22] Zheng, J.H., Wang, S., Huang, Z., *Some properties of Fatou and Julia sets of transcendental meromorphic functions,* Bull. Aust. Math. Soc., 66 (2002), 1-8.

Accepted: 18.07.2015