EXISTENCE AND UNIQUENESS OF $\Psi$-BOUNDED SOLUTIONS FOR NONLINEAR MATRIX DIFFERENCE EQUATIONS

T. Srinivasa Rao$^1$
G. Suressh Kumar
Ch. Vasavi

*Konuru Lakshmaiah University*
Department of mathematics
Vaddeswaram, Guntur dt., A.P.
India
e-mails: tsrao011@gmail.com
drgsk006@kluniversity.in
vasavi.klu@gmail.com

M.S.N. Murty

Professor (RETD), Sainivas
D.No:21-47, Opp. State Bank Of India
Bank Street, NUZVID
Krishna dt., A.P.
India
e-mail: drmsn2002@gmail.com

Abstract. Sufficient conditions are established for the existence and uniqueness of $\Psi$-bounded solutions for nonlinear vector difference equation on $\mathbb{Z}$, using Banach contraction principle. Further, we obtain sufficient conditions for the existence and uniqueness of $\Psi$-bounded solutions for nonlinear matrix difference equation on $\mathbb{Z}$, using Kronecker product of matrices.

Keywords: $\Psi$-bounded solutions, Banach contraction principle, nonlinear difference equations.


1. Introduction

The aim of this paper is to give sufficient conditions for the nonlinear matrix difference equation

\[(1.1) \quad X(n+1) = A(n)X(n)B(n) + F(n, X(n))\]

$^1$Corresponding author.
has a unique $Ψ$-bounded solution on $\mathbb{Z}$, where $A \in \mathbb{R}^{m \times m}$, $F : \mathbb{R} \times \mathbb{R}^{m \times m} \rightarrow \mathbb{R}^{m \times m}$ and $F(n, O) = O$. Here $Ψ$ is an invertible matrix function on $\mathbb{Z}$. The basic problem under consideration is to determine sufficient conditions for the existence of a solution with some specified boundedness conditions. Classical results of this type, for linear and nonlinear differential equations were given by Coppel [2] and for linear and nonlinear difference equations were given by Agarwal [1]. The problem of $Ψ$-bounded solutions for the system of linear difference equations have been studied by many authors [3], [4], [6]. Recently, Suresh kumar et al. [7], [8], [9] studied $Ψ$-bounded solutions for linear matrix difference equations. In [3], Diamandescu proved a necessary and sufficient condition for the existence of $Ψ$-bounded solutions for the nonhomogeneous linear difference equation on $\mathbb{Z}$. Suresh kumar et al. [7] extended these results to matrix difference equations, using technique of Kronecker product of matrices.

In this paper, we present sufficient conditions for the existence and uniqueness of $Ψ$-bounded solutions for the nonlinear difference equation

\begin{equation}
    x(n + 1) = A(n)x(n) + f(n, x(n))
\end{equation}

on $\mathbb{Z}$. Further, we established sufficient conditions for the existence and uniqueness of $Ψ$-bounded solutions for nonlinear matrix difference equation (1.1) on $\mathbb{Z}$ with the help of Kronecker product of matrices.

2. Preliminaries

Denote $\mathbb{R}^n$ the Euclidean $n$-space. $\mathbb{R}$, $\mathbb{Z}$ and $\mathbb{N}$ denote the set of all real, integers and nonnegative integers respectively, denote $N(n_0) = \{n_0, n_0 + 1, n_0 + 2, \ldots \}$. $I_m$ and $O_m$ denote the unit matrix and zero matrix of order $m$ respectively. For $x = \{x_1, x_2, \ldots, x_m\} \in \mathbb{R}^m$ and $A = [a_{ij}] \in \mathbb{R}^{m \times m}$, we use the following vector and matrix norms

\[ \|x\| = \max\{|x_1|, |x_2|, \ldots, |x_m|\} \quad \text{and} \quad |A| = \sup_{\|x\| \leq 1} \|Ax\|. \]

Let $Ψ_i : \mathbb{Z} \rightarrow (0, \infty)$, $i = 1, 2, \ldots, m$ be functions, and define a matrix function

\[ Ψ = \text{diag}[Ψ_1, Ψ_2, \ldots, Ψ_m]. \]

Then $Ψ(n)$ is an invertible matrix function on $\mathbb{Z}$.  

**Definition 2.1** [5] Let $A \in \mathbb{R}^{p \times q}$ and $B \in \mathbb{R}^{r \times s}$ then the Kronecker product of $A$ and $B$ written $A \otimes B$ is defined to be the partitioned matrix

\[ A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \ldots & a_{1q}B \\ a_{21}B & a_{22}B & \ldots & a_{2q}B \\ \vdots & \vdots & \ddots & \vdots \\ a_{p1}B & a_{p2}B & \ldots & a_{pq}B \end{bmatrix} \]

is an $pr \times qs$ matrix and is in $\mathbb{R}^{pr \times qs}$. 
Existence and Uniqueness of $\psi$-bounded ... 455

**Definition 2.2** [5] Let $A = [a_{ij}] \in \mathbb{R}^{p \times q}$, then the vectorization operator $Vec : \mathbb{R}^{p \times q} \to \mathbb{R}^{pq}$, defined and denote by

$$\hat{A} = VecA = \begin{bmatrix} A_1 \\ A_2 \\ \vdots \\ A_q \end{bmatrix}, \text{ where } A_j = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{pj} \end{bmatrix} (1 \leq j \leq q).$$

**Lemma 2.1** [7] The vectorization operator $Vec : \mathbb{R}^{m \times m} \to \mathbb{R}^{m^2}$, is a linear and one-to-one operator. In addition, $Vec$ and $Vec^{-1}$ are continuous operators.

Regarding properties and rules for Kronecker product of matrices we refer to [5].

Now, by applying the Vec operator to the nonlinear matrix difference equation (1.1) and using Kronecker product properties, we have

$$(2.1) \quad \hat{X}(n+1) = G(n) \hat{X}(n) + \hat{F}(n, \hat{X}(n)),$$

where $G(n) = B^T(n) \otimes A(n)$ is a $m^2 \times m^2$ matrix and $\hat{F}(n, \hat{X}(n)) = VecF(n, X(n))$ is a column matrix of order $m^2$. The equation (2.1) is called the Kronecker product difference equation associated with (1.1). It is clear that, if $X(n)$ is a solution of (1.1) if and only if $\hat{X}(n) = Vec X(n)$ is a solution of (2.1).

The corresponding homogeneous difference equation of (2.1) is

$$(2.2) \quad \hat{X}(n+1) = G(n)\hat{X}(n).$$

**Definition 2.3** A function $\varphi : \mathbb{Z} \to \mathbb{R}^m$ is said to be $\Psi$-bounded on $\mathbb{Z}$ if $\Psi \varphi$ is bounded on $\mathbb{Z}$ (i.e., there exists $M > 0$ such that $\|\Psi(n)\varphi(n)\| < M$, for all $n \in \mathbb{Z}$).

Extend this definition to matrix functions.

**Definition 2.4** A matrix sequence $F : \mathbb{Z} \to \mathbb{R}^{m \times m}$ is said to be $\Psi$-bounded on $\mathbb{Z}$ if $\Psi F$ is bounded on $\mathbb{Z}$ (i.e., there exists $L > 0$ such that $|\Psi(n)F(n)| \leq L$, for all $n \in \mathbb{Z}$).

The following lemmas play a vital role in the proof of main results.

**Lemma 2.2** The matrix function $F(n)$ is $\Psi$-bounded on $\mathbb{Z}$ if and only if the vector function $VecF(n)$ is $I_m \otimes \Psi$-bounded on $\mathbb{Z}$.

**Proof.** From the proof of Lemma 2.1, it follows that

$$\frac{1}{m} |A| \leq \|VecA\|_{\mathbb{R}^{m^2}} \leq |A|,$$

for every $A \in \mathbb{R}^{m \times m}$.

Put $A = \Psi(n)F(n)$ in the above inequality, we have

$$(2.3) \quad \frac{1}{m} |\Psi(n)F(n)| \leq \|(I_m \otimes \Psi(n)).VecF(n)\|_{\mathbb{R}^{m^2}} \leq |\Psi(n)F(n)|,$$

for all matrix functions $F(n), n \in \mathbb{Z}$. The proof easily follows from inequality (2.3). \qed
Consider the linear difference equation

\[ x(n + 1) = A(n)x(n), \tag{2.4} \]

where \( A(n) \) is an invertible square matrix of order \( m \) on \( \mathbb{Z} \). The following lemma is well-known and is given in [3].

**Lemma 2.3** Let \( Y(n) \) be the fundamental matrix and satisfies \( Y(0) = I_m \), then

(i) \( Y(n) = \begin{cases} A(n - 1)A(n - 2)\ldots A(1)A(0), & n > 0 \\ I_m, & n = 0 \\ [A(-1)A(-2)\ldots A(n - 1)A(n)]^{-1}, & n < 0 \end{cases} \)

(ii) \( Y(n + 1) = A(n)Y(n) \), for all \( n \in \mathbb{Z} \).

(iii) the solution of \( (2.4) \) with the initial condition \( x(0) = x_0 \) is

\[ x(n) = Y(n)x_0, \quad n \in \mathbb{Z}. \]

(iv) \( Y(n) \) is invertible for each \( n \in \mathbb{Z} \) and

\[ Y^{-1}(n) = \begin{cases} A^{-1}(0)A^{-1}(1)\ldots A^{-1}(n - 2)A^{-1}(n - 1), & n > 0 \\ I_m, & n = 0 \\ A(-1)A(-2)\ldots A(n - 1)A(n), & n < 0 \end{cases} \]

**Lemma 2.4** [7] Let \( Y(n) \) and \( Z(n) \) be the fundamental matrices for the matrix difference equations

\[ X(n + 1) = A(n)X(n) \tag{2.5} \]

and

\[ X(n + 1) = B^T(n)X(n), \tag{2.6} \]

\( n \in \mathbb{Z} \) respectively. Then the matrix \( Z(n) \otimes Y(n) \) is fundamental matrix of \( (2.2) \).

**Lemma 2.5** Let \( Y(n) \) be an invertible matrix function on \( \mathbb{N} \) and let \( P \) be a projection. If there exists a positive constant \( L > 1 \) such that

\[ \sum_{k=\min_0}^{n-1} |\Psi(k+1)Y(k+1)P^{-1}(k+1)\Psi^{-1}(k+1)| \leq L, \quad \text{for all } n \in \mathbb{N}(n_0), \tag{2.7} \]

then there exists a constant \( L_1 > 0 \) such that

\[ |\Psi(n)Y(n)P| \leq L_1 \left( \frac{L^2 - 1}{L} \right)^{n-n_0}, \quad \text{for all } n \in \mathbb{N}(n_0). \tag{2.8} \]
**Proof.** Let \( a(n) = |\Psi(n + 1)Y(n + 1)P|^{-1} \). From the identity
\[
\Psi(n)Y(n)P \left( \sum_{k=n_0}^{n-1} a(k) \right) = \sum_{k=n_0}^{n-1} \Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)\Psi(k+1)Y(k+1)Pa(k),
\]

it follows that
\[
|\Psi(n)Y(n)P| \left( \sum_{k=n_0}^{n-1} a(k) \right) \leq \sum_{k=n_0}^{n-1} |\Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)| |\Psi(k+1)Y(k+1)P|a(k) \leq L.
\]

Setting \( b(n) = \sum_{k=n_0}^{n-1} a(k) \), we obtain \( b(n) - b(n-1) = |\Psi(n)Y(n)P|^{-1} \) for \( n \in \mathbb{N}(n_0 + 1) \).

After substituting in (2.9), we have
\[
b(n) - b(n-1) \geq \frac{b(n)}{L} \quad \text{and} \quad b(n) \geq \frac{L}{L-1} b(n-1),
\]
which implies that
\[
b(n) \geq \left( \frac{L}{L-1} \right)^{n-n_0} b(n_0 + 1), \quad \text{for all } n \in \mathbb{N}(n_0 + 1).
\]

From (2.9), we get
\[
|\Psi(n)Y(n)P|b(n) \leq L,
\]
which implies that
\[
|\Psi(n)Y(n)P| \leq L(b(n))^{-1} \leq L \left( \frac{L-1}{L} \right)^{n-n_0-1} b^{-1}(n_0 + 1)
= L \left( \frac{L-1}{L} \right)^{n-n_0-1} |\Psi(n_0 + 1)Y(n_0 + 1)P|.
\]

If we choose \( L_1 = \max \left\{ |\Psi(n_0)Y(n_0)P|, \frac{L^2}{L-1} |\Psi(n_0 + 1)Y(n_0 + 1)P| \right\} \), then (2.8) follows.

**Lemma 2.6** Let \( Y(n) \) be an invertible matrix which is defined on \( \mathbb{N} \) and let \( P \) be a projection. If there exists a constant \( L > 0 \) such that
\[
\sum_{k=n}^{\infty} |\Psi(n)Y(n)PY^{-1}(k+1)\Psi^{-1}(k+1)| \leq L, \quad \text{for all } n \in \mathbb{N},
\]
then for any vector \( \xi \in \mathbb{R}^m \) such that \( P\xi \neq 0 \),
\[
\limsup_{n \to \infty} \|\Psi(n)Y(n)P\xi\| = \infty.
\]
Proof. For any \( n \in \mathbb{N}(n_0) \), we have \( \|\Psi(n + 1)Y(n + 1)P\xi\| > 0 \). Then, from
\[
\sum_{k=n}^{n_1} \|\Psi(k+1)Y(k+1)P\xi\|^{-1}\Psi(n)Y(n)P\xi
\]
and (2.10), we get
\[
\|\Psi(k+1)Y(k+1)P\xi\| \sum_{k=n}^{n_1} \|\Psi(k+1)Y(k+1)P\xi\|^{-1} \leq L, \quad \text{for } n_1 \geq n, n, n_1 \in \mathbb{N}(n).
\]
Therefore, \( \sum_{k=n}^{\infty} \|\Psi(k+1)Y(k+1)P\xi\|^{-1} \) exists and so
\[
\limsup_{n \to \infty} \|\Psi(n+1)Y(n+1)P\xi\|^{-1} = 0, \quad \text{or } \limsup_{n \to \infty} \|\Psi(n+1)Y(n+1)P\xi\| = \infty. \]

3. Main result

In this section, first we obtain sufficient conditions for the existence and uniqueness of \( \Psi \)-bounded solution of the nonlinear difference equation (1.2), using Banach contraction principle.

Let the vector space \( \mathbb{R}^m \) be represented as a direct sum of three subspaces \( X_-, X_0, X_+ \) such that a solution \( y(n) \) of (2.4) is \( \Psi \)-bounded on \( \mathbb{Z} \) if and only if \( y(0) \in X_0 \) and \( \Psi \)-bounded on \( \mathbb{Z} \) if and only if \( y(0) \in X_- \oplus X_0 \). Also let \( P_-, P_0, P_1 \) denote the corresponding projections of \( \mathbb{R}^m \) onto \( X_-, X_0, X_+ \) respectively.

In the general case where \( (P_0 \neq 0) \), the solution for (1.2) is as follows
\[
x(n) = \sum_{k=-\infty}^{n-1} Y(n)P_-Y^{-1}(k+1)f(k, x(k))
\]
(3.1)
\[
+ \sum_{k=0}^{n-1} Y(n)P_0Y^{-1}(k+1)f(k, x(k)) - \sum_{k=n}^{\infty} Y(n)P_1Y^{-1}(k+1)f(k, x(k)).
\]

For simplicity, assume that the linear equation (2.4) has no nontrivial \( \Psi \)-bounded solution \( (P_0 = 0) \).

Theorem 3.1 (Existence and Uniqueness) Suppose that there exist supplementary projections \( P_-, P_1 \) and a positive constant \( K \) such that
\[
\sum_{k=-\infty}^{n-1} |\Psi(n)Y(n)P_-Y^{-1}(k+1)\Psi^{-1}(k)|
\]
(3.2)
\[
+ \sum_{k=n}^{\infty} |\Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)| \leq K.
\]
Let $f(n, x)$ be a vector function such that

$$
(3.3) \quad \|\Psi(n)[f(n, x) - f(n, y)]\| \leq \alpha \|\Psi(n)(x - y)\|,
$$

for $n \in \mathbb{Z}$, $\|\Psi x\| \leq \rho$, $\|\Psi y\| \leq \rho$, where $\alpha K < 1$, then the equation (1.2) has a unique $\Psi$-bounded solution $x(n)$ for which $\|\Psi x\| \leq \rho$.

**Proof.** From Lemmas 2.5 and 2.6, the condition (3.2) implies that $|\Psi(n)Y(n)P_{-1}\xi|$ is unbounded for $n \leq 0$ if $P_{-1}\xi \neq 0$ and bounded for $n \geq 0$, and that $|\Psi(n)Y(n)P_{1}\xi|$ is unbounded for $n \geq 0$ if $P_{1}\xi \neq 0$ and bounded for $n \leq 0$. Hence the linear equation (2.4) has no nontrivial $\Psi$-bounded solution.

Let $x(n)$ be the solution of (1.2), then from (3.2) and (3.3) the function

$$
\begin{align*}
y(n) &= x(n) - \sum_{k=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(k+1)f(k, x(k)) \\
&\quad + \sum_{k=n}^{\infty} Y(n)P_{1}Y^{-1}(k+1)f(k, x(k)),
\end{align*}
$$

exists and is $\Psi$-bounded for all $n \in \mathbb{Z}$. Moreover, it follows that

\[
y(n + 1) = x(n + 1) - \sum_{k=-\infty}^{n} Y(n + 1)P_{-1}Y^{-1}(k+1)f(k, x(k)) \\
&\quad + \sum_{k=n+1}^{\infty} Y(n + 1)P_{1}Y^{-1}(k+1)f(k, x(k)) \\
&= Ax(n) + f(n, x(n)) - Y(n + 1)P_{-1}Y^{-1}(n+1)f(n, x(n)) \\
&\quad - \sum_{k=-\infty}^{n-1} A(n)Y(n)P_{-1}Y^{-1}(k+1)f(k, x(k)) \\
&\quad - Y(n + 1)P_{1}Y^{-1}(n+1)f(n, x(n)) \\
&\quad + \sum_{k=n}^{\infty} A(n)Y(n)P_{1}Y^{-1}(k+1)f(k, x(k)) \\
&= A(n) \left[ x(n) - \sum_{k=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(k+1)f(k, x(k)) \\
&\quad - \sum_{k=n}^{\infty} Y(n)P_{1}Y^{-1}(k+1)f(k, x(k)) \right] + f(n, x(n)) - f(n, x(n)) \\
&= A(n)y(n).
\]

Therefore, $y(n)$ is a $\Psi$-bounded solution of (1.2). Thus $y(n) = 0$ that is

$$
(3.5) \quad x(n) = \sum_{k=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(k+1)f(k, x(k)) - \sum_{k=n}^{\infty} Y(n)P_{1}Y^{-1}(k+1)f(k, x(k)).
$$
Define $C_\Psi = \{ x : \mathbb{R} \to \mathbb{R}^m : x \text{ is } \Psi\text{-bounded functions on } \mathbb{Z} \text{ such that } \|\Psi x\| \leq \rho \}$ and $\|x\|_\Psi = \sup_{n \in \mathbb{Z}} \|\Psi(n)x(n)\|$. Clearly, this defines a norm on $C_\Psi$ and $(C_\Psi, \|\cdot\|_\Psi)$ is a Banach space. Let $T$ be a mapping defined by

\begin{equation}
Tx(n) = \sum_{k=-\infty}^{n-1} Y(n)P_{-1}Y^{-1}(k+1)f(k, x(k)) - \sum_{k=n}^{\infty} Y(n)P_1Y^{-1}(k+1)f(k, x(k)),
\end{equation}

for all $x \in C_\Psi$. Consider

\[
\|\Psi(n)Tx(n)\| = \left\| \sum_{k=-\infty}^{n-1} \Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k, x(k)) \\
- \sum_{k=n}^{\infty} \Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k, x(k)) \right\|
\leq \left[ \sum_{k=-\infty}^{n-1} \|\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\| \right] \|\Psi(k)f(k, x(k))\| \\
\leq K\alpha\|\Psi(k)x(k)\| \\
\leq K\alpha\rho < \rho.
\]

which implies $Tx(n) \in C_\Psi$ and hence $T : C_\Psi \to C_\Psi$.

Now, we show that $T$ is a contraction mapping on $C_\Psi$. Consider

\[
\|\Psi(n)(Tx - Ty)(n)\| = \left\| \sum_{k=-\infty}^{n-1} \Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k, x(k)) \\
- \sum_{k=n}^{\infty} \Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k, x(k)) - \sum_{k=-\infty}^{n-1} \Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k, y(k)) \\
+ \sum_{k=n}^{\infty} \Psi(n)Y(n)P_1Y^{-1}(k+1)\Psi^{-1}(k)\Psi(k)f(k, y(k)) \right\|
\leq \left[ \sum_{k=-\infty}^{n-1} \|\Psi(n)Y(n)P_{-1}Y^{-1}(k+1)\Psi^{-1}(k)\| \right] \|\Psi(k)(f(k, x(k)) - f(k, y(k)))\| \\
\leq K\alpha\|\Psi(k)(x(k) - y(k))\|.
\]

Thus $\|Tx - Ty\|_\Psi \leq K\alpha\|x - y\|_\Psi$. 

Therefore \( T \) is a contraction mapping on \( C_\Psi \). Hence by Banach contraction principle, \( T \) has a unique fixed point \( x(n) \) on \( C_\Psi \). Thus, the nonlinear difference equation (1.2) has a unique fixed point for which \( \|\Psi x\| \leq \rho \).

Conversely, if \( x(n) \) is a solution of (1.2) such that \( \|\Psi x\| \leq \rho \), then \( y = x - Tx \) is a \( \Psi \)-bounded solution of the linear equation (2.4), therefore \( y = 0 \).

**Example 3.1** Consider the nonlinear difference equation (1.2) with

\[
A(n) = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad f(n, x(n)) = \frac{1}{10} \begin{bmatrix} 3^{-|n|} \tan^{-1}(x_1(n)) \\ 5^{-|n|} x_2(n) \end{bmatrix}.
\]

The fundamental matrix of (2.4) is

\[
Y(n) = \begin{bmatrix} 2^n & 0 \\ 0 & 3^{-n} \end{bmatrix}.
\]

Consider

\[
\Psi(n) = \begin{bmatrix} 3^{-n} & 0 \\ 0 & 5^n \end{bmatrix}, \quad \text{for all } n \in \mathbb{Z}.
\]

Clearly, the linear equation (2.4) has no nontrivial \( \Psi \)-bounded solution on \( \mathbb{Z} \).

There exist supplementary projections

\[
P_{-1} = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \quad \text{and} \quad P_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},
\]

such that

\[
\sum_{k=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(k + 1)\Psi^{-1}(k)| = \left( \frac{1}{3} \right) \sum_{k=-\infty}^{n-1} \left( \frac{2}{3} \right)^{k-(n+1)} = 1,
\]

\[
\sum_{k=n}^{\infty} |\Psi(n)Y(n)P_1Y^{-1}(k + 1)\Psi^{-1}(k)| = (3) \sum_{k=n}^{\infty} \left( \frac{5}{3} \right)^{n-k} = 7.5,
\]

which implies

\[
\sum_{k=-\infty}^{n-1} |\Psi(n)Y(n)P_{-1}Y^{-1}(k + 1)\Psi^{-1}(k)| + \sum_{k=n}^{\infty} |\Psi(n)Y(n)P_1Y^{-1}(k + 1)\Psi^{-1}(k)| = 8.5.
\]

And also

\[
\|\Psi(n)[f(n, x) - f(n, y)]\| \leq \frac{1}{10} \|\Psi(n)[x - y]\|.
\]

Therefore, all the conditions of Theorem 3.1 are satisfied with \( \alpha = 1/10 \) and \( K = 8.5 \). Hence the nonlinear difference equation (1.2) has a unique \( \Psi \)-bounded solution

\[
x(n) = \frac{1}{10} \left[ \sum_{k=-\infty}^{n-1} (2)^{n-(k+1)} 3^{-|k|} \tan^{-1}(x_1(k)) \right] + \sum_{k=n}^{\infty} 3^{k+1-n} 5^{-|k|} x_2(k) \frac{1+k^2}{1+k^2}
\]

on \( \mathbb{Z} \).
Now, we obtain sufficient conditions for the existence and uniqueness of the nonlinear matrix difference equation (1.1), using Theorem 3.1 and the technique of Kronecker product of matrices.

Let the matrix space $\mathbb{R}^{m\times m}$ be represented as a direct sum of three subspaces $Y_-, Y_0, Y_+$ such that a solution $V(n)$ of

$$X(n+1) = A(n)X(n)B(n)$$

is a $\Psi$-bounded solution on $\mathbb{Z}$ if and only if $V(0) \in Y_0$ and $\Psi$-bounded on $\mathbb{N}$ if and only if $V(0) \in Y_\gamma Y_0$. Also, let $R_{-1}, R_0, R_1$ denote the corresponding projections of $\mathbb{R}^{m\times m}$ onto $Y_-, Y_0, Y_+$ respectively.

Then, the vector space $\mathbb{R}^{m^2}$ represents direct sum of three subspaces $S_-, S_0, S_+$ such that a solution $\hat{V}(n) = VecV(n)$ of (2.2) is $(I_m \otimes \Psi)$-bounded on $\mathbb{Z}$ if and only if $\hat{V}(0) \in S_0$ and $(I_m \otimes \Psi)$-bounded on $\mathbb{N}$ if and only if $\hat{V}(0) \in S_- \oplus S_0$. Also, let $Q_{-1}, Q_0, Q_1$ denote the corresponding projections of $\mathbb{R}^{m^2}$ onto $S_-, S_0, S_+$ respectively.

In the general case where $(Q_0 \neq 0)$, the solution for (2.1) is as follows

$$\hat{X}(n) = \sum_{k=-\infty}^{n-1} (Z(n) \otimes Y(n))Q_{-1}(Z^{-1}(k+1) \otimes Y^{-1}(k+1))\hat{F}(k, \hat{X}(k))$$

$$+ \sum_{k=0}^{n-1} (Z(n) \otimes Y(n))Q_0(Z^{-1}(k+1) \otimes Y^{-1}(k+1))\hat{F}(k, \hat{X}(k))$$

$$- \sum_{k=n}^{\infty} (Z(n) \otimes Y(n))Q_1(Z^{-1}(k+1) \otimes Y^{-1}(k+1))\hat{F}(k, \hat{X}(k)).$$

For simplicity, assume that the linear equation (2.2) has no nontrivial $(I_m \otimes \Psi)$-bounded solution $(Q_0 = 0)$.

**Theorem 3.2** Suppose that there exist supplementary projections $Q_{-1}, Q_1$ and a positive constant $M$ such that

$$\sum_{k=-\infty}^{n-1} |(Z(n) \otimes \Psi(n)Y(n))Q_{-1}(Z^{-1}(k+1) \otimes (Y^{-1}(k+1)\Psi^{-1}(k)))|$$

$$+ \sum_{k=n}^{\infty} |(Z(n)\otimes\Psi(n)Y(n))Q_1(Z^{-1}(k+1)\otimes(Y^{-1}(k+1)\Psi^{-1}(k)))| \leq M.$$

Let $F(n, X)$ be a matrix function such that

$$|\Psi(n)(F(n, U) - F(n, V))| \leq \beta|\Psi(n)(U - V)|$$

for $n \in \mathbb{Z}$, $|\Psi U| \leq \gamma$, $|\Psi V| \leq \gamma$, where $m\beta M < 1$, then the equation (1.1) has a unique $\Psi$-bounded solution $X(n)$ from which $|\Psi X| \leq \gamma$. 
Proof. Let $F(n, X)$ be a matrix function satisfies (3.10). From inequality (2.3), we have

$$|(I_m \otimes \Psi(n + 1))(\hat{F}(n, \hat{U}) - \hat{F}(n, \hat{V}))| \leq |\Psi(n + 1)(F(n, U) - F(n, V))|$$

$$\leq \beta|\Psi(n)(U - V)|$$

$$\leq m\beta|(I_m \otimes \Psi(n))(\hat{U} - \hat{V})|,$$

for $\hat{U}, \hat{V} \in \mathbb{R}^{m^2}$. Also

$$|(I_m \otimes \Psi(n))\hat{U}| \leq |\Psi(n)U(n)| \leq \gamma,$$

$$|(I(n) \otimes \Psi(n))\hat{V}| \leq |\Psi(n)V(n)| \leq \gamma.$$

From Kronecker product properties, equations (3.9) and (3.10), we have that the fundamental matrix of (2.2) satisfies condition (3.2) of Theorem 3.1, and the vectorization function $\hat{F}(n, \hat{X})$ satisfies condition (3.3). Therefore, from Theorem 3.1, the Kronecker product difference equation (2.1) has a unique $(I_m \otimes \Psi)$-bounded solution on $\mathbb{Z}$. From Lemma 2.2, the matrix difference equation (1.1) has a unique $\Psi$-bounded solution on $\mathbb{Z}$.

Example 3.2 Consider the nonlinear matrix difference equation (1.1) with

$$A(n) = \begin{bmatrix} 5 & 0 \\ 0 & 1 \end{bmatrix}, \quad B(n) = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}, \quad F(n, X(n)) = \frac{1}{30} \begin{bmatrix} 5^{-|n|}x_1(n) \\ 1+|n| \\ 2 \end{bmatrix} \begin{bmatrix} 3^{-|n|} \tan^{-1} x_3(n) \\ 3^{-|n|} \tan^{-1} x_4(n) \end{bmatrix}$$

Then $Y(n) = \begin{bmatrix} 5^n & 0 \\ 0 & 3^{-n} \end{bmatrix}$ and $Z(n) = \begin{bmatrix} 3^{-n} & 0 \\ 0 & 2^n \end{bmatrix}$ are fundamental matrices for (2.5) and (2.6) respectively.

Let $\Psi(n) = \begin{bmatrix} 5^{-n} & 0 \\ 0 & 3^n \end{bmatrix}$, for all $n \in \mathbb{Z}$. Then, their exist supplementary projections

$$Q_{-1} = \begin{bmatrix} I_2 & O_2 \\ O_2 & O_2 \end{bmatrix} \quad \text{and} \quad Q_1 = \begin{bmatrix} O_2 & O_2 \\ O_2 & I_2 \end{bmatrix},$$

such that condition (3.9) and (3.10) satisfied with $M = 15/2$ and $\beta = 1/30$. Moreover, $m\beta M = 1/2 < 1$. Therefore all the conditions of Theorem 3.2 are satisfied. Hence nonlinear difference equation (1.1) has a unique $\Psi$-bounded solution on $\mathbb{Z}$.

References


Accepted: 01.07.2015