SOME GENERAL NUMERICAL RADIUS INEQUALITIES
FOR THE OFF-DIAGONAL PARTS
OF $2 \times 2$ OPERATOR MATRICES

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Abstract. We give some sharp inequalities involving powers of the numerical radii for the off-diagonal parts of $2 \times 2$ operator matrices. These inequalities, which are based on some classical convexity inequalities for the nonnegative real numbers, generalize earlier numerical radius inequalities.

Keywords: numerical radius, operator matrix, off-diagonal part, Cartesian decomposition, Jensen's inequality, mixed Schwarz inequality.

2000 Mathematics Subject Classification: 47A12, 47A30, 47A63, 47B15.

1. Introduction

Let $B(H)$ denote the $C^*$-algebra of all bounded linear operators on a complex Hilbert space $H$ with inner product $(\cdot, \cdot)$. For $A \in B(H)$, let $\omega(A)$ and $\|A\|$ denote the numerical radius and the usual operator norm of $A$, respectively. It is well known that $\omega(\cdot)$ defines a norm on $B(H)$, which is equivalent to the usual operator norm $\| \cdot \|$. In fact, for every $A \in B(H)$,

$$\frac{1}{2} \|A\| \leq \omega(A) \leq \|A\|. \tag{1.1}$$

The inequalities in (1.1) are sharp. The first inequality becomes an equality if $A^2 = 0$. The second inequality becomes an equality if $A$ is normal. For basic properties of the numerical radius, we refer to [4] and [6]. The inequalities in (1.1) have been improved considerably by Kittaneh. It has been shown in [10] and [11], respectively, that if $A \in B(H)$, then

$$\omega(A) \leq \frac{1}{2} \| |A| + |A^*| \| \leq \frac{1}{2} \left( \|A\| + \|A^2\|^{\frac{1}{2}} \right). \tag{1.2}$$
where $|A| = (A^*A)^{\frac{1}{2}}$ is the absolute value of $A$, and

\begin{equation}
\frac{1}{4} \| A^*A + AA^* \| \leq \omega^2(A) \leq \frac{1}{2} \| A^*A + AA^* \|.
\end{equation}

(1.3)

The inequalities in (1.2), which refine the second inequality in (1.1), have been utilized in [10] to derive an estimate for the numerical radius of the Frobenius companion matrix. Such an estimate can be employed to give new bounds for the zeros of polynomials (see, e.g., [9],[10], and references therein).

If $A = B + iC$ is the Cartesian decomposition of $A$, then $B$ and $C$ are self-adjoint, and $A^*A + AA^* = 2(B^2 + C^2)$. Thus, the inequalities in (1.3) can be written as

\begin{equation}
\frac{1}{2} \| B^2 + C^2 \| \leq \omega^2(A) \leq \| B^2 + C^2 \|.
\end{equation}

(1.4)

The purpose of this paper is to establish a general inequalities involving powers of the numerical radii for the off-diagonal parts of $2 \times 2$ operator matrices that are based on the classical convexity inequalities for nonnegative real numbers and some operator inequalities.

Other recent numerical radius inequalities have been obtained by Dragomir [3], El-Haddad [5], and Yamazaki [12]. The inequalities in [3] are related to the Euclidean radius of two Hilbert space operators, the inequalities in [5] involving powers of the numerical radii for Hilbert space operators, and those in [12] involve the Aluthge transform.

2. Main results

To prove our generalized numerical radius inequalities for the off-diagonal parts of $2 \times 2$ operator matrices, we need several well known lemmas. The first lemma is a simple consequence of the classical Jensen’s inequality concerning the convexity or the concavity of certain power functions. It is a special case of Schlömilch’s inequality for weighted means of nonnegative real numbers (see, e.g., [7, p. 26]).

\textbf{Lemma 2.1} \textit{For $a, b \geq 0$, $0 < \alpha < 1$, and $r \neq 0$, let $M_r(a, b, \alpha) = (a^{\alpha}r + (1 - \alpha)b^{\alpha})^{\frac{1}{r}}$ and let $M_0(a, b, \alpha) = a^{\alpha}b^{1 - \alpha}$. Then}

\begin{equation}
M_r(a, b, \alpha) \leq M_s(a, b, \alpha) \quad \text{for} \quad r \leq s.
\end{equation}

The second lemma is another application of Jensen’s inequality (see, e.g., [7, p. 28]).

\textbf{Lemma 2.2} \textit{For $a, b \geq 0$, and $r > 0$, let $N_r(a, b) = (a^r + b^r)^{\frac{1}{r}}$. Then}

\begin{equation}
N_s(a, b) \leq N_r(a, b) \quad \text{for} \quad s \geq r > 0.
\end{equation}

The third lemma follows from the spectral theorem for positive operators and Jensen’s inequality (see, e.g., [8]).
Lemma 2.3 Let $A \in B(H)$ be positive, and let $x \in H$ be any unit vector. Then

(a) $\langle Ax, x \rangle^r \leq \langle A^r x, x \rangle$ for $r \geq 1$.

(b) $\langle A^r x, x \rangle \leq \langle Ax, x \rangle^r$ for $0 < r \leq 1$.

The fourth lemma is an immediate consequence of the spectral theorem for self-adjoint operators. For generalizations of this lemma, we refer to [8].

Lemma 2.4 Let $A \in B(H)$ be self-adjoint, and let $x \in H$ be any vector. Then

$$| \langle Ax, x \rangle | \leq \langle |A| x, x \rangle.$$

The fifth lemma is a generalized for the mixed Schwarz inequality which has been proved by Kittaneh [8].

Lemma 2.5 Let $T$ be an operator in $B(H)$ and let $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$. Then

$$| \langle Tx, y \rangle | \leq \| f(|T|) x \| \| g(|T^*|) x \| \quad \text{for all } x, y \in H.$$  

The sixth lemma contains two parts. Part (a) is well known and can be found in [2, p. 10]. Part (b) is also known and can be found in [1].

Lemma 2.6 Let $X, Y \in B(H)$. Then

(a) $\omega\left(\begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix}\right) = \max\{\omega(X), \omega(Y)\}.$

(b) $\omega\left(\begin{bmatrix} X & Y \\ Y & X \end{bmatrix}\right) = \max\{\omega(X + Y), \omega(X - Y)\}.$

In particular,

$$\omega\left(\begin{bmatrix} 0 & Y \\ Y & 0 \end{bmatrix}\right) = \omega(Y).$$

Our first result is a generalization of the first inequality in (1.2).

Theorem 2.7 Let $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ be a $2 \times 2$ operator matrix in $B(H_1 \oplus H_2)$, and let $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, and $r \geq 1$. Then

$$\omega^r(S) \leq \frac{1}{2} \max\{\|f^{2r}(|C| + g^{2r}(|B^*|))\|, \|f^{2r}(|B|) + g^{2r}(|C^*|)|\|\}. (2.1)$$
Proof. For every unit vector $X = \left( \begin{array}{c} x_1 \\ x_2 \end{array} \right) \in (H_1 \oplus H_2)$, by using Lemma 2.5, Lemma 2.1, and Lemma 2.3(a) we have

$$| \langle SX, X \rangle | \leq \| f (| S |) X \| \| g (| S^* |) X \|$$

which equals

$$= (f^2 (| S |) X, X) \frac{1}{2} (g^2 (| S^* |) X, X) \frac{1}{2}$$

$$= \left\langle f^2 \left( \begin{bmatrix} | C | & 0 \\ 0 & | B | \end{bmatrix} \right) X, X \right\rangle \frac{1}{2} \left\langle g^2 \left( \begin{bmatrix} | B^* | & 0 \\ 0 & | C^* | \end{bmatrix} \right) X, X \right\rangle \frac{1}{2}$$

$$\leq \frac{1}{2} \left\langle \left( f^2 (| C |) \begin{bmatrix} 0 & 0 \\ 0 & f^2 (| B |) \end{bmatrix} \right) X, X \right\rangle + \left\langle \left( g^2 (| B^* |) \begin{bmatrix} 0 & 0 \\ 0 & g^2 (| C^* |) \end{bmatrix} \right) X, X \right\rangle \right\rangle \frac{1}{2}$$

$$\leq \frac{1}{2} \left\langle f^{2r} (| C |) + g^{2r} (| B^* |) f^{2r} (| B |) + g^{2r} (| C^* |) \right\rangle \frac{1}{2} X, X \right\rangle.$$

Thus,

$$| \langle SX, X \rangle | \leq \frac{1}{2} \left\langle f^{2r} (| C |) + g^{2r} (| B^* |) f^{2r} (| B |) + g^{2r} (| C^* |) \right\rangle X, X \right\rangle,$$

and so

$$\omega^r (S) = \sup \{| \langle SX, X \rangle | : X \in (H_1 \oplus H_2), \| X \| = 1 \}$$

$$\leq \frac{1}{2} \sup \left\{ \left\langle \begin{bmatrix} \lambda & 0 \\ 0 & \mu \end{bmatrix} X, X \right\rangle : X \in (H_1 \oplus H_2), \| X \| = 1 \right\}$$

$$= \frac{1}{2} \max \{ \| \lambda \|, \| \mu \| \},$$

where

$$\lambda = f^{2r} (| C |) + g^{2r} (| B^* |)$$

and

$$\mu = f^{2r} (| B |) + g^{2r} (| C^* |),$$

as required.

Inequality (2.1) includes several numerical radius inequalities for operator matrices. Samples of inequalities are demonstrated in the following remarks.
Remark 2.8 For $f(t) = t^\alpha$ and $g(t) = t^{1-\alpha}$, $\alpha \in (0, 1)$, in inequality (2.1), we get the following inequality

$$\omega^r(S) \leq \frac{1}{2} \max \{\| C \|^2r \alpha + \| B^* \|^2r(1-\alpha), \| B \|^2r \alpha + \| C^* \|^2r(1-\alpha)\}.$$ 

Remark 2.9 If $B = C$ in the above Remark, and by using Lemma 2.6(b), then

$$\omega^r(B) = \omega^r(S) \leq \frac{1}{2} \| B \|^2r \alpha + \| B^* \|^2r(1-\alpha),$$

and this inequality is given in Theorem 1 in [5].

Now, the second result is a generalization of the second inequality in (1.3).

Theorem 2.10 Let $S = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ be a $2 \times 2$ operator matrix in $B(H_1 \oplus H_2)$, and let $f$ and $g$ be nonnegative functions on $[0, \infty)$ which are continuous and satisfying the relation $f(t)g(t) = t$ for all $t \in [0, \infty)$, and $r \geq 1$ and $0 < k < 1$. Then

$$\omega^{2r}(S) \leq \max \{|kf \frac{2}{r}x(|C|) + (1-k)|g \frac{2}{r}x(|B|)\|, \| kf \frac{2}{r}x(|B^*|) + (1-k)|g \frac{2}{r}x(|C^*|)\|\}.$$ 

Proof. For every unit vector $X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (H_1 \oplus H_2)$, by using Lemma 2.5, Lemma 2.3(b), Lemma 2.1, and Lemma 2.3(a), we have

$$| \langle SX, X \rangle | \leq | \langle f^2(|S||X)X, X \rangle \langle g^2(|S^*||X)X, X \rangle |$$

$$= \left\langle \begin{bmatrix} f^2(| C |) & 0 \\ 0 & f^2(| B |) \end{bmatrix} \right\rangle X, X \left\langle \begin{bmatrix} g^2(| B^* |) & 0 \\ 0 & g^2(| C^* |) \end{bmatrix} \right\rangle X, X$$

$$\leq \left\langle \begin{bmatrix} f^2(| C |) & 0 \\ 0 & f^2(| B |) \end{bmatrix} \right\rangle X, X \left\langle \begin{bmatrix} g^2(| B^* |) & 0 \\ 0 & g^2(| C^* |) \end{bmatrix} \right\rangle X, X \right\rangle^{1-k}$$

$$\leq \left( k \left\langle \begin{bmatrix} f^2(| C |) & 0 \\ 0 & f^2(| B |) \end{bmatrix} \right\rangle X, X \right\rangle^{1-k}$$

$$+ (1-k) \left\langle \begin{bmatrix} g^2(| B^* |) & 0 \\ 0 & g^2(| C^* |) \end{bmatrix} \right\rangle X, X \right\rangle^{1-k}$$

$$\leq \left( k \left\langle \begin{bmatrix} f^2(| C |) & 0 \\ 0 & f^2(| B |) \end{bmatrix} \right\rangle X, X \right\rangle^{1-k}$$

$$+ (1-k) \left\langle \begin{bmatrix} g^2(| B^* |) & 0 \\ 0 & g^2(| C^* |) \end{bmatrix} \right\rangle X, X \right\rangle^{1-k}.$$
Thus,
\[
| \langle SX, X \rangle |^{2r} \\
\leq \left( k \left[ \begin{array}{cc} f^{2r}(|C|) & 0 \\ 0 & f^{2r}(|B|) \end{array} \right] + (1 - k) \left[ \begin{array}{cc} g^{2r}(|B^*|) & 0 \\ 0 & g^{2r}(|C^*|) \end{array} \right] \right) X, X \\
= \left[ \begin{array}{cc} k f^{2r}(|C|) + (1 - k) g^{2r}(|B^*|) & 0 \\ 0 & k f^{2r}(|B|) + (1 - k) g^{2r}(|C^*|) \end{array} \right] X, X,
\]
and so
\[
\omega^{2r}(S) = \sup \{ | \langle SX, X \rangle |^{2r} : X \in (H_1 \oplus H_2), \| X \| = 1 \}
\leq \sup \left\{ \left[ \begin{array}{cc} \beta & 0 \\ 0 & \gamma \end{array} \right] X, X : X \in (H_1 \oplus H_2), \| X \| = 1 \right\}
= \max \{ \| \beta \|, \| \gamma \| \},
\]
where
\[
\beta = k f^{2r}(|C|) + (1 - k) g^{2r}(|B^*|)
\]
and
\[
\gamma = k f^{2r}(|B|) + (1 - k) g^{2r}(|C^*|),
\]
as required.

Now, Theorem 2.10 includes several numerical radius inequalities for operator matrices, and so we give some inequalities in the following remarks.

**Remark 2.11** If \( f(t) = t^k \) and \( g(t) = t^{1-k} \), \( k \in (0, 1) \), in Theorem 2.10, then we get the following inequality
\[
\omega^{2r}(S) \leq \max \{ \| k \| | C |^{2r} + (1 - k) \| B^* \|^{2r}, \| k \| | B |^{2r} + (1 - k) \| C^* \|^{2r} \}.
\]

**Remark 2.12** If \( B = C \) in the above Remark, and by using Lemma (2.6)b, then we have
\[
\omega^{2r}(B) = \omega^{2r}(S) \leq \| k \| | B |^{2r} + (1 - k) \| B^* \|^{2r},
\]
and this inequality can be found in Theorem 2 in [5].

**Remark 2.13** If we take \( r = 1 \) and \( k = \frac{1}{2} \) in the last Remark, we find
\[
\omega^2(B) \leq \frac{1}{2} \| | B |^2 + | B^* |^2 \|,
\]
which is the second inequality in (1.3).

Our next results are generalizations of the second inequality in (1.4).
**Theorem 2.14** Let \( R = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix} \) be a \( 2 \times 2 \) operator matrix in \( B(H_1 \oplus H_2) \), with the Cartesian decomposition \( R = S + iT \) and \( 1 \leq r \leq 2 \). Then

\[
\omega^r(R) \leq \frac{1}{2^r} \max \{ \| C + B^* \|^r + \| C - B^* \|^r, \| B + C^* \|^r + \| B - C^* \|^r \}.
\]

**Proof.** For every unit vector \( X = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in (H_1 \oplus H_2) \), and for \( 1 \leq r \leq 2 \), we have

\[
| \langle RX, X \rangle | = | \langle (S + iT)X, X \rangle | = | \langle SX, X \rangle + i \langle TX, X \rangle | = \sqrt{(SX, X)^2 + (TX, X)^2} \leq \sqrt{(\| SX \|_r + \| TX \|_r) \| X \|_r} \quad \text{by Lemma 2.2}
\]

\[
\leq \sqrt{(\| T \| X, X)^r + (\| T \| X, X)^r} \quad \text{by Lemma 2.4}
\]

\[
\leq \sqrt{(\| S \| X, X)^r + (\| T \| X, X)^r} \quad \text{by Lemma (2.3)a}
\]

\[
= \sqrt{(\| S \| X, X)^r + (\| T \| X, X)^r}.
\]

Thus,

\[
| \langle RX, X \rangle |^r \leq (\| S \|_r + \| T \|_r) X, X
\]

and so,

\[
\omega^r(R) = \sup \{ | \langle RX, X \rangle |^r : X \in (H_1 \oplus H_2), \| X \| = 1 \} \leq \sup \{ (\| S \|_r + \| T \|_r) X, X : X \in (H_1 \oplus H_2), \| X \| = 1 \}
\]

\[
= \frac{1}{2^r} \max \{ \| C + B^* \|^r + \| C - B^* \|^r, \| B + C^* \|^r + \| B - C^* \|^r \},
\]

as required. \( \blacksquare \)

**Remark 2.15** Let \( B = C \) and \( r = 2 \) in Theorem (2.14). Then we get

\[
\omega^2 \left( \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right) = \omega^2(B) \quad \text{(by Lemma (2.6)b)}
\]

\[
\leq \frac{1}{4} \max \{ \| B+B^* \|^2 + \| B-B^* \|^2, \| B+B^* \|^2 + \| B-B^* \|^2 \}
\]

\[
= \frac{1}{4} \| B+B^* \|^2 + \| B-B^* \|^2
\]

\[
= \frac{1}{2} \| B^* B + B B^* \|,
\]

which is the second inequality in (1.3).
Theorem 2.16 Let $R = \begin{bmatrix} 0 & B \\ C & 0 \end{bmatrix}$ be a $2 \times 2$ operator matrix in $B(H_1 \oplus H_2)$, with the Cartesian decomposition $R = S + iT$ and $r \geq 2$. Then

$$\omega^r(R) \leq 2^{\frac{-r}{2}} \max \{ \| C + B^* \|^r + \| C - B^* \|^r, \| B + C^* \|^r + \| B - C^* \|^r \}.$$  

**Proof.** For every unit vector $X = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \in (H_1 \oplus H_2)$, we have

$$\frac{1}{\sqrt{2}} | \langle RX, X \rangle | = \frac{1}{\sqrt{2}} | \langle (S + iT)X, X \rangle |$$

$$= \frac{1}{\sqrt{2}} | \langle SX, X \rangle + i \langle TX, X \rangle |$$

$$= \sqrt{\langle SX, X \rangle^2 + \langle TX, X \rangle^2}$$

$$\leq \sqrt{\frac{1}{2} \left( | \langle SX, X \rangle |^r + | \langle TX, X \rangle |^r \right)} (\text{by Lemma 2.1})$$

$$\leq 2^{\frac{-1}{2}} (| \langle S | X, X \rangle |^r + | \langle T | X, X \rangle |^r)^{\frac{1}{2}} (\text{by Lemma 2.4})$$

$$\leq 2^{\frac{-1}{2}} (| \langle S |^r X, X \rangle | + | \langle T |^r X, X \rangle |)^{\frac{1}{2}} (\text{by Lemma (2.3)a})$$

$$= 2^{\frac{-1}{2}} \langle | \langle S |^r + | T |^r \rangle X, X \rangle^{\frac{1}{2}}$$

$$= 2^{\frac{-1}{2}} \left[ \begin{bmatrix} \eta & 0 \\ 0 & \theta \end{bmatrix} X, X \right]^{\frac{1}{2}}.$$ 

Thus,

$$| \langle RX, X \rangle |^r \leq 2^{\frac{-r}{2}} \left[ \begin{bmatrix} \eta & 0 \\ 0 & \theta \end{bmatrix} X, X \right].$$ 

and so,

$$\omega^r(R) = \sup \{ | \langle RX, X \rangle |^r : X \in (H_1 \oplus H_2), \| X \| = 1 \}$$

$$\leq 2^{\frac{-r}{2}} \sup \left\{ \left[ \begin{bmatrix} \eta & 0 \\ 0 & \theta \end{bmatrix} X, X \right] : X \in (H_1 \oplus H_2), \| X \| = 1 \right\}$$

$$= 2^{\frac{-r}{2}} \max \{ \| \eta \|, \| \theta \| \},$$ 

where

$$\eta = | C + B^* |^r + | C - B^* |^r$$

and

$$\theta = | B + C^* |^r + | B - C^* |^r,$$

as required. $\blacksquare$
Remark 2.17 Let $B = C$ and $r = 2$ in Theorem (2.16). Then we get

$$
\omega^2 \left( \begin{bmatrix} 0 & B \\ B & 0 \end{bmatrix} \right) = \omega^2(B) \quad \text{(by Lemma (2.6)b)}
$$

$$
\leq \frac{1}{4} \max \{ \| B + B^* \| + |B - B^*|, \| B + B^* \| + |B - B^*| \}
$$

$$
= \frac{1}{4} \| B + B^* \| + |B - B^*| \|
$$

$$
= \frac{1}{2} \| B^*B + BB^* \|,
$$

which is the second inequality in (1.3).

Acknowledgment. This work was supported by the deanship of scientific research and graduated studies at Yarmouk University.

References


Accepted: 26.06.2015