GENERALIZED CUBIC SOFT SETS AND THEIR APPLICATIONS TO ALGEBRAIC STRUCTURES

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Abstract. In this paper, we introduce the concepts of generalized cubic soft sets, generalized cubic soft $\mathcal{AG}$-subgroupoids and generalized cubic soft left (resp., right) ideals to study the algebraic structures and properties of $\mathcal{AG}$-groupoids. We also give some examples of generalized cubic soft $\mathcal{AG}$-subgroupoids and generalized cubic soft left (resp., right) ideals. Moreover, we characterize intra-regular $\mathcal{AG}$-groupoids using the properties of generalized cubic soft sets and generalized cubic soft right ideals.

Keywords: cubic soft set, $\mathcal{AG}$-groupoid, left invertive law, medial law, paramedical law and $(\in_{(\widetilde{\gamma}_1,\gamma_2)},\in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft right ideal.

2000 Mathematics Subject Classification: 20M10, 20N99.

1. Introduction

The concept of a fuzzy set was introduced by Zadeh in 1965 [14], which is now a days used in almost all branches of science. In fact, a fuzzy set is a suitable tool for modeling because there exist a lot of ambiguities in crisp models while handling problems in different fields of science like artificial intelligence, computer science, control engineering, decision theory, expert system, logic management science, operations research, robotics and many others. In other words the concept of a fuzzy set is used to control uncertainty problems arising in models representing real life phenomenon. This concept is also used in business, medical and related health sciences. Maiers and Sherif [12], reviewed the literature on fuzzy industrial controllers and provided an index of applications of fuzzy set theory to twelve subject areas including decision making, economics, engineering and operations research.

Molodtsov [11], in 1999 introduced the fundamental concept of a soft set which is now used in several basic notions of algebra. Ali et al. [6], introduced several new algebraic operations on soft sets. Cagman and Enginoglu [2], developed the uni-int decision making method in virtue of soft sets. Feng et al. [4], investigated soft semirings by using soft set theory. Aktas and Cagman [1], defined the notion of soft groups and derived some related properties. This initiated an important research direction concerning algebraic properties of soft sets in miscellaneous kinds of algebras such as BCK/BCI-algebras, d-algebras, semirings, rings, Lie algebras and K-algebras. Feng and Li [5], ascertained the relationship among five different types of soft subsets and considered the free soft algebras associated with soft product operations. It has been shown that soft sets have some nonclassical algebraic properties which are distinct from those of crisp sets and fuzzy sets. Further, Jun discussed the applications of soft sets in ideal theory of BCK/BCI-algebras and in d-algebras respectively. Recently, combining cubic sets and soft sets, Muhiuddin and Al-roqi [9], introduced the notions of (external, internal) cubic soft sets, P-cubic (resp., R-cubic) soft subsets, R-union (resp., R-intersection, P-union and P-intersection) of cubic soft sets and the complement of a cubic soft set. They investigated several related properties and applied the notion of cubic soft sets to BCK/BCI-algebras.

In [10], Khan et al. introduced the generalized version of Jun's cubic set and applied it to the ideal theory of semigroups. Here the purpose of this paper is to deal with the algebraic structure of $\mathcal{AG}$-groupoids by applying the generalized version of a soft set.

The idea of generalization of a commutative semigroup (which we call the left almost semigroup) was introduced by M. A. Kazim and M. Naseeruddin in 1972 [8]. It is also known as an Abel-Grassmann groupoid ($\mathcal{AG}$-groupoid).

A groupoid $\mathcal{S}$ is called an $\mathcal{AG}$-groupoid if it satisfies the left invertive law, that is $(ab)c = (cb)a$, $\forall\ a, b, c \in \mathcal{S}$. In an $\mathcal{AG}$-groupoid medial law: $(ab)(cd) = (ac)(bd)$, holds $\forall\ a, b, c, d \in \mathcal{S}$ [8]. If an $\mathcal{AG}$-groupoid $\mathcal{S}$ contains left identity then it satisfies the paramedial law: $(ab)(cd) = (db)(ca)$, $\forall\ a, b, c, d \in \mathcal{S}$. Moreover, if an $\mathcal{AG}$-groupoid $\mathcal{S}$ contains the left identity, then the following law holds.

$$a(bc) = b(ac), \forall a, b, c \in \mathcal{S}.$$

Our interest in this paper is to introduce the concept of $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft sets, then we use this concept and introduced $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft $\mathcal{AG}$-subgroupoid, $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft left (resp., right, two-sided) ideals in an $\mathcal{AG}$-groupoid. Moreover, we discuss some characterizations of intra-regular $\mathcal{AG}$-groupoids using the properties of $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft sets and $(\in_{(\widetilde{\gamma}_1,\gamma_2)}, \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)})$-cubic soft right ideals.

2. Preliminaries notes

In this section, we define some basic definitions that are required for next sections.

Let $\mathcal{G}$ be an $\mathcal{AG}$-groupoid. By an $\mathcal{AG}$-subgroupoid of $\mathcal{G}$ we mean a non-empty subset A of $\mathcal{G}$ such that $A^2 \subseteq A$. A non-empty subset I of $\mathcal{G}$ is called left (resp., right) ideal of $\mathcal{G}$ such that $SI \subseteq I$ (resp., $IS \subseteq I$). An ideal I of $\mathcal{G}$ is called two sided ideal of $\mathcal{G}$ if it is both left and right ideal of $\mathcal{G}$. An $\mathcal{AG}$-subgroupoid B of $\mathcal{G}$ is called bi-ideal if $B(\mathcal{G}B) \subseteq B$. A non-empty subset Q of $\mathcal{G}$ is called quasi-ideal of $\mathcal{G}$ if $Q\mathcal{G} \cap \mathcal{G}Q \subseteq Q$.

A fuzzy subset f of an $\mathcal{AG}$-groupoid $\mathcal{G}$ is defined as a mapping from $\mathcal{G}$ into $[0, 1]$, where $[0, 1]$ is the usual closed interval of real numbers. Let f and g be any two fuzzy subsets of $\mathcal{G}$. Then, its product is denoted by $f \circ g$ and is defined as

$$(f \circ g)(x) = \begin{cases} \bigvee\limits_{x=yz} \{f(y) \wedge g(z)\} & \text{if there exist } x, y \in \mathcal{G}, \text{ such that } x = yz, \\ 0 & \text{otherwise.} \end{cases}$$

Next, we recall the concept of interval valued fuzzy sets. By an interval number, we mean a closed subinterval $\widetilde{a} = [a^-, a^+]$ of the closed interval $I = [0, 1]$, where $0 \leqslant a^- \leqslant a^+ \leqslant 1$. Let $D[0, 1]$ denote the family of all closed subintervals of $[0, 1]$, i.e.,

$$D[0, 1] = \{\widetilde{a} = [a^-, a^+] : a^- \leqslant a^+, \text{ for all } a^-, a^+ \in [0, 1]\},$$

where the elements in $D[0, 1]$ are called the interval numbers on $[0, 1]$, $\widetilde{0} = [0, 0]$ and $\widetilde{1} = [1, 1]$. For any two elements in $D[0, 1]$ the redefined minimum and redefined maximum, respectively, denoted by $r \min$ and $r \max$, and the symbols "$\preceq$","$\succeq$","$=$" are defined. We consider two elements $\widetilde{a} = [a^-, a^+]$ and $\widetilde{b} = [b^-, b^+]$ in $D[0, 1]$. Then,

$$\begin{aligned}
\widetilde{a} &\succeq \widetilde{b} \text{ if and only if } a^- \geq b^- \text{ and } a^+ \geq b^+ \\
\widetilde{a} &\preceq \widetilde{b} \text{ if and only if } a^- \leqslant b^- \text{ and } a^+ \leqslant b^+ \\
\widetilde{a} &= \widetilde{b} \text{ if and only if } a^- = b^- \text{ and } a^+ = b^+ \\
r\min\{\widetilde{a}, \widetilde{b}\} &= \min\{a^-, b^-\}, \min\{a^+, b^+\} \\
r\max\{\widetilde{a}, \widetilde{b}\} &= \max\{a^-, b^-\}, \max\{a^+, b^+\}.
\end{aligned}$$

Let $\widetilde{a}_i \in D\,[0,1]$, where $i \in \Pi$, we define

$$r\inf_{i\in\Pi} \widetilde{a}_i = \left[\inf a_i^-, \inf a_i^+\right]_{i\in\Pi} \text{ and } r\sup_{i\in\Pi} \widetilde{a}_i = \left[\sup a_i^-, \sup a_i^+\right]_{i\in\Pi}.$$

Let X be a non-empty set. A cubic set $\widetilde{\mu}_A$ on X is defined as

$$\widetilde{\mu}_A = \left\{\left\langle x, [\mu_A^-(x), \widetilde{\mu}_A^+(x)]\right\rangle : x \in X\right\},$$

where $\mu_A^-(x) \leq \mu_A^+(x)$, for all $x \in X$. Then the ordinary fuzzy sets $\mu_A^-(x) : X \to [0,1]$ and $\mu_A^+(x) : X \to [0,1]$ are called a lower fuzzy sets and upper fuzzy sets of $\widetilde{\mu}_A$ respectively. Let $\widetilde{\mu}_A(x) = [\mu_A^-(x), \widetilde{\mu}_A^+(x)]$ then

$$A = \{\langle x, \widetilde{\mu}_A(x)\rangle : x \in X\}, \text{ where } \widetilde{\mu}_A : X \to D[0,1].$$

To avoid symbols complications, we use the symbols $\Gamma = (\widetilde{\gamma}_1, \gamma_2)$ and $\Delta = (\widetilde{\delta}_1, \delta_2)$ for rest of the study.

3. $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-Cubic sets

Jun et al. [7], introduced the concept of cubic sets defined on a non-empty set X as objects having the form

$$\Xi = \left\{\left\langle x, \widetilde{\Psi}_\Xi(x), \eta_\Xi(x)\right\rangle : x \in X\right\},$$

which is briefly denoted by $\Xi = \left\langle \widetilde{\Psi}_\Xi, \eta_\Xi \right\rangle$, where the functions $\widetilde{\Psi}_\Xi : X \to D[0,1]$ and $\eta_\Xi : X \to [0,1]$.

Let $\Xi = \left\langle \widetilde{\Psi}_\Xi, \eta_\Xi \right\rangle$ and $\digamma = \left\langle \widetilde{\Psi}_\digamma, \eta_\digamma \right\rangle$ be two cubic sets of $\mathcal{S}$. Then,

$$\Xi \cap \digamma = \left\{\left\langle x, r\min\left\{\widetilde{\Psi}_\Xi(x), \widetilde{\Psi}_\digamma(x)\right\}, \max\{\eta_\Xi(x), \eta_\digamma(x)\}\right\rangle : x \in \mathcal{S}\right\}$$

and

$$\Xi \odot \digamma = \left\{\left\langle \left(\widetilde{\Psi}_\Xi \circ \widetilde{\Psi}_\eta\right)(x), \left(\eta_\Xi \circ \eta_\digamma\right)(x)\right\rangle : x \in \mathcal{S}\right\},$$

where,

$$\left(\widetilde{\Psi}_\Xi \circ \widetilde{\Psi}_\digamma\right)(x) = \begin{cases} r\sup\limits_{x=yz}\{r\min\left\{\widetilde{\Psi}_\Xi(y), \widetilde{\Psi}_\digamma(z)\right\} & \text{if } x = yz, \\ [0,0] & \text{otherwise.} \end{cases}$$

and

$$(\eta_\Xi \circ \eta_F)(x) = \begin{cases} \inf\limits_{x=yz} \{\max\{\eta_\Xi(y), \eta_F(z)\}\} & \text{if } x = yz, \\ 1 & \text{otherwise.} \end{cases}$$

Let $\mathcal{C}(\mathcal{S})$ denote the family of all cubic sets in $\mathcal{S}$. Then, it becomes an $\mathcal{AG}$-groupoid.

Let $\widetilde{\alpha} \in D(0,1]$ and $\beta \in [0,1)$ be such that $\widetilde{0} \prec \widetilde{\alpha}$ and $\beta < 1$. Then, by cubic point (CP) we mean $x_{(\widetilde{\alpha},\beta)}(y) = \langle x_{\widetilde{\alpha}}(y), x_\beta(y)\rangle$, where

$$x_{\widetilde{\alpha}}(y) = \begin{cases} \widetilde{\alpha} & \text{if } x = y, \\ \widetilde{0} & \text{otherwise.} \end{cases}$$

and

$$x_\beta(y) = \begin{cases} \beta & \text{if } x = y, \\ 1 & \text{otherwise.} \end{cases}$$

Here we give the generalized version of a Jun's cubic set [10]. For any cubic set $\Xi = \left\langle \widetilde{\Psi}_\Xi, \eta_\Xi \right\rangle$ and for a cubic point $x_{(\widetilde{\alpha},\beta)}$, with the condition that $[\alpha,\beta]+[\alpha,\beta] = [2\alpha, 2\beta]$ such that $0 \leq 2\alpha < 1$ and $2\beta \leq 1$, we have

(i) $x_{(\widetilde{\alpha},\beta)} \in_\Gamma \Xi$ if $\widetilde{\Psi}_\Xi(x) \succeq \widetilde{\alpha} \succ \widetilde{\gamma}_1$ and $\eta_\Xi(x) \leq \beta < \gamma_2$.

(ii) $x_{(\widetilde{\alpha},\beta)} q_\Delta \Xi$ if $\widetilde{\Psi}_\Xi(x) + \widetilde{\alpha} \succ 2\widetilde{\delta}_1$ and $\eta_\Xi(x) + \beta < 2\delta_2$.

(iii) $x_{(\widetilde{\alpha},\beta)} \in_\Gamma \vee q_\Delta \Xi$ if $x_{(\widetilde{\alpha},\beta)} \in_\Gamma \Xi$ or $x_{(\widetilde{\alpha},\beta)} q_\Delta \Xi$.

(iv) $x_{(\widetilde{\alpha},\beta)} \in_\Gamma \wedge q_\Delta \Xi$ if $x_{(\widetilde{\alpha},\beta)} \in_\Gamma \Xi$ and $x_{(\widetilde{\alpha},\beta)} q_\Delta \Xi$.

Next, we define the generalized characteristic function.

Definition 1 Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid. Then, the cubic characteristic function $\mathcal{X}_\Xi^{(\Gamma,\Delta)} = \left\langle \widetilde{\Psi}_{\mathcal{X}_\Xi^{(\Gamma,\Delta)}}, \eta_{\mathcal{X}_\Xi^{(\Gamma,\Delta)}} \right\rangle$ of $\Xi = \left\langle \widetilde{\Psi}_\Xi, \eta_\Xi \right\rangle$ is defined as

$$\widetilde{\Psi}_{\mathcal{X}_\Xi^{(\Gamma,\Delta)}}(x) \succeq \begin{cases} \widetilde{\delta}_1 = [1,1] & \text{if } x \in \Xi, \\ \widetilde{\gamma}_1 = [0,0] & \text{if } x \notin \Xi, \end{cases}$$

and

$$\eta_{\mathcal{X}_\Xi^{(\Gamma,\Delta)}}(x) \leq \begin{cases} \delta_2 = 0 & \text{if } x \in \Xi, \\ \gamma_2 = 1 & \text{if } x \notin \Xi, \end{cases}$$

where $\widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.

Now, we introduce a new relation on $\mathcal{C}(\mathcal{S})$ denoted by " $\subseteq \vee q_{(\Gamma,\Delta)}$" as follows.

Let $\Xi = \left\langle \widetilde{\Psi}_\Xi, \eta_\Xi \right\rangle, F = \left\langle \widetilde{\Psi}_F, \eta_F \right\rangle \in \mathcal{C}(\mathcal{S})$, by $\Xi \subseteq \vee q_{(\Gamma,\Delta)} F$ we mean that $x_{(\widetilde{\alpha},\beta)} \in_\Gamma \Xi$ implies that $x_{(\widetilde{\alpha},\beta)} \in_\Gamma \vee q_\Delta F$ for all $x \in \mathcal{S}$. Moreover, Ξ and F are said to be (Γ,Δ)-equal if $\Xi \subseteq \vee q_{(\Gamma,\Delta)} F$ and $F \subseteq \vee q_{(\Gamma,\Delta)} \Xi$. The above definitions can be found in [10].

Lemma 1 [*10*] *Let* $\Xi = \left\langle \widetilde{\Psi}_\Xi, \eta_\Xi \right\rangle, F = \left\langle \widetilde{\Psi}_F, \eta_F \right\rangle \in \mathcal{C}(\mathcal{S})$. *Then,* $\Xi \subseteq \vee q_{(\Gamma,\Delta)} F$ *if and only if*

$$r\max\left\{\widetilde{\Psi}_F(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_\Xi(a), \widetilde{\delta}_1\right\} \text{ and } \min\{\eta_F(a), \gamma_2\} \leq \max\{\eta_\Xi(a), \delta_2\}.$$

Lemma 2 [*10*] *Let* $\Xi = \left\langle \widetilde{\Psi}_\Xi, \eta_\Xi \right\rangle, F = \left\langle \widetilde{\Psi}_F, \eta_F \right\rangle, \Omega = \left\langle \widetilde{\Psi}_\Omega, \eta_\Omega \right\rangle \in \mathcal{C}(\mathcal{S})$. *If* $\Xi \subseteq \vee q_{(\Gamma,\Delta)} F$ *and* $F \subseteq \vee q_{(\Gamma,\Delta)} \Omega$. *Then,* $\Xi \subseteq \vee q_{(\Gamma,\Delta)} \Omega$.

From Lemmas 1 and 2 we say that " $=_{(\Gamma,\Delta)}$ " is an equivalence relation on $\mathcal{C}(\mathcal{S})$.

Lemma 3 *Let* $\mathcal{S}$ *be an* $\mathcal{AG}$*-groupoid and* $\Xi = \left\langle \widetilde{\Psi}_\Xi, \eta_\Xi \right\rangle, F = \left\langle \widetilde{\Psi}_F, \eta_F \right\rangle$ *be the cubic sets of* $\mathcal{S}$. *Then, we have*

(i) $\Xi \subseteq F$ *if and only if* $\mathcal{X}_\Xi^{(\Gamma,\Delta)} \subseteq \vee q_{(\Gamma,\Delta)} \mathcal{X}_F^{(\Gamma,\Delta)}$.

(ii) $\mathcal{X}_\Xi^{(\Gamma,\Delta)} \cap \mathcal{X}_F^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_{\Xi\cap F}^{(\Gamma,\Delta)}$.

(iii) $\mathcal{X}_\Xi^{(\Gamma,\Delta)} \odot \mathcal{X}_F^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_{\Xi F}^{(\Gamma,\Delta)}$.

4. Cubic soft sets

In this section, we recall some fundamental concepts of soft sets. For further details and background, see [11], [13], [9].

We introduced the concept of an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set which is actually the generalization of a soft set.

Definition 2 [9] Let U be an initial universal set and E be set of parameters under consideration. Let $\mathcal{C}^U$ denotes the set of all cubic subsets of U. Let $A \subseteq E$. A pair (F, A) is called cubic soft set over U, where F is a mapping given by $F : A \to \mathcal{C}^U$. Note that the the pair (F, A) can be expressed as the following set:

$$(F, A) := \{F(\epsilon) : \epsilon \in A\}, \text{ where } F(\epsilon) = \left\langle \widetilde{\Psi}_{F(\epsilon)}, \eta_{F(\epsilon)} \right\rangle.$$

In general, for every $\epsilon \in A$, $F(\epsilon)$ is a cubic set of U and it is called cubic value set of parameter ϵ. The set of all cubic soft sets over U with parameters from E is called cubic soft class and is denoted by $\mathcal{F}_\rho(U, E)$.

Definition 3 [13] Let (F, A) and (G, B) be two cubic soft sets over U. Then (F, A) is called cubic soft set of (G, B) and write $(F, A) \subset (G, B)$ if

(i) $A \subseteq B$.

(ii) For any $\epsilon \in A$, $F(\epsilon) \subseteq G(\epsilon)$.

(F, A) and (G, B) are said to be cubic soft equal and write $(F, A) = (G, B)$ if $(F, A) \subset (G, B)$ and $(G, B) \subset (F, A)$.

Definition 4 [13] The union of two cubic soft sets (F, A) and (G, B) over U is called cubic soft set and is denoted by (H, C), where $C = A \cup B$ and

$$H(\epsilon) = \begin{cases} F(\epsilon), & \text{if } \epsilon \in A - B, \\ G(\epsilon), & \text{if } \epsilon \in B - A, \\ F(\epsilon) \cup G(\epsilon), & \text{if } \epsilon \in A \cap B, \end{cases}$$

for all $\epsilon \in C$. This is denoted by $(H, C) = (F, A)\widetilde{\cup}(G, B)$.

Definition 5 [13] The intersection of two cubic soft sets (F, A) and (G, B) over U is called cubic soft set and is denoted by (H, C), where $C = A \cup B$ and

$$H(\epsilon) = \begin{cases} F(\epsilon), & \text{if } \epsilon \in A - B, \\ G(\epsilon), & \text{if } \epsilon \in B - A, \\ F(\epsilon) \cap G(\epsilon), & \text{if } \epsilon \in A \cap B, \end{cases}$$

for all $\epsilon \in C$. This is denoted by $(H, C) = (F, A)\widetilde{\cap}(G, B)$.

Here, we introduced the concept of generalized version of cubic soft set.

Definition 6 Let $V \subseteq U$. A cubic soft set (F, A) over U is said to be relative whole (Γ, Δ)-cubic soft set (with respect to universe set V and parameter set A), denoted by $\Sigma(V, A)$, if

$$F(\epsilon) = \mathcal{X}_V^{(\Gamma,\Delta)} \text{ for all } \epsilon \in A.$$

Definition 7 Let (F, A) and (G, B) be two cubic soft sets over U. We say that (F, A) is an (Γ, Δ)-cubic soft subset of (G, B) and write $(F, A) \subset_{(\Gamma,\Delta)} (G, B)$ if

(i) $A \subseteq B$.

(ii) For any $\epsilon \in A$, $F(\epsilon) \subseteq \vee q_{(\Gamma,\Delta)} G(\epsilon)$.

(F, A) and (G, B) are said to be (Γ, Δ)-cubic soft equal and write $(F, A) \asymp_{(\Gamma,\Delta)} (G, B)$ if $(F, A) \subset_{(\Gamma,\Delta)} (G, B)$ and $(G, B) \asymp_{(\Gamma,\Delta)} (F, A)$.

The product of two cubic soft sets (F, A) and (G, B) over an $\mathcal{AG}$-groupoid $\mathcal{S}$, denoted by $(F \circ G, C)$, where $C = A \cup B$ and

$$(F \circ G)(\epsilon) = \begin{cases} F(\epsilon), & \text{if } \epsilon \in A - B, \\ G(\epsilon), & \text{if } \epsilon \in B - A, \\ F(\epsilon) \circ G(\epsilon), & \text{if } \epsilon \in A \cap B, \end{cases}$$

for all $\epsilon \in C$. This is denoted by $(F \circ G, C) = (F, A) \odot (G, B)$.

5. $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-Cubic soft ideals over an $\mathcal{AG}$-groupoid

Here, we will introduce the concepts of $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft $\mathcal{AG}$-subgroupoids and $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left (resp., right,two-sided) ideals over an $\mathcal{AG}$-groupoid $\mathcal{S}$ and investigate the fundamental properties and relationships of $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft sets and $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideals.

Definition 8 A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{S}$ if it satisfies:

$$(F, A) \odot (F, A) \subset_{(\Gamma,\Delta)} (F, A).$$

Definition 9 A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left (resp., right) ideal over $\mathcal{S}$ if it satisfies:

$$\Sigma(\mathcal{S}, A) \odot (F, A) \subset_{(\Gamma,\Delta)} (F, A)(resp., (F, A) \odot \Sigma(\mathcal{S}, A) \subset_{(\Gamma,\Delta)} (F, A)).$$

A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft ideal over $\mathcal{S}$ if it is both an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left ideal and an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal over $\mathcal{S}$.

Theorem 1 *Let (F, A) be a cubic soft set over an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity. Then, (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{S}$ if and only if*

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(xy), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\left\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\Psi}_{F(\epsilon)}(y)\right\}, \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(xy), \gamma_2\} \leq \max\{\{\eta_{F(\epsilon)}(x), \eta_{F(\epsilon)}(y)\}, \delta_2\},$$

for all $x, y \in \mathcal{S}$, where $\epsilon \in A, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$.

Proof. Let (F, A) be a cubic soft set over an $\mathcal{AG}$-groupoid $\mathcal{S}$. Assume that (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{S}$. Let $x, y \in \mathcal{S}$, $\epsilon \in A$, $\widetilde{t}_1, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0, 1], t_2, \delta_2, \gamma_2 \in [0, 1)$ such that

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(xy), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r\min\left\{\left\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\Psi}_{F(\epsilon)}(y)\right\}, \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(xy), \gamma_2\} > t_2 \geq \max\{\{\eta_{F(\epsilon)}(x), \eta_{F(\epsilon)}(y)\}, \delta_2\}.$$

Then,

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(xy), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \text{ implies that } \widetilde{\Psi}_{F(\epsilon)}(xy) \prec \widetilde{t}_1 \prec \widetilde{\gamma}_1$$

and

$$\min\{\eta_{F(\epsilon)}(xy), \gamma_2\} > t_2 \text{ implies that } \eta_{F(\epsilon)}(xy) > t_2 > \gamma_2.$$

Thus, $(xy)_{(\widetilde{t}_1,t_2)}\overline{\in_\Gamma \vee q_\Delta}F(\epsilon)$. On the other hand, if

$$\widetilde{t}_1 \preceq r\min\left\{\left\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\Psi}_{F(\epsilon)}(y)\right\}, \widetilde{\delta}_1\right\}$$

and

$$t_2 \geq \max\{\{\eta_{F(\epsilon)}(x), \eta_{F(\epsilon)}(y)\}, \delta_2\}$$

we have $\widetilde{\Psi}_{F(\epsilon)}(x) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$, $\widetilde{\Psi}_{F(\epsilon)}(y) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\eta_{F(\epsilon)}(x) \leq t_2 < \gamma_2$, $\eta_{F(\epsilon)}(y) \leq t_2 < \gamma_2$, then $(x)_{(\widetilde{t}_1,t_2)} \in_\Gamma F(\epsilon)$ and $(y)_{(\widetilde{t}_1,t_2)} \in_\Gamma F(\epsilon)$ but $(xy)_{(\widetilde{t}_1,t_2)}\overline{\in_\Gamma \vee q_\Delta} F(\epsilon)$. This is contradiction to the hypothesis. Hence,

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(xy), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\left\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\Psi}_{F(\epsilon)}(y)\right\}, \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(xy), \gamma_2\} \leq \max\{\{\eta_{F(\epsilon)}(x), \eta_{F(\epsilon)}(y)\}, \delta_2\},$$

Conversely, let there exist $x \in \mathcal{S}$, $\epsilon \in A, \widetilde{t}, \widetilde{t}_1 \in D(0,1], s, s_1 \in [0,1)$ such that $(x)_{(\widetilde{t},s)} \in_\Gamma F(\epsilon), y \in \mathcal{S}$ such that $(y)_{(\widetilde{t}_1,s_1)} \in_\Gamma F(\epsilon)$. This shows that, $\widetilde{\Psi}_{F(\epsilon)}(x) \succeq \widetilde{t} \succ \widetilde{\gamma}_1, \eta_{F(\epsilon)}(x) \leq s < \gamma_2$ and $\widetilde{\Psi}_{F(\epsilon)}(y) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1, \eta_{F(\epsilon)}(y) \leq s_1 < \gamma_2$. So

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(xy), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\left\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\Psi}_{F(\epsilon)}(y)\right\}, \widetilde{\delta}_1\right\} \succeq r\min\{\{\widetilde{t}, \widetilde{t}_1\}, \widetilde{\delta}_1\}$$

and

$$\min\{\eta_{F(\epsilon)}(xy), \gamma_2\} \leq \max\{\{\eta_{F(\epsilon)}(x), \eta_{F(\epsilon)}(y)\}, \delta_2\} \leq \max\{\{s, s_1\}, \delta_2\}.$$

Now we discuss the following cases:

(1) If $r\min\{\widetilde{t}, \widetilde{t}_1\} \preceq \widetilde{\delta}_1$ and $\max\{s, s_1\} \geq \delta_2$, then $\widetilde{\Psi}_{F(\epsilon)}(xy) \succeq r\min\{\widetilde{t}, \widetilde{t}_1\} \succ \widetilde{\gamma}_1$ and $\eta_{F(\epsilon)}(xy) \leq \max\{s, s_1\} < \gamma_2$. This shows that $(xy)_{\langle r\min\{\widetilde{t},\widetilde{t}_1\},\max\{\{s,s_1\}\rangle} \in_\Gamma F(\epsilon)$.

(2) If $r\min\{\widetilde{t}, \widetilde{t}_1\} \succ \widetilde{\delta}_1$ and $\max\{s, s_1\} < \delta_2$, then $\widetilde{\Psi}_{F(\epsilon)}(xy) + r\min\{\widetilde{t}, \widetilde{t}_1\} \succ 2\widetilde{\delta}_1$ and $\eta_{F(\epsilon)}(xy) + \max\{\{s, s_1\} < 2\delta_2$. This shows that $(xy)_{(r\min\{\widetilde{t},\widetilde{t}_1\},\max\{s,s_1\})} q_\Delta F(\epsilon)$.

From both cases, we get $(xy)_{(r\min\{\widetilde{t}_1,\widetilde{t}_2\},\max\{s_1,s_2\})} \in_\Gamma \vee q_\Delta F(\epsilon)$. Hence, (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{S}$. ∎

Definition 10 A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity is called an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left (resp., right) ideal over $\mathcal{S}$ if for all $x, y \in \mathcal{S}$, $y_{(\widetilde{t},s)} \in_\Gamma F(\epsilon)$ implies $(xy)_{(\widetilde{t},s)} \in_\Gamma \vee q_\Delta F(\epsilon)$ (resp., $x_{(\widetilde{t},s)} \in_\Gamma F(\epsilon)$ implies $(xy)_{(\widetilde{t},s)} \in_\Gamma \vee q_\Delta F(\epsilon)$), where $\epsilon \in A, \widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$ and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.

Theorem 2 *Let A be a non-empty subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$, then for an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left (resp., right) ideal the following are equivalent.*

(1) $\Sigma(\mathcal{S}, A) \odot (F, A) \subset_{(\Gamma,\Delta)} (F, A)$

(2) $x_{(\widetilde{\alpha},\, \beta)} \in_\Gamma F(\epsilon) \Longrightarrow (yx)_{(\widetilde{\alpha},\, \beta)} \in_\Gamma F(\epsilon)$.

(3) $r\max\left\{(\widetilde{\Psi}_{F(\epsilon)}(yx), \widetilde{\gamma}_1\right\} \succeq r\min\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\delta}_1\}$ *and* $\min\{\eta_{F(\epsilon)}(yx), \gamma_2\} \leq \max\{\eta_{F(\epsilon)}(x), \delta_2\}$.

Proof. (1) $\Longrightarrow$ (2): Let $x_{(\widetilde{\alpha},\,\beta)} \in_{(\widetilde{\gamma}_1,\gamma_2)} F(\epsilon)$, then $\widetilde{\Psi}_{F(\epsilon)}(x) \succeq \widetilde{\alpha} \succeq \widetilde{\gamma}_1$ and $\eta_{F(\epsilon)}(x) \le \beta < \gamma_2$. Then,

$$\begin{aligned} r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(yx), \widetilde{\gamma}_1\right\} &\succeq r\min\{(\widetilde{\Psi}_{\mathcal{X}_{\mathcal{S}}^{(\Gamma,\Delta)}} \circ \widetilde{\Psi}_{F(\epsilon)})(yx),\}, \widetilde{\delta}_1\} \\ &= r\min\{r\sup_{yx=ab}\{r\min\{\widetilde{\Psi}_{\mathcal{X}_{\mathcal{S}}^{(\Gamma,\Delta)}}(a), \widetilde{\Psi}_{F(\epsilon)}(b)\}\}, \widetilde{\delta}_1\} \\ &= r\sup_{yx=ab}[r\min\{\widetilde{\Psi}_{\mathcal{X}_{\mathcal{S}}^{(\Gamma,\Delta)}}(a), \widetilde{\Psi}_{F(\epsilon)}(b), \widetilde{\delta}_1\} \\ &\succeq r\min\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\delta}_1\} \\ &\succeq r\min\{\widetilde{\alpha}, \widetilde{\delta}_1\} \succeq \widetilde{\gamma}_1. \end{aligned}$$

Thus, $\widetilde{\Psi}_{F(\epsilon)}(yx) \succeq r\min\{\widetilde{\alpha}, \widetilde{\delta}_1\} \succeq \tilde{\gamma}_1$ and $\widetilde{\alpha} \preceq \tilde{\gamma}_1$.

Case 1. $\widetilde{\alpha} \preceq \widetilde{\delta}_1$, this gives that $\widetilde{\Psi}_{F(\epsilon)}(yx) \succeq \widetilde{\alpha} \succ \tilde{\gamma}_1$.

Case 2. $\widetilde{\alpha} \succ \widetilde{\delta}_1$, thus $\widetilde{\Psi}_{F(\epsilon)}(yx) + \widetilde{\alpha} \succ 2\widetilde{\delta}_1$. Moreover,

$$\begin{aligned} \min\left\{(\eta_{F(\epsilon)}(yx), \gamma_2\right\} &\le \max\{(\widetilde{\Psi}_{\mathcal{X}_{\mathcal{S}}^{(\Gamma,\Delta)}} \circ \eta_{F(\epsilon)})(yx),\}, \delta_2\} \\ &= \max\{\min\{\max_{yx=ab}\{\widetilde{\Psi}_{\mathcal{X}_{\mathcal{S}}^{(\Gamma,\Delta)}}(a), \eta_{F(\epsilon)}(b)\}\}, \delta_2\} \\ &= \max_{yx=ab}[\min\{\eta_{F(\epsilon)}(b), \delta_2\} \\ &\le \max[\min\{\eta_{F(\epsilon)}(x), \delta_2\}] \\ &\le \max\{\beta, \delta_2\} < \gamma_2. \end{aligned}$$

Thus, $\eta_{F(\epsilon)}(yx) \le \max\{\beta, \delta_2\}$.

Case 3. If $\beta \le \delta_2$, then $\eta_{F(\epsilon)}(yx) \le \delta_2$.

Case 4. $\beta > \delta_2$, then $\eta_{F(\epsilon)}(yx) + \beta < 2\delta_2$.

Hence $(yx)_{(\widetilde{\alpha},\,\beta)} \in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)} F(\epsilon)$.

(2) $\Longrightarrow$ (3) : is similar as in [10].

(3) $\Longrightarrow$ (1): Assume that on contrary that $\Sigma(\mathcal{S}, A) \odot (F, A) \overline{\subset_{(\Gamma,\Delta)}} (F, A)$. Then there exist $\epsilon \in A$ and $x_{(\widetilde{\alpha},\,\beta)} \in_{(\widetilde{\gamma}_1,\gamma_2)} (\widetilde{\Psi}_{\mathcal{X}_{\mathcal{S}}^{(\Gamma,\Delta)}} \circ F)(\epsilon)$ such that

$$(x)_{(\widetilde{\alpha},\,\beta)} \overline{\in_{(\widetilde{\gamma}_1,\gamma_2)} \vee q_{(\widetilde{\delta}_1,\delta_2)}} F(\epsilon),$$

where $F(\epsilon) = \langle \widetilde{\Psi}_{F(\epsilon)}, \eta_{F(\epsilon)} \rangle$. Thus,

$$\begin{gathered} (x)_{(\widetilde{\alpha},\,\beta)} \overline{\in_{(\widetilde{\gamma}_1,\gamma_2)}} F(\epsilon) \text{ and } (x)_{(\widetilde{\alpha},\,\beta)} \overline{q_{(\widetilde{\delta}_1,\delta_2)}} F(\epsilon), \text{ therefore} \\ \widetilde{\Psi}_{F(\epsilon)} \preceq \widetilde{\alpha} \succ \widetilde{\gamma}_1 \ \&\ \eta_{F(\epsilon)} \ge \beta \text{ and } \widetilde{\Psi}_{F(\epsilon)} + \widetilde{\alpha} \preceq 2\widetilde{\delta}_1 \ \&\ \eta_{F(\epsilon)} + \beta > 2\delta_1. \end{gathered}$$

Therefore,

$$\widetilde{\Psi}_{F(\epsilon)} + \widetilde{\alpha} \preceq 2\widetilde{\delta}_1 \text{ and } \widetilde{\Psi}_{F(\epsilon)} - \widetilde{\alpha} \preceq \widetilde{\mathbf{0}}, \text{ also } \eta_{F(\epsilon)} - \beta \ge 1 \text{ and } \eta_{F(\epsilon)} + \beta > 2\delta_1$$

Thus we get,

$$\widetilde{\Psi}_{F(\epsilon)} \preceq \widetilde{\delta}_1 \text{ and } \eta_{F(\epsilon)} > \delta_1.$$

Therefore,

$$\begin{aligned}\widetilde{\delta}_1 &\succeq r\max\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\gamma}_1\} = r\max\{r\min\{\widetilde{\Psi}_{F(\epsilon)}(ab), \widetilde{\gamma}_1\}, \widetilde{\delta}_1\} \\ &= r\min\{r\max\{\widetilde{\Psi}_{F(\epsilon)}(ab), \widetilde{\gamma}_1\}, \widetilde{\delta}_1\} \\ &\succeq r\min\{r\min\{\widetilde{\Psi}_{F(\epsilon)}(b), \widetilde{\delta}_1\}, \widetilde{\delta}_1\} \\ &= r\min\{\widetilde{\Psi}_{F(\epsilon)}(b), \widetilde{\delta}_1\}.\end{aligned}$$

Thus, $r\max\{\widetilde{\Psi}_{F(\epsilon)}(x), \widetilde{\gamma}_1\} \succeq \widetilde{\Psi}_{F(\epsilon)}(b)$. Therefore

$$\begin{aligned}\widetilde{\alpha} &\preceq (\widetilde{\Psi}_{\mathcal{X}_{\mathcal{S}}^{(\Gamma,\Delta)}} \circ \widetilde{\Psi}_{F(\epsilon)})(x) = r\max\{r\min\{\widetilde{\Psi}_{\mathcal{X}_{\mathcal{S}}^{(\Gamma,\Delta)}}(a), \widetilde{\Psi}_{F(\epsilon)}(b)\}\} \\ &= r\max\{\widetilde{\Psi}_{F(\epsilon)}(b)\} \preceq r\max\{\widetilde{\Psi}_{F(\epsilon)}(b), \widetilde{\gamma}_1\}.\end{aligned}$$

This is a contradiction.

Hence, $\Sigma(\mathcal{S}, A) \odot (F, A) \subset_{(\Gamma,\Delta)} (F, A)$. ■

Lemma 4 *Let $\emptyset \neq A \subseteq \mathcal{S}$. Then, A is a left (right) ideal of $\mathcal{S}$ if and only if cubic characteristic function $\mathcal{X}_A^{(\Gamma,\Delta)} = \left\langle \widetilde{\Psi}_{\mathcal{X}_A^{(\Gamma,\Delta)}}, \eta_{\mathcal{X}_A^{(\Gamma,\Delta)}} \right\rangle$ of $A = \left\langle \widetilde{\Psi}_A, \eta_A \right\rangle$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic left (right) ideal of $\mathcal{S}$, where $\widetilde{\delta}_1, \widetilde{\gamma}_1 \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1]$ such that $\delta_2 < \gamma_2$.*

Proof. It is same as in [10]. ■

Let us now define the $\in_\Gamma \vee q_\Delta$-cubic level set for the cubic set $\Xi = \left(\widetilde{\psi}_\Xi, \eta_\Xi\right)$ as

$$[\Xi]_{(\widetilde{t},\delta)} = \{x \in \mathcal{S} : x_{(\widetilde{t},\delta)} \in_\Gamma \vee q_\Delta \Xi\}.$$

Theorem 3 *A cubic set $\Xi = \left(\widetilde{\Psi}_\Xi, \eta_\Xi\right)$ is said to be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic $\mathcal{AG}$-subgroupoid (left ideal, right ideal) of $\mathcal{S}$ if and only if $\emptyset \neq [\Xi]_{(\widetilde{t},\delta)}$ is a $\mathcal{AG}$-subgroupoid (left ideal, right ideal) of $\mathcal{S}$.*

Proof. It is same as in [10]. ■

Theorem 4 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid and $P \subseteq \mathcal{S}$. Then, P is a left (resp., right) ideal of $\mathcal{S}$ if and only if $\Sigma(P, A)$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft left (resp., right) ideal over $\mathcal{S}$ for any $P \subseteq E$.*

Proof. It follows from Theorem 3. ■

Example 1 Let $\mathcal{S}= \{a, b, c\}$ and the binary operation " $\cdot$ " on $\mathcal{S}$ be defined as follows:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

Then, $\mathcal{S}$ is an $\mathcal{AG}$-groupoid. For $A = \{e_1, e_2, e_3\} \subseteq E$, the cubic soft set $(F, A) = \{F(e_1), F(e_2), F(e_3)\}$ over $\mathcal{S}$ is defined as follows:

$$\begin{aligned} F(e_1) &= \{\langle a, [0.2, 0.3), 0.1\rangle, \langle b, [0.3, 0.4), 0.2\rangle, \langle c, [0.4, 0.5), 0.3\rangle\}, \\ F(e_2) &= \{\langle a, [0.15, 0.16), 0.2\rangle, \langle b, [0.16, 0.18), 0.3\rangle, \langle c, [0.18, 0.2), 0.3\rangle\}, \\ F(e_3) &= \{\langle a, [0.1, 0.12), 0.1\rangle, \langle b, [0.12, 0.14), 0.2\rangle, \langle c, [0.14, 0.16), 0.2\rangle\}, \end{aligned}$$

such that $\widetilde{\gamma}_1 = [0.1, .18) \prec \widetilde{\delta}_1 = [0.3, 0.4)$ and $\delta_2 = 0.3 < \gamma_2 = 0.4$. Then, it is easy to see that (F, A) is an $(\in_{([0.1,.18),0.4)}, \in_{([0.1,.18),0.4)} \vee q_{([0.3,0.4),0.3)})$-cubic soft $\mathcal{AG}$-subgroupoid over $\mathcal{S}$.

Example 2 Let $\mathcal{S}= \{a, b, c\}$ and the binary operation " $\cdot$ " on $\mathcal{S}$ be defined as follow:

$\cdot$	a	b	c
a	a	a	a
b	a	a	c
c	a	a	a

Then, $\mathcal{S}$ is an $\mathcal{AG}$-groupoid. For $A = \{e_1, e_2, e_3\} \subseteq E$, the cubic soft set $(F, A) = \{F(e_1), F(e_2), F(e_3)\}$ over $\mathcal{S}$ is defined as follows.

$$\begin{aligned} F(e_1) &= \{\langle a, [0.3, 0.4), 0.5\rangle, \langle b, [0.3, 0.4), 0.5\rangle, \langle c, [0.4, 0.5), 0.4\rangle\}, \\ F(e_2) &= \{\langle a, [0.2, 0.15), 0.4\rangle, \langle b, [0.15, 0.18), 0.3\rangle, \langle c, [0.18, 0.2), 0.1\rangle\}, \\ F(e_3) &= \{\langle a, [0.1, 0.15), 0.3\rangle, \langle b, [0.15, 0.20), 0.3\rangle, \langle c, [0.20, 0.25), 0.2\rangle\}, \end{aligned}$$

such that $\widetilde{\gamma}_1 = [0.1, .18) \prec \widetilde{\delta}_1 = [0.19, 0.2)$ and $\delta_2 = 0.53 < \gamma_2 = 0.54$. Then, it is easy to see that (F, A) is an $(\in_{([0.1,.18),0.54)}, \in_{([0.1,.18),0.54)} \vee q_{([0.19,0.2),0.53)})$-cubic soft right ideal over $\mathcal{S}$.

Definition 11 A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime over $\mathcal{S}$ if for all $x \in \mathcal{S}$, $\epsilon \in A, \widetilde{t}_1 \in D(0, 1]$ and $s_1 \in [0, 1)$ we have $x^2_{(\widetilde{t}_1, s_1)} \in_\Gamma F(\epsilon)$ implies that $x_{(\widetilde{t}_1, s_1)} \in_\Gamma F(\epsilon)$, where $\widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$.

Theorem 5 *A cubic soft set (F, A) over an $\mathcal{AG}$-groupoid $\mathcal{S}$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime if and only if for all $a \in \mathcal{S}$*

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(a^2), \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(a), \gamma_2\} \leq \min\{\eta_{F(\epsilon)}(a^2), \delta_2\},$$

where $\widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0, 1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0, 1)$ such that $\delta_2 < \gamma_2$.

Proof. Let (F, A) be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime. Assume that there exist $a \in \mathcal{S}$,$\epsilon \in A, \widetilde{t}_1, \widetilde{\delta}_1 \in D(0, 1], t_2, \delta_2 \in [0, 1)$, such that

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(a), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(a^2), \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(a), \gamma_2\} > t_2 \geq \min\{\eta_{F(\epsilon)}(a^2), \delta_2\},$$

Then,

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(a), \widetilde{\gamma}_1\right\} \prec \widetilde{t}_1 \text{ implies that } \widetilde{\Psi}_{F(\epsilon)}(a) \prec \widetilde{t}_1 \prec \widetilde{\gamma}_1$$

and

$$\min\{\eta_{F(\epsilon)}(a), \gamma_2\} > t_2 \text{ implies that } \eta_{F(\epsilon)}(a) > t_2 > \gamma_2.$$

Thus, $(a)_{(\widetilde{t}_1, t_2)}\overline{\in_\Gamma \vee q_\Delta}F(\epsilon)$. On the other hand if

$$\widetilde{t}_1 \preceq r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(a^2), \widetilde{\delta}_1\right\}$$

and

$$t_2 \geq \max\{\eta_{F(\epsilon)}(a^2), \delta_2\}$$

we have, $\widetilde{\Psi}_{F(\epsilon)}(a^2) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\eta_{F(\epsilon)}(a^2) \leq t_2 < \gamma_2$ this implies that, $(a^2)_{(\widetilde{t}_1, t_2)} \in_\Gamma \vee q_\Delta F(\epsilon)$ but $(a)_{(\widetilde{t}_1, t_2)}\overline{\in_\Gamma \vee q_\Delta}F(\epsilon)$. This is contradiction to the hypothesis. Hence,

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(a^2), \widetilde{\delta}_1\right\}$$

and

$$\min\{\eta_{F(\epsilon)}(a), \gamma_2\} \leq \max\{\{\eta_{F(\epsilon)}(a^2), \delta_2\}.$$

Conversely, let there exist $x \in \mathcal{S}$, $\epsilon \in A, \widetilde{t}_1 \in D(0,1], t_2 \in [0,1)$ such that $(a^2)_{(\widetilde{t}_1, t_2)} \in_\Gamma F(\epsilon)$. This implies that, $\widetilde{\Psi}_{F(\epsilon)}(a^2) \succeq \widetilde{t} \succ \widetilde{\gamma}_1$and $\eta_{F(\epsilon)}(a^2) \leq t_2 < \gamma_2$. So

$$r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(a^2), \widetilde{\delta}_1\right\} \succeq r\min\{\widetilde{t}_1, \widetilde{\delta}_1\}$$

and

$$\min\{\eta_{F(\epsilon)}(a), \gamma_2\} \leq \max\{\eta_{F(\epsilon)}(a^2), \delta_2\} \leq \max\{t_2, \delta_2\}.$$

Now we have the following cases:

(1) If $\widetilde{t}_1 \preceq \widetilde{\delta}_1$ and $t_2 \geq \delta_2$, then $\widetilde{\Psi}_{F(\epsilon)}(a) \succeq \widetilde{t}_1 \succ \widetilde{\gamma}_1$ and $\eta_{F(\epsilon)}(a) \leq t_2 < \gamma_2$. This implies that, $(a)_{(\widetilde{t}_1, t_2)} \in_\Gamma F(\epsilon)$.

(2) If $\widetilde{t}_1 \succ \widetilde{\delta}_1$ and $t_2 < \delta_2$, then $\widetilde{\Psi}_{F(\epsilon)}(a) + \widetilde{t}_1 \succ 2\widetilde{\delta}_1$ and $\eta_{F(\epsilon)}(a) + t_2 < 2\delta_2$. This implies that, $(a)_{(\widetilde{t}_1, t_2)} q_\Delta F(\epsilon)$.

From both cases we get, $(a)_{(\widetilde{t}_1, t_2)} \in_\Gamma \vee q_\Delta F(\epsilon)$.

Hence, (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime. ■

Example 3 Let $\mathcal{S}= \{1, 2, 3\}$ and the binary operation " $\cdot$ " on $\mathcal{S}$ be defined as follows:

$\cdot$	1	2	3
1	2	2	2
2	2	2	2
3	1	2	2

Then, $\mathcal{S}$ is an $\mathcal{AG}$-groupoid. For $A = \{e_1, e_2\} \subseteq E$, the cubic soft set $(F, A) = \{F(e_1), F(e_2)\}$ over $\mathcal{S}$ is defined as follows:

$$\begin{aligned} F(e_1) &= \{\langle 1, [0.5, 0.8), 0.6\rangle, \langle 2, [0.1, 0.7), 0.8\rangle, \langle 3, [0.2, 0.6), 0.9\rangle\}, \\ F(e_2) &= \{\langle 1, [0.2, 0.5), 0.5\rangle, \langle 2, [0.5, 0.6), 0.3\rangle, \langle 3, [0.6, 0.7), 0.1\rangle\}, \end{aligned}$$

such that $\widetilde{\gamma}_1 = [0.1, .18) \prec \widetilde{\delta}_1 = [0.19, 0.2)$ and $\delta_2 = 0.53 < \gamma_2 = 0.54$. Then, it is easy to verify that (F, A) is an $(\in_{([0.1,.18),0.54)}, \in_{([0.1,.18),0.54)} \vee q_{([0.19,0.2),0.53)})$-cubic soft semiprime ideal over $\mathcal{S}$.

Theorem 6 *For a right ideal R of an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, the following conditions are equivalent:*

(i) *R is semiprime.*

(ii) *$\mathcal{X}_R^{(\Gamma,\Delta)}$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime, where $\widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.*

Proof. (i) $\Rightarrow$ (ii) : Let R be a semiprime ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$. Let a be an arbitrary element of $\mathcal{S}$ such that $a \in R$. Then $a^2 \in R$. Hence, $\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a) \succeq \widetilde{\delta}_1$, $\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a) \leq \delta_2$ and $\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \succeq \widetilde{\delta}_1, \eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \leq \delta_2$. This implies that,

$$r\max\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \widetilde{\delta}_1\right\}$$

and

$$\min\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \gamma_2\right\} \leq \max\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \delta_2\right\}.$$

Now let $a \notin R$. Since R is semiprime we have $a^2 \notin R$. This implies that,

$$\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a) \preceq \widetilde{\gamma}_1, \widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \preceq \widetilde{\gamma}_1$$

and

$$\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a) \geq \gamma_2, \eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \geq \gamma_2.$$

Hence,

$$\begin{aligned} r\max\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \widetilde{\gamma}_1\right\} &\succeq r\min\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \widetilde{\delta}_1\right\} \\ \min\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \gamma_2\right\} &\leq \max\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \delta_2\right\}, \end{aligned}$$

for all $a \in \mathcal{S}, \widetilde{\delta}_1, \widetilde{\gamma}_1, \in D(0,1]$ such that $\widetilde{\gamma}_1 \prec \widetilde{\delta}_1$, and $\delta_2, \gamma_2 \in [0,1)$ such that $\delta_2 < \gamma_2$.

(ii) $\Rightarrow$ (i) : Let $\mathcal{X}_R^{(\Gamma,\Delta)}$ be cubic soft semiprime. If $a^2 \in R$, for some $a \in \mathcal{S}$, then $\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \succeq \widetilde{\delta}_1$ and $\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2) \leq \delta_2$. Since $\mathcal{X}_R^{(\Gamma,\Delta)}$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime, we have

$$r\max\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \widetilde{\gamma}_1\right\} \succeq r\min\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \widetilde{\delta}_1\right\}$$

and

$$\min\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a), \gamma_2\right\} \leq \max\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a^2), \delta_2\right\}.$$

Therefore, $r\max\left\{\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a),\widetilde{\gamma}_1\right\}\succeq\widetilde{\delta}_1$ and $\min\left\{\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a),\gamma_2\right\}\leq\delta_2$. But $\widetilde{\gamma}_1\prec\widetilde{\delta}_1$ and $\delta_2<\gamma_2$, so $\widetilde{\Psi}_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a)\succeq\widetilde{\delta}_1$ and $\eta_{\mathcal{X}_R^{(\Gamma,\Delta)}}(a)\leq\delta_2$. Thus $a\in R$. Hence, R is semiprime. ∎

6. Characterizations of intra-regular $\mathcal{AG}$-groupoids

An element r of an $\mathcal{AG}$-groupoid $\mathcal{S}$ is called **intra-regular** if there exist elements $s,t\in\mathcal{S}$ such that $r=(sr^2)t$ and $\mathcal{S}$ is called **intra-regular**, if every element of $\mathcal{S}$ is intra-regular.

Theorem 7 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity. Then, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is intra-regular.*

(ii) *For a right ideal R of an $\mathcal{AG}$-groupoid $\mathcal{S}$, $R\subseteq R^2$ and R is semiprime.*

(iii) *For an $(\in_\Gamma,\in_\Gamma\vee q_\Delta)$-cubic soft right ideal (F,A) over $\mathcal{S}$,*
$(F,A)\subseteq\vee q_{(\Gamma,\Delta)}(F,A)\odot(F,A)$ and
(F,A) is an $(\in_\Gamma,\in_\Gamma\vee q_\Delta)$-cubic soft semiprime.

Proof. (i) $\Rightarrow$ (iii) : Let (F,A) be an $(\in_\Gamma,\in_\Gamma\vee q_\Delta)$-cubic soft right ideal of intra-regular $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity. Since $\mathcal{S}$ is intra-regular so for any $r\in\mathcal{S}$ there exist $s,t\in\mathcal{S}$ such that $r=(sr^2)t$. Since $\mathcal{S}=\mathcal{S}^2$, so for each t in $\mathcal{S}$ there exists t_1,t_2 in $\mathcal{S}$ such that $t=t_1t_2$, then

$$r=(sr^2)t=(sr^2)(t_1t_2)=(t_2t_1)(r^2s)=r^2[(t_2t_1)s]=[r(t_2t_1)](rs).$$

If $r=bc$. Then,

$$\begin{aligned}
&r\max\left\{(\widetilde{\Psi}_{F(\epsilon)}\circ\widetilde{\Psi}_{F(\epsilon)})(r),\widetilde{\gamma}_1\right\}\\
&\quad=r\max\left[r\sup_{r=bc}\left\{r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(b),\widetilde{\Psi}_{F(\epsilon)}(c)\right\}\right\},\widetilde{\gamma}_1\right]\\
&\quad\succeq r\max\left[r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(r(t_2t_1)),\widetilde{\Psi}_{F(\epsilon)}(rs)\right\},\widetilde{\gamma}_1\right]\\
&\quad=r\min\left[r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(r(t_2t_1)),\widetilde{\gamma}_1\right\},r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(rs),\widetilde{\gamma}_1\right\}\right]\\
&\quad\succeq r\min\left[r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(r),\widetilde{\delta}_1\right\},r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(r),\widetilde{\delta}_1\right\}\right]\\
&\quad=r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(r),\widetilde{\delta}_1\right\}
\end{aligned}$$

and

$$\begin{aligned}
\min\left\{(\eta_{F(\epsilon)}\circ\eta_{F(\epsilon)})(r),\gamma_2\right\}&=\min\left[r\min_{r=bc}\left\{\max\left\{\eta_{F(\epsilon)}(b),\eta_{F(\epsilon)}(c)\right\}\right\},\gamma_2\right]\\
&\leq\min\left[\max\left\{\eta_{F(\epsilon)}(r(t_2t_1)),\eta_{F(\epsilon)}(rs)\right\},\gamma_2\right]\\
&=\max\left[\min\left\{\eta_{F(\epsilon)}(r(t_2t_1)),\gamma_2\right\},\min\left\{\eta_{F(\epsilon)}(rs),\gamma_2\right\}\right]\\
&\leq\max\left[\max\left\{\eta_{F(\epsilon)}(r),\delta_2\right\},\max\left\{\eta_{F(\epsilon)}(r),\delta_2\right\}\right]\\
&=\max\left\{\eta_{F(\epsilon)}(r),\delta_2\right\}.
\end{aligned}$$

Thus, by Lemma 1, $(F, A) \subseteq \vee q_{(\Gamma,\Delta)}(F, A) \odot (F, A)$. Next we show that (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime. Since

$$\begin{aligned} r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\gamma}_1\right\} &= r \max\left\{\widetilde{\Psi}_{F(\epsilon)}(r^2[(t_2t_1)s]), \widetilde{\gamma}_1\right\} \\ &\succeq r \min\left\{\widetilde{\Psi}_{F(\epsilon)}(r^2), \widetilde{\delta}_1\right\}. \end{aligned}$$

and

$$\begin{aligned} \min\{\eta_{F(\epsilon)}(r), \gamma_2\} &= \min\{\eta_{F(\epsilon)}(r^2[(t_2t_1)s]), \gamma_2\} \\ &\leq \max\{\eta_{F(\epsilon)}(r^2), \delta_2\}. \end{aligned}$$

Hence, (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime.

(iii) $\Rightarrow$ (ii) : Let R be a right ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$, then by Theorem 4, $\Sigma(R, E)$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal over $\mathcal{S}$ by (iii) $\mathcal{X}_R^{(\Gamma,\Delta)}$ is cubic soft semiprime. Now by using assumption we have $\Sigma(R, E) \subset_{(\Gamma,\Delta)} \Sigma(R, E) \odot \Sigma(R, E)$. Now by using Lemma 3, we have

$$\mathcal{X}_R^{(\Gamma,\Delta)} = \mathcal{X}_{R\cap R}^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_R^{(\Gamma,\Delta)} \cap \mathcal{X}_R^{(\Gamma,\Delta)} \subseteq \vee q_{(\Gamma,\Delta)} \mathcal{X}_R^{(\Gamma,\Delta)} \odot \mathcal{X}_R^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_{R^2}^{(\Gamma,\Delta)}$$

Hence, by Lemma 3, $R \subseteq R^2$.

(ii) $\Rightarrow$ (i) : It is easy. ∎

Theorem 8 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity. Then, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is intra-regular.*

(ii) *For any right ideal R and for any subset A of an $\mathcal{AG}$-groupoid $\mathcal{S}$, $R \cap A \subseteq RA$ and R is semiprime.*

(iii) *For any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal (F, A) and for any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set (G, B) over $\mathcal{S}$, $(F, A)\widetilde{\cap}(G, B) \subseteq \vee q_{(\Gamma,\Delta)}(F, A) \odot (G, B)$ and (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime.*

Proof. (i) $\Rightarrow$ (iii) : Let $\mathcal{S}$ be intra-regular, (F, A) be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal and (G, B) be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set over $\mathcal{S}$, respectively. Now let a be any element of $\mathcal{S}$, $\epsilon \in A \cup B$ and $(F, A)\widetilde{\cap}(G, B) = (H, A \cup B)$. We consider the following cases.

Case 1: $\epsilon \in A - B$. Then $H(\epsilon) = F(\epsilon) = (F \circ G)(\epsilon)$.

Case 2: $\epsilon \in B - A$. Then $H(\epsilon) = G(\epsilon) = (F \circ G)(\epsilon)$.

Case 3: $\epsilon \in A \cap B$. Then $H(\epsilon) = F(\epsilon) \cap G(\epsilon)$ and $(F \circ G)(\epsilon) = F(\epsilon) \circ G(\epsilon)$. Now we show that $F(\epsilon) \cap G(\epsilon) \subseteq \vee q_{((\Gamma,\Delta))} F(\epsilon) \circ G(\epsilon)$. Since $\mathcal{S}$ is intra-regular, then for any $r \in \mathcal{S}$ there exist $s, t \in \mathcal{S}$ such that $r = (sr^2)t$. Then,

$$r = (sr^2)t = [(s(rr))t] = [(r(sr))t] = [(t(sr))]r.$$

$$\begin{aligned} t(sr) &= [t\{s((sr^2)t)\}] = [t\{(sr^2)(st)\}] = [(sr^2)(st^2)] \\ &= [(t^2s)(r^2s)] = r^2(t^2s^2). \end{aligned}$$

Thus, $r = [r^2(t^2s^2)]r$. If $r = pq$. Then

$$\begin{aligned} & r\max\left\{(\widetilde{\Psi}_{F(\epsilon)} \circ \widetilde{\Psi}_{G(\epsilon)})(r), \widetilde{\gamma}_1\right\} \\ &\quad = r\max\left[r\sup_{r=pq}\left\{r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(p), \widetilde{\Psi}_{G(\epsilon)}(q)\right\}\right\}, \widetilde{\gamma}_1\right] \\ &\quad \succeq r\max\left[r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(r^2(t^2s^2)), \widetilde{\Psi}_{G(\epsilon)}(r)\right\}, \widetilde{\gamma}_1\right] \\ &\quad = r\min\left[r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(r^2(t^2s^2)), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\gamma}_1\right\}\right] \\ &\quad \succeq r\min\left[r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\delta}_1\right\}\right] \\ &\quad = r\min\left\{(\widetilde{\Psi}_{F(\epsilon)} \cap \widetilde{\Psi}_{G(\epsilon)})(r), \widetilde{\delta}_1\right\} \end{aligned}$$

and

$$\begin{aligned} \min\left\{(\eta_{F(\epsilon)} \circ \eta_{G(\epsilon)})(r), \gamma_2\right\} &= \min\left[r\min_{r=pq}\{\max\left\{\eta_{F(\epsilon)}(p), \eta_{G(\epsilon)}(q)\right\}\}, \gamma_2\right] \\ &\leq \min\left[\max\left\{\eta_{F(\epsilon)}(r^2(t^2s^2)), \eta_{G(\epsilon)}(r)\right\}, \gamma_2\right] \\ &= \max\left[\min\left\{\eta_{F(\epsilon)}(r^2(t^2s^2)), \gamma_2\right\}, \min\left\{\eta_{G(\epsilon)}(r), \gamma_2\right\}\right] \\ &\leq \max\left[\max\left\{\eta_{F(\epsilon)}(r), \delta_2\right\}, \max\left\{\eta_{G(\epsilon)}(r), \delta_2\right\}\right] \\ &= \max\left\{(\eta_{F(\epsilon)} \cap \eta_{G(\epsilon)})(r), \delta_2\right\}. \end{aligned}$$

Thus, by Lemma 1, $(F, A)\widetilde{\cap}(G, B) \subseteq \vee q_{(\Gamma,\Delta)}(F, A) \odot (G, B)$. The rest of the proof is similar as in Theorem 7.

(iii) $\Rightarrow$ (ii) : Let R be a right ideal and A be any subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$, then by Theorem 4, $\Sigma(R, E)$ and $\Sigma(A, E)$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal and $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set over $\mathcal{S}$, respectively. Now by using assumption we have $\Sigma(R, E) \cap \Sigma(A, E) \subset_{(\Gamma,\Delta)} \Sigma(R, E) \odot \Sigma(A, E)$. By using Lemma 3, we have

$$\mathcal{X}_{R\cap A}^{(\Gamma,\Delta)} = {}_{(\Gamma,\Delta)}\mathcal{X}_R^{(\Gamma,\Delta)} \cap \mathcal{X}_A^{(\Gamma,\Delta)} \subseteq \vee q_{(\Gamma,\Delta)} \mathcal{X}_R^{(\Gamma,\Delta)} \odot \mathcal{X}_A^{(\Gamma,\Delta)} = {}_{(\Gamma,\Delta)}\mathcal{X}_{RA}^{(\Gamma,\Delta)}.$$

By Lemma 3, $R \cap A \subseteq RA$.

(ii) $\Rightarrow$ (i) : It is easy. ∎

Theorem 9 *Let $\mathcal{S}$ be an $\mathcal{AG}$-groupoid with left identity. Then, the following conditions are equivalent.*

(i) *$\mathcal{S}$ is intra-regular.*

(ii) *For any right ideal R and for any subset A of an $\mathcal{AG}$-groupoid $\mathcal{S}$, $R\cap A \subseteq AR$ and R is semiprime.*

(iii) *For any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal (F, A) and for any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set (G, B) over $\mathcal{S}$, $(F, A)\widetilde{\cap}(G, B) \subseteq \vee q_{(\Gamma,\Delta)}(G, B) \odot (F, A)$ and (F, A) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime.*

Proof. (i) $\Rightarrow$ (iii) : Let (F,A) be an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal and (G,B) be any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set over an intra-regular $\mathcal{AG}$-groupoid $\mathcal{S}$. Now, let a be any element of $\mathcal{S}$, $\epsilon \in A \cup B$ and $(F,A)\widetilde{\cap}(G,B) = (H, A \cup B)$. We consider the following cases.

Case 1: $\epsilon \in A - B$. Then $H(\epsilon) = F(\epsilon) = (G \circ F)(\epsilon)$.

Case 2: $\epsilon \in B - A$. Then $H(\epsilon) = G(\epsilon) = (G \circ F)(\epsilon)$.

Case 3: $\epsilon \in A \cap B$. Then $H(\epsilon) = G(\epsilon) \cap F(\epsilon)$ and $(G \circ F)(\epsilon) = G(\epsilon) \cap F(\epsilon)$. Now we show that $F(\epsilon) \cap G(\epsilon) \subseteq \vee_{q(\Gamma,\Delta)} G(\epsilon) \circ F(\epsilon)$. Since $\mathcal{S}$ is intra-regular, then for any $r \in \mathcal{S}$ there exist $s,t \in \mathcal{S}$ such that $r = (sr^2)t$. Now we obtain

$$\begin{aligned} r &= (sr^2)t = (sr^2)(t_1t_2) = (t_2t_1)(r^2s) = r^2[(t_2t_1)s] \\ &= [s((t_2t_1)]r^2 = r[\{s(t_2t_1)\}r] = r[s\{(t_2t_1)\}\{(sr^2)t\}] \\ &= r[(sr^2)[\{s(t_2t_1)\}t]] = r[[t\{s(t_2t_1)\}](r^2s)] \\ &= r[r^2([t\{s(t_2t_1)\}]s] = r[(s[t\{s(t_2t_1)\}]r^2] \\ &= r[(s[t\{s(t_2t_1)\}])(rr)] \\ &= r[r((s[t\{s(t_2t_1)])r)] = r(ru), \text{ where } u = (s[t\{s(t_2t_1)\}])r. \end{aligned}$$

If $r = pq$, then

$$\begin{aligned} &r\max\left\{(\widetilde{\Psi}_{G(\epsilon)} \circ \widetilde{\Psi}_{F(\epsilon)})(r), \widetilde{\gamma}_1\right\} \\ &\qquad = r\max\left[r\sup_{r=pq}\left\{r\min\left\{\widetilde{\Psi}_{G(\epsilon)}(p), \widetilde{\Psi}_{F(\epsilon)}(q)\right\}\right\}, \widetilde{\gamma}_1\right] \\ &\qquad \succeq r\max\left[r\min\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\Psi}_{F(\epsilon)}(ru)\right\}, \widetilde{\gamma}_1\right] \\ &\qquad = r\min\left[r\max\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(ru), \widetilde{\gamma}_1\right\}\right] \\ &\qquad \succeq r\min\left[r\min\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\delta}_1\right\}\right] \\ &\qquad = r\min\left\{(\widetilde{\Psi}_{F(\epsilon)} \cap \widetilde{\Psi}_{G(\epsilon)})(r), \widetilde{\delta}_1\right\}, \end{aligned}$$

and

$$\begin{aligned} \min\left\{(\eta_{G(\epsilon)} \circ \eta_{F(\epsilon)})(r), \gamma_2\right\} &= \min\left[r\min_{r=pq}\max\left\{\eta_{G(\epsilon)}(p), \eta_{F(\epsilon)}(q)\right\}\}, \gamma_2\right] \\ &\leq \min\left[\max\left\{\eta_{G(\epsilon)}(r), \eta_{F(\epsilon)}(u)\right\}, \gamma_2\right] \\ &= \max\left[\min\left\{\eta_{G(\epsilon)}(r), \gamma_2\right\}, \min\left\{\eta_{F(\epsilon)}(ru), \gamma_2\right\}\right] \\ &\leq \max\left[\max\left\{\eta_{G(\epsilon)}(r), \delta_2\right\}, \max\left\{\eta_{F(\epsilon)}(r), \delta_2\right\}\right] \\ &= \max\left\{(\eta_{F(\epsilon)} \cap \eta_{G(\epsilon)})(r), \delta_2\right\}. \end{aligned}$$

Hence, by Lemma 1, $(F,A)\widetilde{\cap}(G,B) \subseteq \vee q_{(\Gamma,\Delta)}(G,B) \odot (F,A)$. For the remaining proof see Theorem 7.

(iii) $\Rightarrow$ (ii) : Let R be a right ideal and A be any subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$, then by Theorem 4, $\Sigma(R,E)$ and $\Sigma(A,E)$ is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal and $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set over $\mathcal{S}$, respectively. Now, by using assumption, we have $\Sigma(R,E) \cap \Sigma(A,E) \subset_{(\Gamma,\Delta)} \Sigma(R,E) \odot \Sigma(A,E)$. By using Lemma 3, we have

$$\begin{aligned}\mathcal{X}_{R\cap A}^{(\Gamma,\Delta)} &= \mathcal{X}_{A\cap R}^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_{A}^{(\Gamma,\Delta)} \cap \mathcal{X}_{R}^{(\Gamma,\Delta)} \\ &\subseteq \vee q_{(\Gamma,\Delta)} \mathcal{X}_{A}^{(\Gamma,\Delta)} \odot \mathcal{X}_{R}^{(\Gamma,\Delta)} =_{(\Gamma,\Delta)} \mathcal{X}_{AR}^{(\Gamma,\Delta)}.\end{aligned}$$

By Lemma 3, $R \cap A \subseteq AR$. The rest of the proof is similar to Theorem 7.

(ii) $\Rightarrow$ (i) : Sine $\mathcal{S}p^2$ is a right ideal of $\mathcal{S}$ containing p^2. By (ii), it is semiprime. Therefore,

$$\begin{aligned}p \in \mathcal{S}p^2 \cap \mathcal{S}p \subseteq (\mathcal{S}p)(\mathcal{S}p^2) &= (p^2\mathcal{S})(p\mathcal{S}) = [(pp)(\mathcal{S}\mathcal{S})](p\mathcal{S}) \\ &= [(\mathcal{S}\mathcal{S})(pp)](p\mathcal{S}) \subseteq (\mathcal{S}p^2)\mathcal{S}.\end{aligned}$$

Hence, $\mathcal{S}$ is intra-regular. ■

Theorem 10 *The following conditions are equivalent for an $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity:*

(i) *$\mathcal{S}$ is intra-regular.*

(ii) *For any subsets A, B and any right ideal R of $\mathcal{S}$, $A \cap B \cap R \subseteq (AB)R$ and R is semiprime ideal.*

(iii) *For any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft sets (F, A), (G, B) and for any $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$ -cubic soft right ideal (H, R) over $\mathcal{S}$, we have $(F, A)\widetilde{\cap}(G, B)\widetilde{\cap}(H, R) \subseteq \vee q_{(\Gamma,\Delta)}[(F, A) \odot (G, B)] \odot (H, R)$ and (H, R) is an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft semiprime ideal over $\mathcal{S}$.*

Proof. (i) $\Rightarrow$ (iii) Let a be any element of an intra-regular $\mathcal{AG}$-groupoid $\mathcal{S}$ with left identity, $[(F, A)\widetilde{\cap}(G, B)]\widetilde{\cap}(H, R) = (K, A \cup B \cup R)$. For any $\epsilon \in A \cup B \cup R$. We consider the following cases.

Case 1: $\epsilon \in A \backslash (B \cap R)$, then $F(\epsilon) = [(F \circ G) \circ H](\epsilon)$.

Case 2: $\epsilon \in B \backslash (A \cap R)$, then $G(\epsilon) = [(F \circ G) \circ H](\epsilon)$.

Case 3: $\epsilon \in R \backslash (A \cap B)$, then $H(\epsilon) = [(F \circ G) \circ H](\epsilon)$

Case 4: $\epsilon \in (A \cap B) \cap R$, then $[(F \circ G) \circ H](\epsilon) = [F(\epsilon) \circ G(\epsilon)] \circ H(\epsilon)$.

Since $\mathcal{S}$ is intra-regular, then for any $r \in \mathcal{S}$ there exist $s, t \in \mathcal{S}$ such that $r = (sr^2)t$. Then,

$$\begin{aligned}r &= (sr^2)t = (t_2t_1)(r^2s) = r^2[(t_2t_1)s] = [s(t_2t_1)]r^2 \\ &= r[\{s(t_2t_1)\}r] = r(pr), \text{ where } s(t_2t_1) = p, \text{ and} \\ pr &= p[(sr^2)t] = (sr^2)(pt) = [(tp)(r^2s)] \\ &= r^2[(tp)s] = [s(tp)](rr) = r[\{(tp)\}r] \\ &= [r(qr)], \text{ where } s(tp) = q, \text{ and} \\ qr &= q[(sr^2)t] = (sr^2)(qt) = (tq)(r^2s) \\ &= r^2[(tq)s].\end{aligned}$$

Thus,

$r = r[r(r^2c)] = r[r^2(rc)] = r^2[r(rc)]$, where $(tq)s = c$ and $s(tp) = q$ and $(t_2t_1) = p$

For any $r \in \mathcal{S}$ there exist $x, y \in \mathcal{S}$ such that $r = xy$, then

$$\begin{aligned}
& r\max\left\{(\widetilde{\Psi}_{F(\epsilon)} \circ \widetilde{\Psi}_{G(\epsilon)}) \circ \widetilde{\Psi}_{H(\epsilon)})(r), \widetilde{r}_1\right\} \\
&\qquad = r\max\left[r\sup_{r=xy}\left\{r\min\left\{\left(\widetilde{\Psi}_{F(\epsilon)} \circ \widetilde{\Psi}_{G(\epsilon)}\right)(x), \widetilde{\Psi}_{H(\epsilon)}(y)\right\}\right\}, \widetilde{\gamma}_1\right] \\
&\qquad \succeq r\max\left[r\min\left\{\left(\widetilde{\Psi}_{F(\epsilon)} \circ \widetilde{\Psi}_{G(\epsilon)}\right)(rr), \widetilde{\Psi}_{H(\epsilon)}(r(rc))\right\}, \widetilde{\gamma}_1\right] \\
&\qquad = r\max\left[r\sup_{rr=pq}\left\{r\min\left\{\begin{array}{c}\left\{\widetilde{\Psi}_{F(\epsilon)}(p), \widetilde{\Psi}_{G(\epsilon)}(q)\right\}, \\ \widetilde{\Psi}_{H(\epsilon)}(r(rc))\end{array}\right\}\right\}, \widetilde{\gamma}_1\right] \\
&\qquad \succeq r\max\left[r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\Psi}_{H(\epsilon)}(r(rc))\right\}, \widetilde{\gamma}_1\right] \\
&\qquad = r\min\left[\begin{array}{c} r\max\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\gamma}_1\right\}, r\max\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\gamma}_1\right\}, \\ r\max\left\{\widetilde{\Psi}_{R(\epsilon)}(r(rc)), \widetilde{\gamma}_1\right\}\end{array}\right] \\
&\qquad \succeq r\min\left[\begin{array}{c} r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\delta}_1\right\}, r\min\left\{\widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\delta}_1\right\}, \\ r\min\left\{\widetilde{\Psi}_{H(\epsilon)}(r), \widetilde{\delta}_1\right\}\end{array}\right] \\
&\qquad = r\min\left[r\min\left\{\widetilde{\Psi}_{F(\epsilon)}(r), \widetilde{\Psi}_{G(\epsilon)}(r), \widetilde{\Psi}_{H(\epsilon)}(r)\right\}, \widetilde{\delta}_1\right] \\
&\qquad = r\min\left\{\left(\widetilde{\Psi}_{F(\epsilon)} \cap \widetilde{\Psi}_{G(\epsilon)}(r) \cap \widetilde{\Psi}_{H(\epsilon)}\right)(r), \widetilde{\delta}_1\right\}
\end{aligned}$$

and

$$\begin{aligned}
& \min\left\{(\eta_{F(\epsilon)} \circ \eta_{G(\epsilon)}) \circ \eta_{H(\epsilon)})(r), r_2\right\} \\
&\qquad = \min\left[r\min_{r=xy}\max\left\{\left(\eta_{F(\epsilon)} \circ \eta_{G(\epsilon)}\right)(x), \eta_{H(\epsilon)}(y)\right\}\}, \gamma_2\right] \\
&\qquad \leq \min\left[\max\left\{\left(\eta_{F(\epsilon)} \circ \eta_{G(\epsilon)}\right)(rr), \eta_{H(\epsilon)}(r(rc))\right\}, \gamma_2\right] \\
&\qquad = \min\left[r\min_{rr=pq}\left\{\max\left\{\left\{\eta_{F(\epsilon)}(p), \eta_{G(\epsilon)}(q)\right\}, \eta_{H(\epsilon)}(r(rc))\right\}\right\}, \gamma_2\right] \\
&\qquad \leq \min\left[\max\left\{\eta_{F(\epsilon)}(r), \eta_{G(\epsilon)}(r), \eta_{H(\epsilon)}(r(rc))\right\}, \gamma_2\right] \\
&\qquad = \max\left[\begin{array}{c}\min\left\{\eta_{F(\epsilon)}(r), \gamma_2\right\}, \min\left\{\eta_{G(\epsilon)}(r), \gamma_2\right\}, \\ \min\left\{\eta_{H(\epsilon)}(r(rc)), \gamma_2\right\}\end{array}\right] \\
&\qquad \leq \max\left[\begin{array}{c}\max\left\{\eta_{F(\epsilon)}(r), \delta_2\right\}, \max\left\{\eta_{G(\epsilon)}(r), \delta_2\right\}, \\ \max\left\{\eta_{H(\epsilon)}(r), \delta_2\right\}\end{array}\right] \\
&\qquad = \max\left[\max\left\{\eta_{F(\epsilon)}(r), \eta_{G(\epsilon)}(r), \eta_{H(\epsilon)}(r)\right\}, \delta_2\right] \\
&\qquad = \max\left\{\left(\eta_{F(\epsilon)} \cap \eta_{G(\epsilon)}(r) \cap \eta_{H(\epsilon)}\right)(r), \delta_2\right\}
\end{aligned}$$

Thus, by Lemma 1,

$$(F, A)\widetilde{\cap}(G, B)\widetilde{\cap}(H, R) \subseteq \vee q_{(\Gamma,\Delta)}((F, A) \odot (G, B)) \odot (H, R).$$

The rest of the proof is similar as in Theorem 7.

(iii) $\Rightarrow$ (ii) : Let R be a right ideal and A, B are any subset of an $\mathcal{AG}$-groupoid $\mathcal{S}$, then by Theorem 4, $\Sigma(R,E)$, $\Sigma(A,E), \Sigma(B,E)$ are an $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft right ideal and $(\in_\Gamma, \in_\Gamma \vee q_\Delta)$-cubic soft set over $\mathcal{S}$, respectively. Now by using assumption we have $[\Sigma(A,E)\widetilde{\cap}\Sigma(B,E)]\widetilde{\cap}\Sigma(R,E) \subset_{(\Gamma,\Delta)} [\Sigma(A,E) \odot \Sigma(B,E)] \odot \Sigma(R,E)$. By using Lemma 3, we have

$$\begin{aligned} \mathcal{X}^{(\Gamma,\Delta)}_{(A\cap B)\cap R} &= {}_{(\Gamma,\Delta)}[\mathcal{X}^{(\Gamma,\Delta)}_{A} \cap \mathcal{X}^{(\Gamma,\Delta)}_{B}] \cap \mathcal{X}^{(\Gamma,\Delta)}_{R} \\ &\subseteq \vee q_{(\Gamma,\Delta)} \left(\mathcal{X}^{(\Gamma,\Delta)}_{A} \odot \mathcal{X}^{(\Gamma,\Delta)}_{B}\right) \odot \mathcal{X}^{(\Gamma,\Delta)}_{R} \\ &= {}_{(\Gamma,\Delta)}\mathcal{X}^{(\Gamma,\Delta)}_{(AB)R}. \end{aligned}$$

By Lemma 3, $(A \cap B) \cap R \subseteq (AB)R$. The rest of the proof is similar as in Theorem 7.

(ii) $\Rightarrow$ (i) : Since $\mathcal{S}p^2$ is a right ideal of an $\mathcal{AG}$-groupoid $\mathcal{S}$ containing p^2, so by (ii) it is semiprime. Therefore, by (ii) we have

$$\mathcal{S}p \cap \mathcal{S}p \cap \mathcal{S}p \subseteq [(\mathcal{S}p)(\mathcal{S}p)](\mathcal{S}p^2) = [(\mathcal{S}\mathcal{S})(pp)](\mathcal{S}p^2) \subseteq (\mathcal{S}p^2)\mathcal{S}.$$

Hence, $\mathcal{S}$ is intra-regular. ∎

Conclusion

In this paper, we introduce a new concept of generalized cubic soft sets and then apply it to the ideals theory of $\mathcal{AG}$-groupoids. We characterized intra-regular $\mathcal{AG}$-groupoids via cubic soft sets. In our future work we will be focusing on other generalized cubic soft ideals for characterizations of regular $\mathcal{AG}$-groupoids.

Acknowledgment. This project is supported by National Natural Science Foundation of China (61074115) and (10971242).

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Accepted: 15.06.2015