MULTIPLICATION COMPONENTS OF GRADED MODULES

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Abstract. Let $G$ be a group and $g \in G$. Let $R$ be a commutative $G$-graded ring and $M$ be a graded $R$-module. In this paper, we study some cases when $R$ is strongly graded ring and the component $M_e$ of $M$ is multiplication $R_e$-module. Also, we prove that if $R$ is strongly graded, then the components $M_g$ of $M$ are multiplication $R_e$-modules if and only if the component $M_e$ is $P$-torsion or $P$-cyclic where $P$ is a prime ideal of the component $R_e$ of $R$.

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1. Introduction

A ring $R$ with unity 1 graded by a group $G$ will means that $R = \bigoplus_{g \in G} R_g$ where $R_g$ is an additive subgroup of $R$ and $R_g R_h \subseteq R_{gh}$ for all $g, h \in G$. If the inclusion is an equality, then the ring is called strongly graded. Clearly, $R_e$ is a subring of $R$ with 1 $\in$ $R_e$. An $R$-module is said to be graded if $M = \bigoplus_{g \in G} M_g$ for a family of subgroups $\{M_g\}_{g \in G}$ of $M$ such that $R_g M_h \subseteq M_{gh}$ for all $g, h \in G$. Clearly, $M_g$ is an $R_e$-module for all $g \in G$. In a similar way, we define a strongly graded module. The ring $R$ is strongly graded if and only if every graded $R$-module is strongly graded. For more details, we refer the readers to [4], as well as [6], and the references therein.

If $M$ is an $R$-module and $N$ is an $R$-submodule of $M$, then the ideal $\{r \in R : rM \subseteq N\}$ of $R$ will be denoted by $(N : M)$. An $R$-module $M$ is said to be multiplication module if for every $R$-submodule $N$ of $M$, there exists an ideal of $R$ such that $N = IM$. Moreover, if $N = IM$ for some ideal of $R$, then $N = (N : M)M$. Given a ring $R$ and a multiplicative subset $S$ of $R$, the ring of fractions $S^{-1}R$ is $\{\frac{r}{s} : r \in R, s \in S\}$. For more details, we refer the readers to [1], [2], [3], as well as [5], and the references therein.

Throughout this paper, unless stated otherwise, $R$ is commutative nontrivially graded ring.

2. Results

In this section, we introduce and prove the main results of the paper.

Theorem 2.1 Let $R$ be a strongly $G$-graded ring and $M$ be a torsion free graded $R$-module. If $M_e$ is a multiplication $R_e$-module and $A$ is a proper ideal of $R_e$ such that $AM_e = M_e$, then $M = \{0\}$. 
Let \( n \in G \), \( x \in M_g \). Then \( R_g^{-1}x \) is an \( R_e \)-submodule of \( M_e \). Since \( M_e \) is multiplication \( R_e \)-module, \( R_g^{-1}x = BM_e \) for some ideal \( B \) of \( R_e \) and then \( R_g^{-1}x = BM_e = BAM_e = ABM_e = AR_g^{-1}x = R_g^{-1}Ax \) and so \( R_e x = Ax \). Now, \( x = 1.x \in R_e x = Ax \) and then there exists \( a \in A \) such that \( (1-a)x = 0 \). Since \( M \) is torsion free, either \( 1 = a \) or \( x = 0 \). If \( 1 = a \in A \), then \( A = R_e \) a contradiction. So, \( x = 0 \), i.e., \( M_g = \{0\} \) for all \( g \in G \) and hence \( M = \{0\} \).

**Theorem 2.2** Let \( R \) be a strongly \( G \)-graded ring and \( M \) be a graded \( R \)-module. If \( M_e \) is a multiplication \( R_e \)-module and \( S \) is a multiplicative subset of \( R_e \), then \( S^{-1}M_g \) is multiplication as an \( S^{-1}R_e \)-module for all \( g \in G \).

**Proof.** Let \( g \in G \) and \( X \) be an \( S^{-1}R_e \)-submodule of \( S^{-1}M_g \). Then \( X = S^{-1}N \) for some \( R_e \)-submodule \( N \) of \( M_g \) and then \( R_g^{-1}N \) is an \( R_e \)-submodule of \( M_g \) and it follows that \( R_g^{-1}N = AM_e \) for some ideal \( A \) of \( R_e \). So, \( X = R_e X = R_g R_g^{-1}S^{-1}N = R_g S^{-1}R_g^{-1}N = R_g S^{-1}AM_e = S^{-1}AR_g M_e = S^{-1}AM_g = S^{-1}AS^{-1}M_g \) where \( S^{-1}A \) is an ideal of \( S^{-1}R_e \) and this completes the proof.

Given a prime ideal \( P \) of \( R_e \), we consider the set

\[
T_P(M_e) = \{ m \in M_e : cm = 0 \text{ for some } c \in R_e - P \}.
\]

It is easy to check that \( T_P(M_e) \) is an \( R_e \)-submodule of \( M_e \). If \( T_P(M_e) = M_e \), then we will say that \( M_e \) is \( P \)-torsion. If there exists \( x \in M_e \) and \( c \in R_e - P \) such that \( cM_e \subseteq R_e x \), we will say that \( M_e \) is \( P \)-cyclic. Now, we introduce the main result of our paper:

**Theorem 2.3** Let \( R \) be a strongly \( G \)-graded ring, and \( M \) be a graded \( R \)-module. Then for \( g \in G \), \( M_g \) is multiplication \( R_e \)-module if and only if for every prime ideal \( P \) of \( R_e \), either \( M_e \) is \( P \)-torsion or \( P \)-cyclic.

**Proof.** Let \( g \in G \). Suppose that \( M_g \) is multiplication \( R_e \)-module and \( P \) is a prime ideal of \( R_e \). Firstly, we consider the case in which \( PM_e = M_e \). Let \( m \in M_e \). Then \( R_m \) is an \( R_e \)-submodule of \( M_g \) and then there exists an ideal \( A \) of \( R_e \) such that \( R_g m = AM_g \). So, \( m = 1.m \in R_e m = R_g R_g m = R_g AM_g = AR_g M_g = AM_e = AP_g M_g = PR_g M_g = PR_g R_g m = PR_m = R_e P m = P m \) and then there exists \( p \in P \) such that \( (1 - p)m = 0 \) and it follows that \( c = 1 - p \in R_e - P \) such that \( cm = 0 \) and therefore, \( M_e = T_P(M_e) \), i.e., \( M_e \) is \( P \)-torsion. Now, we consider that \( PM_e \neq M_e \). Then there exists \( x \in M_e - PM_e \) and since \( R_g x \) is an \( R_e \)-submodule of \( M_g \), there exists an ideal \( B \) of \( R_e \) such that \( R_g x = BM_g \). If \( B \subseteq P \), then \( x = 1.x \in R_e x = R_g R_g x = R_g BM_g = BM_g \) is a contradiction. Therefore, \( B \nsubseteq P \), then there exists \( c \in B - P \) such that \( cM_e = R_g cM_e \subseteq R_g BM_g = R_g R_g x = R_e x \), i.e., \( M_e \) is \( P \)-cyclic. Conversely, let \( g \in G \) and \( N \) be an \( R_e \)-submodule of \( M_g \). Suppose that \( A = (R_g^{-1}N : M_e) \), \( n \in R_g^{-1}N \) and \( K = (AM_e : R_e n) \). Assume that \( K \neq R_e \). Then there exists a maximal ideal \( P \) of \( R_e \) containing \( K \). If \( M_e \) is \( P \)-torsion, then there exists \( c \in R_e - P \) such that \( cm = 0 \) and it follows that \( c \in K - P \) a contradiction. So, \( M_e \) is \( P \)-cyclic, i.e., there exists \( x \in M_e \) and \( c \in R_e - P \) such that \( cM_e \subseteq R_e x \). Thus, \( R_g^{-1}cN \) is an \( R_e \)-submodule of \( R_e x \), and then \( cN \) is \( R_e \)-submodule of \( R_g x \).
but $R_gx$ is multiplication because it is cyclic, hence there exists $J = (cN : R_gx)$ such that $cN = Jx$. It holds that
\[
cJM_e = JcM_e \subseteq JR_ex = JR_{g^{-1}}R_gx = R_{g^{-1}}JR_gx \subseteq R_{g^{-1}}cN
\]
and hence, $cJ \subseteq A$. Now, the element $c^2n \in c^2N = cJx \subseteq Ax \subseteq AM_e$. As a result, $c^2 \in K \subseteq P$ which is a contradiction. It follows that $K = R_e$ and then $R_e = (AM_e : R_{g^{-1}}N)$ and therefore, $R_{g^{-1}}N \subseteq (R_{g^{-1}}N : M_e)M_e$ and then $N \subseteq (N : M_e)M_g$. Since the other inclusion is always true, the proof ends.

**Corollary 2.4** Let $R$ be a strongly $G$-graded ring, and $M$ be a graded $R$-module. Then for $g \in G$, $M_g$ is multiplication $R_e$-module if and only if for every prime ideal $P$ of $R_e$, either $M_e$ is $P$-torsion or there exists an $R_e$-submodule $N$ of $M_e$ and $c \in R_e - P$ such that $cM_e \subseteq N$.

**Proof.** To prove the sufficiency, let $P$ be a prime ideal of $R_e$ and suppose that $M_e$ is not $P$-torsion. Then by hypothesis, there exists $c \in R_e - P$ such that $cM_e \subseteq N$ where $N$ is an $R_e$-submodule of $M_e$. Since $M_e$ is not $P$-torsion, $N$ is not $P$-torsion. By Theorem 2.3, there exists $x \in N$ and $r \in R_e - P$ such that $rN \subseteq R_ex$. Thus, $crM_e = rcm_e \subseteq rN \subseteq R_ex$ and so $M_e$ is $P$-cyclic and therefore, by Theorem 2.3, $M_g$ is multiplication $R_e$-module for any $g \in G$. The necessity is obvious by Theorem 2.3.

**Corollary 2.5** Let $R$ be a strongly $G$-graded ring, and $M$ be a graded $R$-module. If $M_e$ is multiplication $R_e$-module and $Ann(M_e) = \{0\}$, then for $g \in G$,

1. $\bigcap_{k \in K}(I_kM_g) = \bigcap_{k \in K}I_kM_g$ for every family $I_k(k \in K)$ of ideals of $R_e$.

2. if $N$ is an $R_e$-submodule of $M_g$ and $A$ is an ideal of $R_e$ such that $N \subseteq AM_g$, then there exists an ideal $B$ of $R_e$ such that $B \subseteq A$ and $N \subseteq BM_g$.

**Proof.** 1. Let $I_k(k \in K)$ be a family of ideals of $R_e$. We call $I = \bigcap_{k \in K}I_k$. Then it is always true that $IM_e \subseteq \bigcap_{k \in K}(I_kM_e)$. Let $x \in \bigcap_{k \in K}(I_kM_e)$ and $H = (IM_e : R_ex)$. Suppose that $H \neq R_e$. Then there exists a prime ideal $P$ of $R_e$ containing $H$. If $x \in T_P(M_e)$, then we find an element in $H - P$, so $x \notin T_P(M_e)$. By Theorem 2.3, $M_e$ is $P$-cyclic and then there exists $m \in M_e$ and $c \in R_e - P$ such that $cM_e \subseteq R_em$ and so $cx \in \bigcap_{k \in K}I_km$. It follows that, for every $k \in K$, there exists $a_k \in I_k$ such that $cx = a_km$. Now, choose $k_0 \in K$ such that $cx \in I_{k_0}m$ and then $cx = a_{k_0}m$. Hence $a_{k_0}m = a_km$, i.e., $(a_{k_0} - a_k)m = 0$ for every $k \in K$. We have, $c(a_{k_0} - a_k)M_e = (a_{k_0} - a_k)cM_e \subseteq (a_{k_0} - a_k)R_em = R_e(a_{k_0} - a_k)m = \{0\}$. Since $Ann(M_e) = \{0\}$, $c(a_{k_0} - a_k) = 0$. Hence, $ca_{k_0} = ca_k \in I_k$ for every $k \in K$. As a consequence, $ca_{k_0} \in I$ and then $c^2x = ca_{k_0}m \in IM_e$ and so $c^2 \in H \subseteq P$ a contradiction. So, $H = R_e$ and hence $x \in IM_e$. Now, let $g \in G$. Then $\bigcap_{k \in K}(I_kM_g) = \bigcap_{k \in K}(I_kR_gM_e) = R_g\bigcap_{k \in K}(I_kM_e) = R_g(\bigcap_{k \in K}I_k)M_e = (\bigcap_{k \in K}I_k)R_gM_e = (\bigcap_{k \in K}I_k)M_g$.

2. Let $g \in G$, $N$ be an $R_e$-submodule of $M_g$ and $A$ be an ideal of $R_e$ such that $N \subseteq AM_g$. Then $R_{g^{-1}}N$ is an $R_e$-submodule of $M_e$ such that $R_{g^{-1}}N \subseteq AM_e$. 
Since $M_e$ is multiplication, $R_{g^{-1}}N = CM_e$ for some ideal $C$ of $R_e$ and then $N = CM_g$. So, $N = AM_g \cap CM_g = (A \cap C)M_g$ by using (1). Hence, choose $B = A \cap C$.

An $R$-module $M$ is said to be finitely cogenerated if for every non-empty family of $R$-submodules $N_k (k \in K)$ of $M$ such that $\bigcap_{k \in K} N_k = \{0\}$, there exists a finite subset $F$ of $K$ such that $\bigcap_{k \in F} N_k = \{0\}$. A ring $R$ is said to be finitely cogenerated if it is finitely cogenerated as an $R$-module. We close the paper with the following result:

Theorem 2.6 Let $R$ be a strongly $G$-graded ring, and $M$ be a graded $R$-module. If $M_e$ is multiplication $R_e$-module and $\text{Ann}(M_e) = \{0\}$, then $M_e$ is finitely cogenerated $R_e$-module if and only if $M_g$ is finitely cogenerated $R_e$-module for all $g \in G$.

Proof. Suppose that $M_e$ is finitely cogenerated $R_e$-module. Firstly, we prove that $R_e$ is finitely cogenerated. Let $I_k (k \in K)$ be a non-empty family of ideals of $R_e$ such that $\bigcap_{k \in K} I_k = \{0\}$. Then by Corollary 2.5, $\bigcap_{k \in K} (I_k M_e) = \{0\}$. Since $M_e$ is finitely cogenerated, there exists a finite subset $F$ of $K$ such that $\bigcap_{k \in F} I_k M_e = \{0\}$ and then by Corollary 2.5, $(\bigcap_{k \in F} I_k) M_e = \{0\}$. Since $\text{Ann}(M_e) = \{0\}$, $\bigcap_{k \in F} I_k = \{0\}$. Therefore, $R_e$ is finitely cogenerated. Now, let $g \in G$ and $N_k (k \in K)$ be a non-empty family of $R_e$-submodules of $M_g$ such that $\bigcap_{k \in K} N_k = \{0\}$. Then $R_{g^{-1}} N_k (k \in K)$ are $R_g$-submodules of $M_e$ and then for $k \in K$, there exists an ideal $A_k$ of $R_e$ such that $R_{g^{-1}} N_k = A_k M_e$, it is clear that $(\bigcap_{k \in K} A_k) M_e = \bigcap_{k \in K} (A_k M_e) = \bigcap_{k \in K} R_{g^{-1}} N_k = R_{g^{-1}} \bigcap_{k \in K} N_k = \{0\}$. Since $\text{Ann}(M_e) = \{0\}$, $\bigcap_{k \in K} A_k = \{0\}$ and since $R_e$ is finitely cogenerated, there exists a finite subset $F$ of $K$ such that $\bigcap_{k \in F} A_k = \{0\}$ and then

$$
\bigcap_{k \in F} N_k = R_e (\bigcap_{k \in F} N_k) = R_g R_{g^{-1}} (\bigcap_{k \in F} N_k)
$$

$$
= R_g (\bigcap_{k \in F} R_{g^{-1}} N_k) = R_g (\bigcap_{k \in F} A_k M_e) = R_g (\bigcap_{k \in F} A_k) M_e = \{0\}.
$$

Hence, $M_g$ is finitely cogenerated. The converse is obvious.

References


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