A NOTE ON SINGULAR VALUE INEQUALITIES FOR COMPACT OPERATORS

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Abstract. In this short note, we present some singular value inequalities for compact operators. Our results are generalizations of ones obtained by Audeh and Kittaneh [Linear Algebra Appl., 437 (2012), 2516-2522].

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1. Introduction

We mainly adopt the notations and terminologies in [1]. For convenience, recall that, as usual, Let $B(H)$ denote the space of all bounded linear operators on a complex separable Hilbert space H, and let $K(H)$ denote the two-sided ideal of compact operators in $B(H)$. For $T \in K(H)$, the singular values of T , denoted by $s_1(T), s_2(T), \cdots$ are the eigenvalues of the positive operator $|T| = (TT^*)^{1/2}$ enumerated as $s_1(T) \geq s_2(T) \cdots$ and repeated according to multiplicity. For $T \in B(H)$, let $T^+ = \dfrac{|T|+T}{2}, T^- = \dfrac{|T|-T}{2}$. Let $S, T \in B(H)$ be self-adjoint, the order relation $S \geq T$ means, as usual, that $S - T$ is positive . We use the direct sum notation $S \oplus T$ for the block-diagonal operator $\begin{bmatrix} S & 0 \\ 0 & T \end{bmatrix}$ defined on $H \oplus H$.

Let $A, B \in K(H)$, where A is self-adjoint and $B \geq 0$. Audeh and Kittaneh have proved in [1] that if $\pm A \leq B$, then

$$s_j(A) \leq \frac{1}{2} s_j((B+A) \oplus (B-A)) \tag{1.1}$$

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for $j = 1, 2, \ldots$. Inequality (1.1) is equivalent to three inequalities considered by Tao [2].

As an application of inequality (1.1), Audeh and Kittaneh have proved in [1] that if $A, B \in K(H)$ and $X \in B(H)$, then

$$s_j(AX^*B^* + BXA^*) \leq \frac{1}{2} s_j \begin{pmatrix} (A|X|A^* + B|X^*|B^* + AX^*B^* + BXA^*) \\ \oplus (A|X|A^* + B|X^*|B^* - AX^*B^* - BXA^*) \end{pmatrix} \tag{1.2}$$

for $j = 1, 2, \ldots$. Bhatia and Kittaneh have proved in [2] that, if $A, B \in K(H)$, then

$$s_j(AB^* + BA^*) \leq s_j((AA^* + BB^*) \oplus (AA^* + BB^*)) \tag{1.3}$$

for $j = 1, 2, \ldots$. Inequality (1.2) is sharper and more general than inequality (1.3). As another application of inequality (1.1), these authors also have proved in [1] that if $A, B \in K(H)$ are self-adjoint operators, then

$$s_j(A + B) \leq s_j\left(\left(A^+ + B^+\right) \oplus \left(A^- + B^-\right)\right) \tag{1.4}$$

for $j = 1, 2, \ldots$.

In this note, we present generalizations of inequalities (1.1), (1.2), and (1.4).

2. Main results

To generalize inequalities (1.1), (1.2), and (1.4), we need the following lemma [4, Theorem 1].

Lemma 2.1. *Let $A, X, B \in B(H)$ such that A and B are compact and positive. Then*

$$s_j(AX - XB) \leq \|X\| s_j(A \oplus B)$$

for $j = 1, 2, \ldots$, where $\|\cdot\|$ denotes the usual operator norm on $B(H)$.

Theorem 2.1. *Let $A, B \in K(H)$ and $X \in B(H)$, where A, B are self-adjoint and $\pm A \leq B$. Then*

$$s_j((B + A)X - X(B - A)) \leq \|X\| s_j((B + A) \oplus (B - A)) \tag{2.1}$$

for $j = 1, 2, \ldots$.

Proof. Since $\pm A \leq B$, we have $B - A \geq 0$ and $B + A \geq 0$. Now, applying Lemma 2.1 to the operators $B + A$ and $B - A$, we get

$$s_j((B + A)X - X(B - A)) \leq \|X\| s_j((B + A) \oplus (B - A))$$

for $j = 1, 2, \ldots$. This completes the proof. ∎

Remark 2.1. Putting $X = I$ in Theorem 2.1, we obtain inequality (1.1). Note that, $B \geq 0$ is no longer the condition of Theorem 2.1.

Theorem 2.2. *Let $A, B \in K(H)$ and $X, Y \in B(H)$. Then*

$$s_j(Y_1Y + YY_1 + Y_2Y - YY_2) \leq \|Y\| \, s_j((Y_2 + Y_1) \oplus (Y_2 - Y_1))$$

for $j = 1, 2, \ldots$, where

$$Y_1 = AX^*B^* + BXA^*, \; Y_2 = A|X|A^* + B|X|B^*.$$

Proof. For $X \in B(H)$, it is known [5, p.15] that

$$\begin{bmatrix} |X| & X^* \\ X & |X^*| \end{bmatrix} \geq 0$$

and so

$$\begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} |X| & X^* \\ X & |X^*| \end{bmatrix} \begin{bmatrix} A^* & 0 \\ B^* & 0 \end{bmatrix} \geq 0$$

That is

$$A|X|A^* + B|X|B^* + AX^*B^* + BXA^* \geq 0.$$

Similarly, we also have

$$A|X|A^* + B|X|B^* - AX^*B^* - BXA^* \geq 0.$$

Now, applying Lemma 2.1 to the operators $Y_2 + Y_1$ and $Y_2 - Y_1$, we get

$$s_j((Y_2 + Y_1)Y - Y(Y_2 - Y_1)) \leq \|Y\| \, s_j((Y_2 + Y_1) \oplus (Y_2 - Y_1))$$

for $j = 1, 2, \ldots$, which is equivalent to

$$s_j(Y_1Y + YY_1 + Y_2Y - YY_2) \leq \|Y\| \, s_j((Y_2 + Y_1) \oplus (Y_2 - Y_1))$$

for $j = 1, 2, \ldots$. This completes the proof. ∎

Remark 2.2. Putting $Y = I$ in Theorem 2.2, we obtain inequality (1.2).

Theorem 2.3. *Let $A, B \in K(H)$ be self-adjoint operators and $X \in B(H)$. Then*

$$\begin{aligned} s_j((A + B)X + X(A + B) + (|A| + |B|)X - X(|A| + |B|)) \\ \leq 2\|X\| \, s_j((A^+ + B^+) \oplus (A^- + B^-)) \end{aligned}$$

for $j = 1, 2, \ldots$.

Proof. Since A and B are Hermitian, it follows that

$$\pm(A + B) \leq |A| + |B|.$$

Let

$$Y_1 = A + B, \; Y_2 = |A| + |B|.$$

Now, applying Lemma 2.1 to the operators $Y_2 + Y_1$ and $Y_2 - Y_1$, we have

$$s_j((Y_2 + Y_1)X - X(Y_2 - Y_1)) \leq \|X\| \, s_j((Y_2 + Y_1) \oplus (Y_2 - Y_1)),$$

which is equivalent to

$$s_j\left((A+B)X+X(A+B)+(|A|+|B|)X-X(|A|+|B|)\right) \\ \leq 2\|X\|\,s_j\left((A^+ + B^+)\oplus(A^- + B^-)\right)$$

for $j=1,2,\ldots$. This completes the proof. ∎

Remark 2.3. Putting $X=I$ in Theorem 2.3, we obtain inequality (1.3). Let $g(t)$ be a polynomial. If $X=g(|A|+|B|)$, then, by Theorem 2.3, we have

$$s_j\left((A+B)X+X(A+B)\right)\leq 2\|X\|\,s_j\left(\left(A^+ + B^+\right)\oplus\left(A^- + B^-\right)\right)$$

for $j=1,2,\ldots$.

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