# $T_1$ CONCEPTS IN FUZZY BITOPOLOGICAL SPACES

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**Abstract.** In this paper, we introduce some notions of fuzzy pairwise-$T_1$ bitopological spaces and find relations among them. We also study some other properties of these concepts.

**Keywords:** quasi-coincidence; Q-neighbourhood; fuzzy bitopological spaces; fuzzy pairwise-$T_1$ bitopological spaces.

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## 1. Introduction

The notion of bitopological spaces was initially introduced by Kelly [7] in 1963. Later in 1991, fuzzy pairwise-$T_i$ (in short $FPTi$, $i = 0, 1, 2$) bitopological spaces have been introduced earlier by Kandil and El-Shafee [5]. Since then several other authors are investigating such concepts. Fuzzy pairwise-$T_1$ separation axioms have also been introduced by Abu Sufiya et al. [1] and Nouh [9]. Here we introduce a definition of fuzzy pairwise-$T_1$ bitopological space and obtain its several properties. Moreover, it will be seen that these two concepts are good extensions in the sense of Lowen [8].

## 2. Priliminaries

For the purpose of the main results, we need to introduce some definitions and notations.

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**Definition 2.1** [13] A fuzzy set $\mu$ in a set $X$ is a function from $X$ into the closed unit interval $I = [0, 1]$. For every $x \in X$, $\mu(x) \in I$ is called the grade of membership of $x$. Throughout this paper, $I^X$ will denote the set of all fuzzy sets from $X$ into the closed unit interval $I$. A member of $I^X$ may also be a called fuzzy subset of $X$.

**Definition 2.2** [3] Let $f$ be a mapping from a set $X$ into a set $Y$ and $u$ be a fuzzy set in $X$. Then the image of $u$, written as $f(u)$, is a fuzzy set in $Y$ whose membership function is given by

$$f(u)(y) = \begin{cases} \sup\{u(x)\}, & \text{if } f^{-1}[\{y\}] \neq \Phi,\ x \in X; \\ 0, & \text{otherwise.} \end{cases}$$

**Definition 2.3** [4] Let $f$ be a mapping from a set $X$ into a set $Y$ and $v$ be a fuzzy set in $Y$. Then the inverse of $v$ written as $f^{-1}(v)$ is a fuzzy set in $X$ which is defined by $f^{-1}(v)(x) = v(f(x))$, for $x \in X$.

**Definition 2.4** [12] A fuzzy set $\mu$ in $X$ is called a fuzzy singleton iff $\mu(x) = r$,$(0 < r \leq 1)$ for a certain $x \in X$ and $\mu(y) = 0$ for all points $y$ of $X$ except $x$. The fuzzy singleton is denoted by $x_r$ and $x$ is its support. We call $x_r$ is a fuzzy point if $0 < r < 1$. The class of all fuzzy singletons in $X$ will be denoted by $S(X)$.

**Definition 2.5** [3] A fuzzy topology $t$ on $X$ is a collection of members of $I^X$ which is closed under arbitrary suprema and finite infima and which contains constant fuzzy sets 1 and 0. The pair $(X, t)$ is called a fuzzy topological space (fts, in short) and members of t are called $t$-open (or simply open) fuzzy sets. A fuzzy set $\mu$ is called a $t$-closed (or simply closed ) fuzzy set if $1 - \mu \in t$.

**Definition 2.6** [10] Let $\lambda$ be a fuzzy set in an fts $(X, t)$. Then the closure of $\lambda$ is denoted by $t - cl(\lambda)$ and defined by $t - cl(\lambda) = \cap\{\mu : \lambda \subseteq \mu \text{ and } \mu^c \in t\}$.

**Definition 2.7** [3] Let $(X, t)$ and $(Y, s)$ be two fuzzy topological spaces. A mapping $f : (X, t) \to (Y, s)$ is called an fuzzy continuous iff for every $v \in s$, $f^{-1}(v) \in t$.

**Definition 2.8** [11] Let $f$ be a real valued function on a topological space $(X, T)$. If $\{x : f(x) > \alpha\}$ is open for every real $\alpha \in I_1$, then $f$ is called lower semi continuous function.

**Definition 2.9** [2] Let $X$ be a non-empty set and $T$ be a topology on $X$. Let $t = w(T)$ be the set of all lower semi continuous (lsc, in short) functions from $(X, T)$ into $I$ (with usual topology) such that $w(T) = \{\mu \in I^X : \mu^{-1}(\alpha, 1] \in T\}$ for each $\alpha \in I_1$. Then $w(T)$ is a fuzzy topology on $X$.

Let $P$ be a property of topological spaces and $FP$ be its fuzzy topology analogue. Then $FP$ is called a good extension of $P$ iff the statement $(X, T)$ has $P$ iff $(X, w(T))$ has $FP$ holds good for every topological space $(X, T)$.

**Definition 2.10** [6] *A fuzzy singleton $x_r$ is said to be quasi-coincident with a fuzzy set $\mu$, denoted by $x_r q \mu$ iff $r + \mu(x) > 1$. If $x_r$ is not quasi-coincident with $\mu$, we write $x_r \bar{q} \mu$.*

**Definition 2.11** [9] A fuzzy set $u$ of $(X, t)$ is called quasi-neighborhood (Q-nbd, in short) of $x_r$ iff there exists $v \in t$ such that $x_r qv$ and $v \subset u$. If $x_r$ is a fuzzy point or a fuzzy singletone, then $N(x_r, t) = \{\mu \in t : x_r \in \mu\}$ is the family of all fuzzy $t$-open neighborhoods ($t$-nbds, in short) of $x_r$ and $N_Q(x_r, t) = \{\mu \in t : x_r q\mu\}$ is the family of all Q-neighborhoods (Q-nbd, in short) of $x_r$.

**Definition 2.12** [6] A fuzzy bitopological space (fbts, in short) is a triple $(X, s, t)$, where $s$ and $t$ are arbitrary fuzzy topologies on $X$.

**Definition 2.13** [7] A bitopological space $(X, S, T)$ is called pairwise-$T_1$ (in short, $PT_1$) if for all $x, y \in X$, $x \neq y$, there exist $U \in S, V \in T$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$.

## 3. The main results

**Definition 3.14** An fbts $(X, s, t)$ is called

($a$) $FPT_1(i)$-space iff for every pair of fuzzy singletons $x_p, y_r$ in $X$ with $x \neq y$, there exist fuzzy sets $u \in s$, $v \in t$ such that $(x_p qu, y_r \cap u = 0)$ and $(y_r qv, x_p \cap v = 0)$.

($b$) [9] $FPT_1(ii)$-space iff for every pair of fuzzy singletons $x_p, y_r$ in $X$ such that $x \neq y$, there exist fuzzy sets $u, v \in s \cup t$ such that $(x_p qu, y_r \bar{q} u)$ and $(y_r qv, x_p \bar{q} v)$.

($c$) [1] $FPT_1(iii)$-space iff for any two distinct fuzzy points $x_p$, $y_r$ in $X$, there exist fuzzy sets $u \in s$, $v \in t$ such that $(x_p \in u, y_r \subseteq u^c)$ and $(y_r \in v, x_p \subseteq v^c)$.

**Theorem 3.1** *Let $(X, s, t)$ be an fbts. Then we have the following implications:*

(i) $(a) \Rightarrow (b)$, *and*

(ii) $(a) \Rightarrow (c)$.

*The arrows are not reversible in general.*

**Proof.** $(i)$ $(a) \Rightarrow (b)$ Let $x_r$, $y_s \in S(X)$ with $x \neq y$. Since $(X, s, t)$ be $FPT_1(i)$-space, there exist fuzzy sets $\mu \in s$, $\eta \in t$ such that $(x_r q\mu,\ y_s \cap \mu = 0)$ and $(y_s q\eta, x_r \cap \eta = 0)$. Now, $y_s \cap \mu = 0$ implies $\mu(y) = 0$. Then $\mu(y) + s \leq 1$. So, $y_s \bar{q} \mu$. Similarly, $x_r \cap \eta = 0$ implies $x_r \bar{q} \eta$. Again since $\mu \in s$, $\eta \in t$, then $\mu$, $\eta \in s \cup t$. Hence $(X, s, t)$ be $FPT_1(ii)$-space.

(ii) $(a) \Rightarrow (c)$ Let $x_r, y_s$ be two distinct fuzzy points in $X$. Choose $r^*, s^* \in (0, 1)$ such that $r^* < 1 - r$ and $s^* < 1 - s$. Since $x_{r^*}$, $y_s$ are two distinct fuzzy points in $X$ and $(X, s, t)$ is $FPT_1(i)$-space, there exist fuzzy sets $u \in s, v \in t$ such that $(x_{r^*} qu,\ y_s \cap u = 0)$ and $(y_s qv, x_{r^*} \cap v = 0)$. Now, $x_{r^*} qu, y_s \cap u = 0$ implies that $u(x) + r^* > 1$ and $u(y) = 0$. For $u(x) + r^* > 1$ and $r^* < 1 - r$, we have $u(x) > r$. So, $x_r \in u$. For $u(y) = 0$, we have $u(y) + s \leq 1$. That is, $s \leq 1 - u(y)$. So, $y_s \subseteq u^c$. Again, since $x_r, y_{s^*}$ are two distinct fuzzy points in $X$ and $(X, s, t)$ is $FPT_1(i)$, there exist fuzzy sets $\mu \in s$, $\lambda \in t$ such that $(y_{s^*} q\mu,\ x_r \cap \mu = 0)$ and $(x_r q\lambda, y_{s^*} \cap \lambda = 0)$. Similarly, $y_{s^*} q\mu, x_r \cap \mu = 0$ implies that $y_s \in \mu$, $x_r \subseteq \mu^c$. Hence, $(X, s, t)$ is $FPT_1(iii)$-space. ∎

The following counter example shows that $(c) \not\Rightarrow (b)$ as well as $(c) \not\Rightarrow (a)$ in general.

**Example.** Let $X = I, t_1 = t_2 = \{0, \lambda : \lambda(x) > 0, \forall x \in X\}$. Let $x_r$ and $y_s$ be distinct fuzzy singletons in $X$ and $\gamma = \min\{1-r, 1-s\}$. Now, if $\gamma \neq 0$, we define $\mu$, $\eta$ as follows

$$\begin{array}{llll} \mu(x) = 1 & \text{and} & \mu(y) = \frac{\gamma}{8}, & \text{if } y \neq x; \\ \eta(y) = 1 & \text{and} & \eta(x) = \frac{\gamma}{4}, & \text{if } x \neq y. \end{array}$$

Again, if $\gamma = 0$, we define $\mu_1, \eta_1$ as follows

$$\begin{array}{llll} \mu_1(x) = 1 & \text{and} & \mu_1(y) = 0.1, & \text{if } y \neq x; \\ \eta_1(y) = 1 & \text{and} & \eta_1(x) = 0.3, & \text{if } x \neq y. \end{array}$$

Then $\mu, \eta, \mu_1, \eta_1 \in t_1 = t_2$. For any pair of distinct fuzzy points $x_r$, $y_s$ in $X$, we see that $(x_r \in \mu,\ y_s \subseteq \mu^c)$ and $(y_s \in \eta,\ x_r \subseteq \eta^c)$. Therefore, $(X, t_1, t_2)$ is $FPT_1(iii)$-space. But if we take $x_1$, $y_1 \in S(X)$, then we see that $(x_1 q \mu_1, y_1 q \mu_1)$ and $(x_1 q \eta_1,\ y_1 q \eta_1)$. Hence $(X, t_1, t_2)$ is not $FPT_1(ii)$-space. Since $(a) \Rightarrow (b)$, then we see that $(c) \not\Rightarrow (a)$. ∎

The following counter example will show that (b) does not imply (c) as well as (b) does not imply (a) in general.

**Example.** Let $X = \{x, y, z\}$ and let $s = \{0, 1, x_1, y_1, x_1 \cup y_1\}$ and $t = \{0, 1, z_1\}$. Then $(X, s, t)$ is $FPT_1(ii)$-space but not $FPT_1(iii)$-space, since for distinct fuzzy points $x_r, y_s$ in $X$, we have $x_1 \in s$ such that $x_r \in x_1$, $y_s \subseteq x_1^c$ and there does not $v \in t$ such that $y_s \in v$, $x_r \subseteq v^c$. Since $(a) \Rightarrow (c)$, then we see that $(b) \not\Rightarrow (a)$.

**Theorem 3.2** *An fbts $(X, s, t)$ is $FPT_1(j)$-space iff for all $x_r \in S(X)$ $x_r = cl(x_r)$, where $j = i, ii, iii$.*

**Proof.** Suppose $(X, s, t)$ is $FPT_1(i)$-space and suppose that $x_r$, $y_s \in S(X)$ with $x \neq y$. Then there exists $\mu \in s$ such that $y_s q \mu$ and $x_r \cap \mu = 0$. Now $y_s q \mu$ implies that $\mu(y) + s > 1$, that is, $1 - \mu(y) < s$. So, $y_s \notin \mu^c$. Again, $x_r \cap \mu = 0$ implies that $\mu(x) = 0$, that is, $\mu(x) + r \leq 1 \quad \Rightarrow 1 - \mu(x) \geq r$. So $x_r \in \mu^c$. Therefore we see that $y_s \notin s - cl(x_r)$. Hence $s - cl(x_r) \subseteq x_r$. But $x_r \subseteq s - cl(x_r)$. Hence $x_r = s - cl(x_r)$. Similarly, we can show that $y_s$ is $t$-closed.

Conversely, suppose that $x_r = s - cl(x_r)$ and $y_s = t - cl(y_s)$ for all $x_r$, $y_s \in S(X)$ with $x \neq y$. Then $y_s \notin s - cl(x_1)$ and $x_r \notin t - cl(y_1)$. So, there exist fuzzy sets $\mu \in s$, $\eta \in t$ such that $(y_s \notin \mu^c,\ x_1 \in \mu^c)$ and $(x_r \notin \eta^c,\ y_1 \in \eta^c)$. Now, $y_s \notin \mu^c$, $x_1 \in \mu^c$ implies that $1 - \mu(y) < s$ and $1 - \mu(x) \geq 1$. That is, $\mu(y) + s > 1$ and $\mu(x) = 0$. That is, $y_s q \mu$ and $\mu(x) = 0$. So, for any $x_r \in S(X)$, we have $x_r \cap \mu = 0$ as $\mu(x) = 0$. Similarly, $x_r \notin \eta^c$, $y_1 \in \eta^c$ implies that $x_r q \eta$, $y_s \cap \eta = 0$. Hence $(X, s, t)$ is $FPT_1(i)$-space.

Next, suppose $(X, s, t)$ is $FPT_1(iii)$ and $x_r$, $y_s$ are two distinct fuzzy points in $X$. Since $x_r, y_{1-s}$ are two distinct fuzzy points in $X$ and $(X, s, t)$ is $FPT_1(iii)$-space, there exists $u \in s$ such that $y_{1-s} \in u$ and $x_r \subseteq u^c$.

Now $y_{1-s} \in u$ implies that $u(y) > 1-s$, that is, $1-u(y) < s$. So, $y_s \notin u^c$. Again, $x_r \subseteq u^c$ implies that $r \leq 1-u(x)$. So $x_r \in u^c$. Therefore we see that $y_s \notin s-cl(x_r)$. Hence $s-cl(x_r) \subseteq x_r$. But $x_r \subseteq s-cl(x_r)$. Hence $x_r = s-cl(x_r)$. Similarly, we can show that $y_s$ is $t$-closed.

Conversely, suppose that $x_r = s-cl(x_r)$ and $y_s = t-cl(y_s)$ for all distinct fuzzy points $x_r$, $y_s$ in $X$. Then $y_{1-s} \notin s-cl(x_r)$ and $x_{1-r} \notin t-cl(y_s)$. So, there exist fuzzy sets $u \in s$, $v \in t$ such that $(y_{1-s} \notin u^c,\ x_r \in u^c)$ and $(x_{1-r} \notin v^c,\ y_s \in v^c)$. Now, $y_{1-s} \notin u^c$, $x_r \in u^c$ implies that $1-u(y) < 1-s$ and $1-u(x) > r$. That is, $u(y) > s$ and $x_r \subseteq u^c$. That is, $y_s \in u$ and $x_r \subseteq u^c$. Similarly, $x_{1-r} \notin v^c$, $y_s \in v^c$ implies that $x_r \in v$ and $y_s \subseteq v^c$. Hence $(X,s,t)$ is $FPT_1(iii)$. For the proof of $j = ii$, see [9].

**Theorem 3.3** *Let $(X,s,t)$ be a fuzzy bitopological space, $A \subset X$ and*

$$s_A = \{u/A : u \in s\}, \quad t_A = \{v/A : v \in t\}.$$

*Then*

(a) *$(X,s,t)$ is $FPT_1(i)$ $\Rightarrow (A,s_A,t_A)$ is $FPT_1(i)$.*

(b) *$(X,s,t)$ is $FPT_1(ii)$ $\Rightarrow (A,s_A,t_A)$ is $FPT_1(ii)$.*

(c) *$(X,s,t)$ is $FPT_1(iii)$ $\Rightarrow (A,s_A,t_A)$ is $FPT_1(iii)$.*

**Proof.** (a) Suppose $(X,s,t)$ is $FPT_1(i)$. We have to show that $(A,s_A,t_A)$ is $FPT_1(i)$. Let $a_r$, $b_p \in S(A)$ with $a \neq b$. Then $a_r$, $b_p \in S(X)$ with $a \neq b$. Since $(X,s,t)$ is $FPT_1(i)$, there exist fuzzy sets $u \in s$, $v \in t$ such that $(a_r q u,\ b_p \cap u = 0)$ and $(b_p q v,\ a_r \cap v = 0)$. Now, $a_r q u$, $b_p \cap u = 0$ implies that $u(a) + r > 1$ and $(b_p \cap u)(x) = 0$ for all $x \in X$. But, $u/A \in s_A$ and $(u/A)(a) = u(a)$. Then $(u/A)(a) + r > 1$. So, $a_r q(u/A)$. Also, for all $x \in X$, we have $(b_p \cap (u/A))(x) = b_p(x) \cap (u/A)(x) = b_p(x) \cap u(x) = (b_p \cap u)(x) = 0$, since $(u/A)(b) = u(b)$. So $b_p \cap (u/A) = 0$. Similarly, $b_p q v$, $a_r \cap v = 0$ implies that $b_p q(v/A)$ and $a_r \cap (v/A) = 0$, where $v/A \in t_A$. Hence $(A,s_A,t_A)$ is $FPT_1(i)$-space. Similarly, (b) and (c) can be proved.

**Theorem 3.4** *Let $(X,T_1,T_2)$ be a bitopological space. Then*

(a) *$(X,T_1,T_2)$ is $PT_1$ $\Leftrightarrow$ $(X,w(T_1),w(T_2))$ is $FPT_1(i)$.*

(b) *$(X,T_1,T_2)$ is $PT_1$ $\Leftrightarrow$ $(X,w(T_1),w(T_2))$ is $FPT_1(i)$.*

(c) *$(X,T_1,T_2)$ is $PT_1$ $\Leftrightarrow$ $(X,w(T_1),w(T_2))$ is $FPT_1(iii)$.*

**Proof.** (a) Suppose that $(X,T_1,T_2)$ is $PT_1$. We have to show that $(X,w(T_1),w(T_2))$ is $FPT_1(i)$-space. Let $x_p$, $y_r \in S(X)$ with $x \neq y$. Since $(X,T_1,T_2)$ is $PT_1$, there exist $U \in T_1$, $V \in T_2$ such that $x \in U$, $y \notin U$ and $y \in V$, $x \notin V$. This implies $(1_U \in N_Q(x_p,w(T_1)),\ (1_U \cap y_r = 0))$ and $(1_V \in N_Q(y_r,w(T_2)),\ (1_V \cap x_p = 0))$. Hence $(X,w(T_1),w(T_2))$ is $FPT_1(i)$-space.

Conversely, suppose that $(X,w(T_1),w(T_2))$ is $FPT_1(i)$-space. We have to show that $(X,T_1,T_2)$ is $PT_1$. Let $x$, $y \in X$ such that $x \neq y$. Since $(X,w(T_1),w(T_2))$ is $FPT_1(i)$, then $(\exists \mu \in N_Q(x_1,w(T_1)),\ y_1 \cap \mu = 0)$ and $(\exists \eta \in N_Q(y_1,w(T_2)),\ x_1 \cap \eta = 0)$. Now, $\mu \in N_Q(x_1,w(T_1))$, $y_1 \cap \mu = 0$ implies that $\mu(x) + 1 > 1$ and $\mu(y) = 0$ that is, $\mu(x) > 0$ and $\mu(y) = 0$. Hence $x \in \mu^{-1}(0,1] \in T_1$ and $y \notin \mu^{-1}(0,1] \in T_1$. Similarly, we can show that $\eta \in N_Q(y_1,w(T_2)), x_1 \cap \eta = 0$ implies that $y \in \eta^{-1}(0,1] \in T_2$ and $x \notin \eta^{-1}(0,1] \in T_2$. Hence $(X,T_1,T_2)$ is $PT_1$.

Proof of (c) is similar, and for the proof (b), see [9].

**Theorem 3.5** *Product of any two $FPT_1(j)$-spaces is $FPT_1(j)$-space, where $j = i, ii, iii$.*

**Proof.** Suppose $(X_1, s_1, t_1)$ and $(X_2, s_2, t_2)$ are $FPT_1(i)$-spaces. We have to show that $(X_1 \times X_2, s_1 \times s_2, t_1 \times t_2)$ is $FPT_1(i)$-space. Let $(x, y)_p$, $(x_1,\, y_1)_q$ be two distinct fuzzy singletons in $X_1 \times X_2$. We can assume without loss of generality that $x \neq x_1$. Since $x_p, (x_1)_q$ are two distinct fuzzy singletons in $X_1$ and $(X_1, s_1, t_1)$ is $FPT_1(i)$-space, then $(\exists \mu \in N_Q(x_p, s_1)$, $((x_1)_q \cap \mu = 0))$ and $(\exists \eta \in N_Q((x_1)_q, t_1)$, $(x_p \cap \eta = 0))$. Now, $\mu \in N_Q(x_p, s_1)$, $((x_1)_q \cap \mu = 0)$ implies that $\mu(x) + p > 1$ and

$$((x_1)_q \cap \mu)(x) = 0 \text{ for all } x \in X. \tag{A}$$

Since $\mu \times X_2 \in s_1 \times s_2$ and $(\mu \times X_2)(x, y) = \min\{\mu(x), X_2(y)\} = \mu(x)$, we have $(\mu \times X_2)(x, y) + p = \mu(x) + p > 1$. So, $(x, y)_p q( \times X_2)$. We know that

$$(x_1, y_1)_q(x, y) = \begin{cases} q & \text{if } (x, y) = (x_1, y_1); \\ 0, & \text{otherwise.} \end{cases} \tag{B}$$

When $(x, y) = (x_1, y_1)$, we have $\mu(x) = 0$, by $(A)$, and

$$\begin{aligned} ((x_1, y_1)_q \cap (\mu \times X_2))(x, y) &= \min\{(x_1, y_1)_q(x, y),\, (\mu \times X_2)(x, y)\} \\ &= \min\{q, \mu(x)\} = \min\{q, 0\} = 0. \end{aligned}$$

Again when $(x, y) \neq (x_1, y_1)$, we have $(x_1, y_1)_q(x, y) = 0$, by $(B)$, and

$$\begin{aligned} ((x_1, y_1)_q \cap (\mu \times X_2))(x, y) &= \min\{(x_1, y_1)_q(x, y), (\mu \times X_2)(x, y)\} \\ &= \min\{0, \mu(x)\} = \min\{0, \mu(x)\} = 0. \end{aligned}$$

Hence, for all $(x, y) \in X_1 \times X_2$, we have $((x_1, y_1)_q \cap (\mu \times X_2))(x, y) = 0$.

Similarly, we can show that $\eta \in N_Q((x_1)_q, t_1)$, $(x_p \cap \eta = 0)$ implies that $(x_1, y_1)_q q(\eta \times X_2)$ and $(x, y)_p \cap (\eta \times X_2) = 0$, where $\eta \times X_2 \in t_1 \times t_2$. Hence $(X_1 \times X_2, s_1 \times s_2, t_1 \times t_2)$ is $FPT_1(i)$.

Other proofs are similar.

**Theorem 3.6** *A bijective mapping from an fts $(X, t)$ to an fts $(Y, s)$ preserves the value of a fuzzy singleton (fuzzy point).*

**Proof.** Let $c_r$ be a fuzzy singleton in $X$. So, there exist a point $a \in Y$ such that $f(c) = a$. Now $f(c_r)(a) = f(c_r)(f(c)) = \sup c_r(c) = c_r(c) = r$, since $f$ is bijective. Hence $a_r$ has same value as $c_r$.

**Note.** Preimage of any fuzzy singleton (fuzzy point) under bijective mapping preserves its value.

The ideas of the following two theorems are taken from [4].

**Theorem 3.7** *Let $(X, s, t)$ and $(Y, s_1, t_1)$ be two fuzzy bitopological spaces and let $f : X \to Y$ be bijective and $FP$-open. Then*

(a) $(X,s,t)$ *is* $FPT_1(i) \Rightarrow (Y,s_1,t_1)$ *is* $FPT_1(i)$.

(b) $(X,s,t)$ *is* $FPT_1(ii) \Rightarrow (Y,s_1,t_1)$ *is* $FPT_1(ii)$.

(c) $(X,s,t)$ *is* $FPT_1(iii) \Rightarrow (Y,s_1,t_1)$ *is* $FPT_1(ii)$.

**Proof.** (a) Suppose $(X,s,t)$ is $FPT_1(i)$. We have to show that $(Y,s_1,t_1)$ is $FPT_1(i)$. Let $a_r$, $b_q$ be two distinct fuzzy singletons in $Y$. Since $f$ is bijective, there exist distinct fuzzy singletons $c_r$, $d_q$ in $X$ such that $f(c)=a$, $f(d)=b$ and $c \neq d$. Again since $(X,s,t)$ is $FPT_1(i)$, there exist fuzzy sets $\mu \in s$, $\eta \in t$ such that $(c_r q\mu,\ d_q \cap \mu = 0)$ and $(d_q q\eta,\ c_r \cap \eta = 0)$.

Now, $c_r q\mu$, $d_q \cap \mu = 0$ implies that $\mu(c)+r>1$ and $(d_q \cap \mu)(x)=0$ for all $x \in X$. But $f(\mu)(a) = f(\mu)(f(c)) = \sup \mu(c) = \mu(c)$, since $f$ is bijective. So $f(\mu)(a)+r>1$, since $\mu(c)+r>1$. Hence $a_r q f(\mu)$. Also,

$$\begin{aligned}(b_q \cap f(\mu))(y) &= (f(d_q) \cap f(\mu))(f(x)) = f(d_q)(f(x)) \cap f(\mu)(f(x)) \\ &= (d_q \cap \mu)(x) = 0.\end{aligned}$$

Similarly, we can prove that $d_q q\eta, c_r \cap \eta = 0$ implies that $b_q q f(\eta)$ and $a_r \cap f(\eta)=0$.

Since $f$ is $FP$-open, then $f(\mu) \in s_1$, $f(\eta) \in t_1$. Hence $(Y,s_1,t_1)$ is $FPT_1(i)$.

Similarly, (b) and (c) can be proved.

**Theorem 3.8** *Let* $(X,s,t)$ *and* $(Y,s_1,t_1)$ *be two fuzzy bitopological spaces and* $f: X \to Y$ *be* $FP$*-continuous and bijective. Then*

(a) $(Y,s_1,t_1)$ *is* $FPT_1(i) \ \Rightarrow\ (X,s,t)$ *is* $FPT_1(i)$.

(b) $(Y,s_1,t_1)$ *is* $FPT_1(ii) \ \Rightarrow\ (X,s,t)$ *is* $FPT_1(ii)$.

(c) $(Y,s_1,t_1)$ *is* $FPT_1(iii) \ \Rightarrow\ (X,s,t)$ *is* $FPT_1(iii)$.

**Proof.** We now prove (a) only. Suppose $(Y,s_1,t_1)$ is $FPT_1(i)$. We have to show that $(X,s,t)$ is $FPT_1(i)$. Let $c_r$, $d_q$ be two distinct fuzzy singletons in $X$. Then there exist distinct fuzzy singletons $a_r$, $b_q$ in $Y$ such that $f(c)=a$, $f(d)=b$ and $a \neq b$, since $f$ is one-one. Again since $(Y,s_1,t_1)$ is $FPT_1(i)$, there exist fuzzy sets $\mu \in s_1$, $\eta \in t_1$ such that $(a_r q\mu,\ b_q \cap \mu = 0)$ and $(b_q q\eta,\ a_r \cap \eta = 0)$. Now, $a_r q\mu$, $b_q \cap \mu = 0$ implies that $\mu(a)+r>1$ and $(b_q \cap \mu)(f(x))=0$ for all $x \in X$. Since $f^{-1}(\mu)(c) = \mu(f(c)) = \mu(a)$, then we have

$$f^{-1}(\mu)(c)+r>1.$$

So, $c_r q f^{-1}(\mu)$. Also,

$$\begin{aligned}(d_q \cap f^{-1}(\mu))(x) &= (f^{-1}(b_q) \cap f^{-1}(\mu))(x) = f^{-1}(b_q)(x) \cap f^{-1}(\mu)(x) \\ &= (b_q \cap \mu)(f(x)) = 0.\end{aligned}$$

Similarly, we can show that $b_q q\eta, a_r \cap \eta = 0$ implies that $d_q q f^{-1}(\eta)$ and $c_r \cap f^{-1}(\eta) = 0$. Since $f$ is $FP$-continuous, then $f^{-1}(\mu) \in s$, $f^{-1}(\eta) \in t$. Hence $(X,s,t)$ is $FPT_1(i)$.

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