COMPOSITE TRAPEZOID RULE FOR THE RIEMANN-STIELTJES INTEGRAL AND ITS RICHARDSON EXTRAPOLATION FORMULA

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Abstract. In this paper, the composite trapezoid rule for the Riemann-Stieltjes integralis presented and its error is investigated. And then, the rationality of the generalization of composite trapezoid rule for Riemann-Stieltjes integral is demonstrated. At last, Richardson extrapolation is applied to the composite trapezoid rule for the Riemann-Stieltjes integral to obtain high accuracy approximations with little computational cost.

Keywords: composite rule; trapezoid rules; Riemann-Stieltjes integral; error term.

2000 Mathematics Subject Classification: 65D30; 65D32.

1. Introduction

In mathematics, the Riemann-Stieltjes integral is a kind of generalization of the Riemann integral, named after Bernhard Riemann and Thomas Stieltjes. It is Stieltjes [1] that first give the definition of this integral in 1894. It serves as an instructive and useful precursor of the Lebesgue integral. It is known that the Riemann-Stieltjes integral has wide applications in the field of probability theory [2]-[3], stochastic process [4] and functional analysis [5], especially in original formulation of F. Riesz's theorem [2], [5] and the spectral theorem for self-adjoint operators in a Hilbert space [2], [5].

Definite integration is one of the most important and basic concepts in mathematics. And it has numerous applications in fields such as physics and engineering.

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In several practical problems, we need to calculate integrals. As is known to all, as for $I = \int_a^b f(x)dx$, once the primitive function $F(x)$ of integrand $f(x)$ is known, the definite integral of $f(x)$ over the interval $[a, b]$ is given by Newton-Leibniz formula, that is,

$$\int_a^b f(x)dx = F(b) - F(a). \tag{1.1}$$

However, the explicit primitive function $F(x)$ is not available or its primitive function is not easy to obtain, such as $e^{\pm x^2}$, $\sin x^2$, $\frac{\sin x}{x}$, etc. Moreover, the integrand $f(x)$ is only available at certain points x_i, $i = 1, 2, ..., n$. How to get high-precision numerical integration formulas becomes one of the challenges in fields of mathematics [6].

In numerical analysis, the trapezoidal rule is the most well known numerical integration rules for approximating the definite integral. Trapezoidal rule with error term for classical Riemann integral is

$$\int_a^b f(x)dx = \frac{b-a}{2}\left(f(a) + f(b)\right) - \frac{(b-a)^3}{12} f''(\xi), \tag{1.2}$$

where $\xi \in (a, b)$.

In spite of the many accurate and efficient methods for numerical integration being available in [7-9], recently Mercer has obtained trapezoid rule for Riemann-Stielsjes integral which engender a generalization of Hadamard's integral inequality [10]. Trapezoidal rule with error term for Riemann-Stieltjes integral is

$$\int_a^b f(t)dg = \left[G - g(a)\right] f(a) + \left[g(b) - G\right] f(b) - \frac{(b-a)^3}{12} f''(\xi) g'(\eta), \tag{1.3}$$

where $G = \dfrac{1}{b-a}\displaystyle\int_a^b g(t)dt$, ξ, $\eta \in (a, b)$.

Then Mercer develops Midpoint and Simpson's rules for Riemann-Stielsjes integral [11] by using the concept of relative convexity. Burg [12] has proposed derivative-based closed Newton-Cotes numerical quadrature which uses both the function value and the derivative value on uniformly spaced intervals. Zhao and Li have proposed midpoint derivative-based closed Newton-Cotes quadrature [13] and numerical superiority has been shown. Then, the derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented by Zhao and Zhang [14], which uses derivative values at the endpoints. Recently, the midpoint derivative-based trapezoid rule for the Riemann-Stieltjes integral is presented by Zhao, Zhang and Ye [15], which only uses derivative values at the midpoint.

The trapezoid rule or the Riemann-Stieltjes integral uses only two points to approximate an integral, certainly not unsuitable for use over large integration intervals. A different direction is to subdivide the interval into smaller intervals and use lower-order schemes, like the trapezoid rule, on these smaller intervals. In numerical analysis, Richardson extrapolation is a sequence acceleration method, used to improve the rate of convergence of a sequence. It is used to generate high-accuracy results while using low-order formulas. It is named after Lewis Fry Richardson, who introduced the technique in 1927 [16].

One motivation for the research presented here lies in construction of composite trapezoid rule for the Riemann-Stieltjes integral, which is generalization of the results in [10], [11], [14]. The other is apply Richardson extrapolation to the composite trapezoid rule for the Riemann-Stieltjes integral to obtain high accuracy approximations with little computational cost.

In this paper, we divide the interval into subintervals and apply an integration rule to each subinterval, so the composite trapezoid rule for the Riemann-Stieltjes integral is presented. These new scheme is investigated in Section 2. In Section 3, the error term is presented. In Section 4, Richardson extrapolation is applied to the composite trapezoid rule for the Riemann-Stieltjes integral to obtain high accuracy approximations. Finally, conclusions are drawn in Section 5.

2. Composite trapezoid rule for the Riemann-Stieltjes integral

In this section, by dividing the interval into subintervals and applying an integration rule to each subinterval, composite trapezoid rule for the Riemann-Stieltjes integral is presented.

Theorem 2.1 *Suppose that $f(t)$ and $g(t)$ are continuous on $[a,b]$ and $g(t)$ is increasing there. Suppose that the interval $[a,b]$ is subdivided into n subintervals $[x_k, x_{k+1}]$ of width $h=\frac{b-a}{n}$ by using the equally spaced nodes $x_k = a+kh$, for $k=0,1,...,n$. The composite trapezoidal rule for n subintervals can be expressed as follows:*

$$\begin{aligned}\int_a^b f(t)dg \approx T_n = &\left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a)\\ &+\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right] f(x_k)\\ &+\left[g(b)-\frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right] f(b).\end{aligned} \tag{2.1}$$

Proof. By (1.3), the trapezoidal rule for Riemann-Stieltjes integral is

$$\int_a^b f(t)dg \approx \left[\frac{1}{b-a}\int_a^b g(t)dt - g(a)\right] f(a) + \left[g(b) - \frac{1}{b-a}\int_a^b g(t)dt\right] f(b). \tag{2.2}$$

Applying formula (2.2) over each subinterval, we obtain

$$\begin{aligned}\int_a^b f(t)dg \approx &\left[\frac{1}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt - g(a)\right] f(a) + \left[g(x_1) - \frac{1}{\frac{b-a}{n}}\int_a^{x_1} g(t)dt\right] f(x_1)\\ +&\left[\frac{1}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt - g(x_1)\right] f(x_1) + \left[g(x_1) - \frac{1}{\frac{b-a}{n}}\int_{x_1}^{x_2} g(t)dt\right] f(x_2) + \cdots\end{aligned}$$

$$+\left[\frac{1}{\frac{b-a}{n}}\int_{x_{k-1}}^{x_k} g(t)dt - g(x_k)\right] f(x_{k-1}) + \left[g(x_k) - \frac{1}{\frac{b-a}{n}}\int_{x_k}^{x_{k+1}} g(t)dt\right] f(x_k) + \cdots$$
$$+\left[\frac{1}{\frac{b-a}{n}}\int_{x_{n-1}}^{b} g(t)dt - g(x_{n-1})\right] f(x_{n-1}) + \left[g(b) - \frac{1}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right] f(b)$$
$$=\left[\frac{n}{b-a}\int_a^{x_1} g(t)\,dt - g(a)\right] f(a) + \frac{n}{b-a}\left[\int_{x_1}^{x_2} g(t)\,dt - \int_a^{x_1} g(t)\,dt\right] f(x_1)$$
$$+\frac{n}{b-a}\left[\int_{x_2}^{x_3} g(t)dt - \int_{x_1}^{x_2} g(t)dt\right] f(x_2) + \cdots$$
$$+\frac{n}{b-a}\left[\int_{x_{n-1}}^{x_n} g(t)\,dt - \int_{x_{n-2}}^{x_{n-1}} g(t)\,dt\right] f(x_{n-1}) + \left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)dt\right] f(b)$$
$$=\left[\frac{n}{b-a}\int_a^{x_1} g(t)dt - g(a)\right] f(a) + \frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)dt - \int_{x_{k-1}}^{x_k} g(t)dt\right] f(x_k)$$
$$+\left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)\,dt\right] f(b). \qquad \blacksquare$$

So, we have the composite trapezoid rule for the Riemann-Stieltjes integral as desired.

3. The error term for Riemann-Stieltjes composite trapezoid rule

In this section, the error term of the composite trapezoid rule for the Riemann-Stieltjes is investigated.

Theorem 3.1 *Suppose that $f''(t)$ and $g'(t)$ are continuous on $[a,b]$ and $g(t)$ is increasing there. Other conditions are the same as Theorem* 2.1. *The composite trapezoid rule for the Riemann-Stieltjes integral with the error term is*

$$\begin{aligned}\int_a^b f(t)dg &= \left[\frac{n}{b-a}\int_a^{x_1} g(t)\,dt - g(a)\right] f(a) \\ &+\frac{n}{b-a}\sum_{k=1}^{n-1}\left[\int_{x_k}^{x_{k+1}} g(t)\,dt - \int_{x_{k-1}}^{x_k} g(t)\,dt\right] f(x_k) \\ &+\left[g(b) - \frac{n}{b-a}\int_{x_{n-1}}^{b} g(t)\,dt\right] f(b) - \frac{(b-a)^3}{12n^2} f''(\xi)g'(\eta),\end{aligned} \tag{3.1}$$

where $\xi, \eta \in (a,b)$. And the error term $R[f]$ of this method is

$$-\frac{(b-a)^3}{12n^2} f''(\xi)g'(\eta). \tag{3.2}$$

Proof. We know from (1.3) that in every subinterval the quadrature error is

$$-\frac{h^3}{12} f''(\xi_k)g'(\eta_k), \tag{3.3}$$

where $\xi_k,\ \eta_k \in (x_{k-1},\ x_k)$, $k = 1, ..., n$.

Hence, the overall error is obtained by summing over n such terms:

$$\sum_{k=1}^{n} -\frac{h^3}{12} f''(\xi_k) g'(\eta_k) = -\frac{nh^3}{12}\left[\frac{1}{n}\sum_{k=1}^{n} f''(\xi_k) g'(\eta_k)\right]. \tag{3.4}$$

Let $M = \dfrac{1}{n}\sum_{k=1}^{n} f''(\xi_k) g'(\eta_k)$.

Clearly, $\min\limits_{x\in[a,b]}\{f''(x)g'(x)\} \le M \le \max\limits_{x\in[a,b]}\{f''(x)g'(x)\}$. Since $f''(t)$ and $g'(t)$ are continuous on $[a,b]$, then there exists two points ξ and η such that $M = f''(\xi)g'(\eta)$. This implies that the error term is $R[f] = -\dfrac{nh^3}{12} f''(\xi)g'(\eta)$.

Recalling that $h = \dfrac{b-a}{n}$, we obtain $R[f] = -\dfrac{(b-a)^3}{12n^2} f''(\xi)g'(\eta)$. ∎

Corollary 3.2 *Conditions are the same as Theorem* 3.1. *When $g(t) = t$, equation* (2.2) *reduces to the composite trapezoid rule for the classical Riemann integral.*

Proof. It is easy to obtain $\dfrac{n}{b-a}\displaystyle\int_a^{x_1} t\,dt - a = \dfrac{b-a}{2n}$, $\displaystyle\int_{x_k}^{x_{k+1}} t\,dt - \int_{x_{k-1}}^{x_k} t\,dt = \left(\dfrac{b-a}{n}\right)^2$, $b - \dfrac{n}{b-a}\displaystyle\int_{x_{n-1}}^{b} t\,dt = \dfrac{b-a}{2n}$ and $g'(t) \equiv 1$. By Theorem 3.1,

$$\begin{aligned}
\int_a^b f(t)dg &= \int_a^b f(t)dt \\
&= \frac{b-a}{2n} f(a) + \frac{n}{b-a}\left(\frac{b-a}{n}\right)^2 \sum_{k=1}^{n-1} f(x_k) + \frac{b-a}{2n} f(b) - \frac{(b-a)^3}{12n^2} f''(\xi) \\
&= \frac{b-a}{2n}\left(f(a) + 2\sum_{k=1}^{n-1} f(x_k) + f(b)\right) - \frac{(b-a)^3}{12n^2} f''(\xi).
\end{aligned}$$

Remark 3.3 From Corollary 3.2, we know that the results in Theorem 3.1 possess the reducibility. When $g(t) = t$, formula (3.1) reduces to the composite trapezoid rule for the classical Riemann integral. So it is a reasonable generalization of composite trapezoid rule for Riemann-Stieltjes integral.

4. Richardson extrapolation formula for the composite trapezoid rule for the Riemann-Stieltjes integral

Extrapolation can be applied whenever it is known that an approximation technique has an error term with a predictable form, one that depends on a parameter, usually the step size h [8].

In this section, we will illustrate how Richardson extrapolation applied to results from the composite trapezoid rule for the Riemann-Stieltjes integral can be used to obtain high accuracy approximations with little computational cost.

Theorem 4.1 *Suppose that $f''(t)$ and $g'(t)$ are continuous on $[a,b]$ and $g(t)$ is increasing there. Other conditions are the same as Theorem* 2.1. *The Richardson extrapolation formula for the composite trapezoid rule for the Riemann-Stieltjes integral is*

$$\int_a^b f(t)dg \approx \bar{T}_n = \frac{4}{3}T_{2n} - \frac{1}{3}T_n. \tag{4.1}$$

Proof. By equation (3.4), it is easy to obtain the error term $R[f]$ of composite trapezoid rule for the Riemann-Stieltjes integral is

$$R[f] = \sum_{k=1}^{n} -\frac{h^3}{12}f''(\xi_k)g'(\eta_k) = -\frac{nh^3}{12}\left[\frac{1}{n}\sum_{k=1}^{n} f''(\xi_k)g'(\eta_k)\right],$$

where $\xi_k,\ \eta_k \in (x_{k-1},\, x_k)\,,\ \ k = 1, ..., n.$

By the definition of Riemann integral, when n is big enough, we obtain

$$-\frac{nh^3}{12}\left[\frac{1}{n}\sum_{k=1}^{n} f''(\xi_k)g'(\eta_k)\right] = -\frac{h^2}{12}\left[h\sum_{k=1}^{n} f''(\xi_k)g'(\eta_k)\right] \approx -\frac{h^2}{12}\int_a^b f''(x)g'(x)\,dx.$$

It means

$$\int_a^b f(t)dg - T_n \approx -\frac{h^2}{12}\int_a^b f''(x)g'(x)\,dx. \tag{4.2}$$

Similarly,

$$\int_a^b f(t)dg - T_{2n} = -\frac{\left(\frac{h}{2}\right)^2}{12}\int_a^b f''(x)g'(x)\,dx. \tag{4.3}$$

Therefore, $\frac{\int_a^b f(t)dg - T_{2n}}{\int_a^b f(t)dg - T_n} \approx \frac{1}{4}$. Thus,

$$\int_a^b f(t)dg - T_{2n} \approx \frac{1}{4}\left(\int_a^b f(t)dg - T_n\right), \tag{4.4}$$

that is,

$$\int_a^b f(t)dg \approx \frac{4}{3}T_{2n} - \frac{1}{3}T_n.$$

Denote $\bar{T}_n = \frac{4}{3}T_{2n} - \frac{1}{3}T_n$, so we have the Richardson extrapolation formula as desired. ∎

Corollary 4.2 *Conditions are the same as Theorem* 4.1. *The Richardson extrapolation formula for the composite trapezoid rule for the Riemann-Stieltjes integral is* 4*th-order accurate.*

Proof. We know that the composite trapezoid rule for the Riemann-Stieltjes integral is second-order accurate (see equation (3.2)).

A more detailed study of the quadrature error reveals that the difference between $\int_a^b f(t)dg$ and T_n can be written as

$$\int_a^b f(t)dg = T_n + c_1 h^2 + c_2 h^4 + \cdots c_k h^{2k} + o\left(h^{2k+2}\right). \tag{4.5}$$

The exact values of the coefficients, c_k, are of no interest to us as long as they do not depend on h.

We can now write a similar quadrature that is based on double the number of points, i.e., T_{2n}.

Therefore,

$$\int_a^b f(t)dg = T_{2n} + c_1 \left(\tfrac{h}{2}\right)^2 + c_2 \left(\tfrac{h}{2}\right)^4 + \cdots + c_k \left(\tfrac{h}{2}\right)^{2k} + o\left(\left(\tfrac{h}{2}\right)^{2k+2}\right). \tag{4.6}$$

4*(4.6)-(4.5), this enables us to eliminate the h^2 error term,

$$3\int_a^b f(t)dg = 4T_{2n} - T_n + \hat{c}_2 h^4 + \cdots + \hat{c}_k h^k + o\left(h^{2k+2}\right),$$

where $\hat{c}_k = \left(2^{2-2k} - 1\right) c_k$, $k = 2, 3, \ldots.$ That is,

$$\begin{aligned} \int_a^b f(t)dg &= \frac{4}{3}T_{2n} - \frac{1}{3}T_n + \bar{c}_2 h^4 + \cdots + \bar{c}_k h^k + o\left(h^{2k+2}\right) \\ &= \bar{T}_n + \bar{c}_2 h^4 + \cdots + \bar{c}_k h^k + o\left(h^{2k+2}\right). \end{aligned} \tag{4.7}$$

where $\bar{c}_k = \dfrac{1}{3}\left(2^{2-2k} - 1\right) c_k, \quad k = 2, 3, \ldots.$

By equation (4.7), it is easy to obtain that the Richardson extrapolation formula for the composite trapezoid rule for the Riemann-Stieltjes integral is 4th-order accurate. ■

5. Conclusions

We briefly summarize our main conclusions in this paper as follows.

1) The composite trapezoid rule for the Riemann-Stieltjes integral is presented and its error term is investigated.
2) The rationality of the generalization of composite trapezoid rule for Riemann-Stieltjes integral is demonstrated.
3) Richardson extrapolation formula for the composite trapezoid rule for the Riemann-Stieltjes integral is obtained.

Acknowledgements. This work is supported by the Scientific Research Foundation of Civil Aviation University of China (No. 2013QD01X), the Fundamental Research Funds for the Central Universities (No. 3122014C023).

References

[1] Gordon, R.A., *The integrals of Lebesgue, Denjoy, Perron and Henstock*, American Mathematical Society, Providence, 1994.

[2] Billingsley, P., *Probability and Measures*, John Wiley and Sons, Inc., New York, 1995.

[3] Egghe, L., *Construction of concentration measures for general Lorenz curves using Riemann-Stieltjes integral*, Mathematical and Computer Modelling, 35 (2002), 1149-1163.

[4] Kopp, P.E., *Martingales and Stochastic Integrals*, Cambridge University Press, Cambridge, 1984.

[5] Rudin, W., *Functional Analysis*, McGraw Hill Science, McGraw, 1991.

[6] Bailey, D.H., Borwein, J.M., *High-precision numerical integration: Progress and challenges*, Journal of Symbolic Computation, 46 (2011), 741-754.

[7] Atkinson, K., *An Introduction to Numerical Analysis*, second ed., Wiley, 1989.

[8] Burden, R.L., Faires, J.D., *Numerical Analysis*, Brooks/Cole, Boston, Mass, USA, 9th edition, 2011.

[9] Isaacson, E., Keller, H.B., *Analysis of Numerical Methods*, John Wiley and Sons, New York, 1966.

[10] Mercer, P.R., *Hadamard's inequality and Trapezoid rules for the Riemann-Stieltjes integral*, Journal of Mathematical Analysis and Applications, 344 (2008), 921-926.

[11] Mercer, P.R., *Relative convexity and quadrature rules for the Riemann-Stieltjes integral*, Journal of Mathematical inequality, 6 (2012), 65-68.

[12] Burg, O.E., *Derivative-based closed Newton-Cotes numerical quadrature*, Applied Mathematics and Computation, 218 (2012), 7052-7065.

[13] Zhao, W.J., Li, H.X., *Midpoint Derivative-Based Closed Newton-Cotes Quadrature*, Abstract and Applied Analysis, Article ID 492507, 2013.

[14] Zhao, W.J., Zhang, Z.N., *Derivative-Based Trapezoid Rule For The Riemann-Stieltjes Integral*, Mathematical Problems in Engineering, Article ID 874651, 2014.

[15] Zhao, W.J., Zhang, Z.N., Ye, Z.J., *Midpoint Derivative-Based Trapezoid Rule for the Riemann-Stieltjes Integral*, Italian Journal of Pure and Applied Mathematics, 33 (2014), 369-376.

[16] Richardson, L.F., Gaunt, J.A., *The deferred approach to the limit*, Philosophical Transactions of the Royal Society, A 226 (1927), 299-349.

Accepted: 02.05.2015