THE WIGNER-POISSON-FOKKER-PLANCK SYSTEM WITH EXCHANGE POTENTIAL IN WEIGHTED L^2 SPACE

Bin Li

Department of Mathematics
Jincheng College of Sichuan University
Chengdu 611731
P.R. China
e-mail: endure2008@163.com

Jie-Qiong Shen

Department of Mathematics
Jincheng College of Sichuan University
Chengdu 611731
P.R. China
and
College of Mathematics
Sichuan University
Chengdu 610064
P.R. China e-mail: jieqiongshen123@163.com

Abstract. This paper is concerned with the Wigner-Poisson-Fokker-Planck(WPFP) system subject to Coulomb and exchange potential. In this work, existence and uniqueness of the local mild solution are established on an appropriately weighted-L^2 space in one dimension. The main difficulties in establishing mild solution are to derive a-priori estimates on the appropriate potential term. The proof is based on contraction mapping principle and parabolic regularization of the quantum Fokker-Planck term.

Keywords: Wigner-Poisson-Fokker-Planck system; exchange potential; contraction mapping principle; parabolic regularization.

AMS 2010 Subject Classification: 35Q40; 47B44; 35K55.

1. Introduction and main result

In quantum mechanics, there is a widespread concern over the following version of the nonlinear Schrödinger-Poisson-Slater (Schrödinger-Poisson-X^α) system, such as [14], [13], [24], namely,

$$(1.1)\quad i\hbar\psi_t(t,x) = -\frac{\hbar^2}{2}\Delta\psi(t,x) + V_{\text{cou}}\psi(t,x) - V_{\text{ext}}\psi(t,x) - \alpha V_{\text{exc}}\psi(t,x),\ x \in R^n,$$

which has been studied by many researchers in recent years. The wave function $\psi(t,x)$ is complex-valued, α denotes a given constant and V_{ext} stands for a given external potential, for example a harmonic potential

$$V_{\text{ext}} = a|x|^2 + b\cdot x + c,\ \ a\in R^+,\ \ b\in R^n,\ \ c\in R. \tag{1.2}$$

The macroscopic density $\rho=\rho(t,x)$ is now given by the zeroth order moment in the kinetic variable v, i.e., by the physical observables from both the wave function ψ and the Winger function w, namely,

$$\rho(t,x) = |\psi|^2 = \int_{R^n} w(t,x,v)dv. \tag{1.3}$$

The Coulomb potential V_{cou} is a solution of the following Poisson equation

$$\Delta V_{\text{cou}} = -(\rho - \varrho), \tag{1.4}$$

where $\varrho(x)\ge 0$ denotes the fixed positively charged background, i.e., the doping profile in semiconductor modeling (see [25], [15] for more details), which is assumed to be given. The exchange potential V_{exc} in (1.1) is due to Slater [17], [28] who replaces it by

$$V_{\text{exc}} = \rho^{\frac{1}{n}},\ \ x\in R^n. \tag{1.5}$$

The Wigner transform of the wave function $\psi(t,x)$ is

$$w(t,x,v) = \frac{1}{(2\pi)^n}\int_{R^n}\psi\left(t,x-\frac{\hbar}{2}y\right)\overline{\psi}\left(t,x+\frac{\hbar}{2}y\right)\exp(iv\cdot y)dy, \tag{1.6}$$

where $\overline{\psi}$ denotes the complex conjugate of the wave function ψ. A direct calculation by applying the transform (1.6) to the Schrödinger-Poisson-Slater system (1.1) shows that $w(t,x,v)$ satisfies the so-called collisionless Wigner-Poisson-X^α system, see e.g. [28], [23],

$$\begin{aligned} w_t + (v\cdot\nabla_x)w - \Theta_\hbar[V_{\text{ext}}]w - \alpha\Theta_\hbar[\rho^{\frac{1}{n}}]w + \Theta_\hbar[V_{\text{cou}}]w &= 0,\\ w(0,x,v) &= w_0(x,v), \end{aligned} \tag{1.7}$$

where the Wigner function $w=w(t,x,v)$ is a probabilistic quasi-distribution function of particles at time $t\ge 0$, located at $x\in R^n$ with velocity $v\in R^n$. The operator $\Theta_\hbar[\varphi]$ in (1.7) is a pseudo-differential operator, as in [4], [6], [9], formally defined by

$$\begin{aligned} \Theta_\hbar[\varphi]w(t,x,v) &= \frac{i}{(2\pi)^n\hbar}\int_{R^n}\int_{R^n}\delta[\varphi](t,x,\eta)w(t,x,v')e^{i(v-v')\eta}dv'd\eta,\\ \delta[\varphi](t,x,\eta) &= \varphi\left(t,x+\frac{\hbar\eta}{2}\right) - \varphi\left(t,x-\frac{\hbar\eta}{2}\right). \end{aligned} \tag{1.8}$$

The system (1.7)–(1.8) is a Hamiltonian system (i.e., the collisionless Wigner-Poisson-X^α equation), which is (almost) equivalent to the Schrödinger-Poisson-X^α

system. This allows for a much simpler analysis, see [14], [13] and therein. In this work, we shall now consider an open quantum system that includes a collision operator $Q[w]$, for example a Fokker-Planck operator, such as [4], [6],

$$Q[w] = \beta \mathrm{div}_v(vw) + \sigma\Delta_v w + 2\gamma \mathrm{div}_x(\nabla_v w) + \kappa\Delta_x w, \tag{1.9}$$

which models diffusive effects (e.g. the electronphonon-interaction), see [20, 22, 29]. A combination of the Wigner-Poisson-X^α system (1.7) and the Fokker-Planck operator (1.9) gives us the Wigner-Poisson-Fokker-Planck system coupled with exchange potential:

$$\begin{aligned} w_t + (v\cdot\nabla_x)w - \Theta_\hbar[V_{\mathrm{ext}}]w - \alpha\Theta_\hbar[\rho^{\frac{1}{n}}]w + \Theta_\hbar[V_{\mathrm{cou}}]w &= Q[w], \\ w(0,x,v) &= w_0(x,v). \end{aligned} \tag{1.10}$$

Such a model is not any more equivalent to a system of Schrödinger-Poisson-X^α system. Therefore, the mathematical analysis must be done on the level of the Wigner function. Moreover, the methods are introduced in our analysis, which refer to [4], [2], [10], [16].

In the literature, two problems linked to the Wigner(-Poisson)-Fokker-Planck system without exchange potential (i.e., $\alpha = 0$) are mainly studied: the long time behavior of the solutions and the existence of stationary solutions of Wigner-Fokker-Planck system (i.e., $V_{cou} = 0$) have studied in [6], [11]; the existence and uniqueness of the solution of the Wigner-Poisson-Fokker-Planck system have established in [4], [2]. Some of these results (as well as considerations about other quantum physical problems) have been reviewed in [5].

The present paper, we aim to establish the existence and uniqueness of the solution for the WPFP system (1.10) coupled Poisson equations (1.4) and exchange potential (1.5) on $\Omega = I\times R_v = [0,1]\times R_v$ in a weighted L^2 space (in symbols X, see (2.1) for detail), with the following boundary conditions:

$$V_{\mathrm{cou}}(0,t) = V_{\mathrm{cou}}(1,t),\ w(0,v,t) = w(1,v,t),\ v\in R_v,\ t\geq 0. \tag{1.11}$$

The analysis will proceed as follows. The natural choice of the functional setting for the study of the Wigner-Poisson or Wigner-Poisson-Fokker-Planck problem is the Hilbert space $L^2(R_x^n\times R_v^n)$, see [18], [26]. However, it can be immediately observed that the density $\rho(t,x)$, given by (1.3), is not well-defined for any $w(t,x,v)$ belonging to this space. In other words, the nonlinear terms $\Theta_\hbar[\rho^{\frac{1}{n}}]w$ and $\Theta_\hbar[V_{\mathrm{cou}}]w$ are not defined pointwise in t on the state space of the Wigner function. Therefore, inspired by [4], [9], [27], we introduce a Hilbert space X in Section 2, such that, the existence of the density $\rho(t,x)$ is granted for any $w\in X$.

With the above notations, the main result of this paper can be described as the following theorem.

Theorem 1. *Let $\kappa\sigma\geq\gamma^2$, for every $w_0\in X$, the WPFP problem* $(1.10)-(1.11)$, *coupled the equations* $(1.4)-(1.5)$, *has a unique mild solution* $w\in C([0,t_{\max}),X)$.

Remark 1. When considering the WPFP problem (1.10) without external potential and exchange potential, i.e., $V_{\text{ext}} = 0$ and $\alpha = 0$, Arnold et al.[2], [4] proved that $t_{\max} = +\infty$ by recovering a priori estimates for $\|w(t)\|_{L^2}, \||v|w\|_{L^2}, \||v|^2 w\|_{L^2}$ for all times in one or three dimensions.

In the present case $\alpha \neq 0$, we do not succeed in repeating the analogous strategy because the operator $\Theta_\hbar[\rho]w$ can be only controlled by $\|w\|_{H^1}$. Precisely speaking, we must use the regularity of the linear WFP equation to obtain existence of a solution, see Lemma 4. Thus, we are still working on results assuring the existence of the solution for t on the whole R^+.

Remark 2. Nevertheless, due to the regularity of the exchange potential, this result does not hold in higher dimensions. Precisely speaking, the symbols $\nabla(\rho^{\frac{1}{n}}), \Delta(\rho^{\frac{1}{n}})$ do not make sense at $\rho = 0$ when $n > 1$, see Lemma 4.

Our paper is structured as follows. In Section 2, we introduce a weighted space for the Wigner function w that allows to define the nonlinear terms $\Theta_\hbar[\rho]w$ and $\Theta_\hbar[V_{\text{cou}}]w$. And we also show that the quantum Fokker-Planck term is a semigroup-generator on the weighted L^2 space via the the Lumer-Phillips theorem [7]. In Section 3, we obtain a local-in-time, mild solution for WPFP with exchange potential in one dimensional using a fixed point argument and the parabolic regularization of the quantum Fokker-Planck term.

2. The functional setting and preliminaries

In this section, we shall discuss the functional analytic preliminaries for studying the nonlinear WPFP problem. First, we shall introduce an appropriate state space for the Wigner function w which allows to control the nonlinear terms $\Theta_\hbar[\rho]w$ and $\Theta_\hbar[V_{\text{cou}}]w$. Next, we shall discuss the linear WFP equation and the dissipativity of its (evolution) generator A defined in (2.15).

2.1. The State space and pseudo-differential operator

In this subsection, we shall analyze the properties of the pseudo-differential operator $\Theta_\hbar[\varphi]w$, which will be considered as a perturbation of the generator A defined in (2.15). This is one of the key ingredients for proving the Theorem 1.

We would show that the pseudo-differential operators $\Theta_\hbar[\rho]w$ and $\Theta_\hbar[V_{\text{cou}}]w$ are (local) bounded in the weighted L^2 space, in symbols:

$$X := L^2(I_x \times R_v, (1+v^2)dxdv), \; I = [0,1], \tag{2.1}$$

endowed with the following scalar product

$$\langle f, g\rangle_X := \int_0^1 \int_{R_v} f(x,v) \cdot \overline{g(x,v)}(1+v^2)dvdx, \; \forall f, g \in X. \tag{2.2}$$

In our calculations, we shall use the following norm:

$$||f||_X^2 := ||f||_{L^2}^2 + ||vf||_{L^2}^2, \tag{2.3}$$

The following proposition motivates our choice of the space X for the analysis.

Lemma 1. *Let $w \in X$ and $\rho(x,t)$ defined in* (1.3), *for all $x \in I$, then ρ belongs to $L^2(I)$ and satisfies*

$$||\rho||_{L^2(I)} \leq \sqrt{\pi}||w||_X, \tag{2.4}$$

And ρ also belongs to $L^1(I)$ and satisfies

$$||\rho||_{L^1(I)} \leq \sqrt{\pi}||w||_X. \tag{2.5}$$

Moreover, ρ belongs to $L^p(I)$ with $1 \leq p \leq 2$ and $\|\rho\|_{L^p(I)} \leq \sqrt{\pi}\|w\|_X$.

Proof. Similar to [4], [2], [9], [10], the first assertion follows directly by using Cauchy–Schwartz inequality in v-integral. On the other hand, by Hölder inequality, we have

$$\begin{aligned}
\|\rho\|_{L^1(I)} &\leq \int_0^1 \left| \int_R w(x,v,t)dv \right| dx \\
&\leq \left[\int_0^1 1^2 dx \right]^{\frac{1}{2}} \left[\int_0^1 \left| \int_R w(x,v,t)dv \right|^2 dx \right]^{\frac{1}{2}} \\
&\leq \sqrt{\pi}\|w\|_X.
\end{aligned}$$

Using the interpolation inequality, we get

$$\|\rho\|_{L^p(I)} \leq \|\rho\|^{\theta}_{L^2(I)} \|\rho\|^{1-\theta}_{L^1(I)} \leq \sqrt{\pi}\|w\|_X. \qquad \blacksquare$$

Now, we introduce the weak version of the Poisson equation (1.4)

$$\int_0^1 \partial_x V_{\text{cou}} \partial_x \phi dx = \int_0^1 \phi(\rho - \varrho)dx, \ \forall \phi \in W_0^{1,2}(I). \tag{2.6}$$

We can summarize the results relevant to the solution of the equation (1.4) in the following lemma and remarks.

Lemma 2. *For all $\rho, \varrho \in L^2(I)$, an unique $V_{cou} \in W_0^{1,2}(I)$ exists which satisfies* (2.7) *and*

$$\|V_{cou}\|_{W_0^{1,2}(I)} \leq C(\|\rho\|_{L^2(I)} + \|\varrho\|_{L^2(I)}).$$

Moreover, $\|V_{cou}\|_{W^{2,2}(I)} \leq C(\|\rho\|_{L^2(I)} + \|\varrho\|_{L^2(I)})$.

Remark 3. In fact, because of the following Sobolev's Theorem

$$W^{2,2}(I) \hookrightarrow \mathcal{C}_B^1(I),$$

we have

$$\|V_{\text{cou}}\|_{\mathcal{C}_B^1(I)} \leq C\|V_{\text{cou}}\|_{W^{2,2}(I)}.$$

So, if we call $V_{\rm cou}$ the extension of the solution with value zero out of the I, we obtain

$$\begin{aligned}\|V_{\rm cou}\|_{W^{1,\infty}(R)} &\leq \max\left\{\sup_{x\in I}|V_{\rm cou}| + \sup_{x\in I}|\partial_x V_{\rm cou}|\right\}\\ &\leq C\|V_{\rm cou}\|_{W^{2,2}(I)}\\ &\leq C(\|\rho\|_{L^2(I)} + \|\varrho\|_{L^2(I)}).\end{aligned}$$

Actually, an extension of the Lemma 2 states that the weak solution $V_{\rm cou}$ is in $W^{3,2}(I)$ if $\rho, \varrho \in W^{1,2}(I)$. Furthermore, we can show that the following estimate is true by extending Lemma 1

$$\|\partial_x \rho\|_{L^2(I)} \leq C\|\partial_x w\|_X. \tag{2.7}$$

Remark 4. Let $T : X \mapsto W^{1,\infty}(R)$ be the map

$$(Tw)(x) = \begin{cases} V_{\rm cou} & : \quad x \in I,\\ 0 & : \quad x \in R \setminus I,\end{cases}$$

where $V_{\rm cou}$ is the continuous solution of the equation (1.4) with boundary conditions (1.11). Then $T \in \mathcal{B}(X; W^{1,\infty}(R))$, moreover, see [19], Theorem 5.4.1,

$$\|Tw\|_{W^{1,\infty}(R)} \leq C\|V_{\rm cou}\|_{W^{1,\infty}(I)} \leq C(\|w\|_X + \|\varrho\|_{L^2(I)}). \tag{2.8}$$

Next, we consider the Lipschitz properties of the pseudo-differential operator $\Theta_\hbar[\varphi]w$ defined by (1.8). But by the definition of it, the $V_{\rm cou}$ and $V_{\rm exc}$ have to be appropriately extended to R_x. However, the Remarks 3-4 enable us to give a less formal definition of the pseudo-differential operator $\Theta_\hbar[\varphi]w$; we show indeed that the operator $\Theta_\hbar[\varphi]w$ is well defined from the space X to itself. On the other hand, the operator $\Theta_\hbar[\varphi]w$ can be rewritten in a more compact form as, see [4], [3],

$$\begin{aligned}\Theta_\hbar\left[\varphi\right]w &= \frac{i}{\hbar}\mathcal{F}^{-1}_{\eta\to v}\left[\delta\varphi\mathcal{F}_{v\to\eta}[w]\right] = \frac{i}{\hbar}(\mathcal{F}^{-1}_{\eta\to v}\left[\delta\varphi\right] *_v w),\\ \mathcal{F}_{v\to\eta}(\Theta_\hbar\left[\varphi\right]w)(x,\eta) &= \frac{i}{\hbar}\delta\varphi(x,\eta)\mathcal{F}^{-1}_{\eta\to v}w(x,\eta),\end{aligned} \tag{2.9}$$

where the symbol $*_v$ is the partial convolution with respect to the variable v, $\mathcal{F}_{v\to\eta}$ is the Fourier transformation with respect to the variable v and $\mathcal{F}^{-1}_{\eta\to v}$ its inverse:

$$\mathcal{F}_{v\to\eta}[f(x,\cdot)](\eta) = \int_R f(x,v)e^{iv\cdot\eta}dv,\ \mathcal{F}^{-1}_{\eta\to v}[g(x,\cdot)](v) = \frac{1}{2\pi}\int_R g(x,v)e^{-iv\cdot\eta}d\eta$$

for suitable functions f and g. Moreover, we can state the following result:

Lemma 3. *The operator $\Theta_\hbar\left[V_{cou}\right]w$ maps X into itself and there exists $C > 0$ such that*

$$||\Theta_\hbar\left[V_{cou}\right]w||_X \leq C(||w||_X + ||\varrho||_{L^2(I)})||w||_X. \tag{2.10}$$

Moreover, the operator $\Theta_\hbar\left[V_{cou}\right]w$ is of class C^∞ in X, and satisfies

$$\begin{aligned}&\|\Theta_\hbar\left[V^1_{cou}\right]w_1 - \Theta_\hbar\left[V^2_{cou}\right]w_2\|_X\\ &\qquad\leq C(\|w_1\|_X + \|w_2\|_X + \|\varrho\|_{L^2(I)})\|w_1 - w_2\|_X.\end{aligned} \tag{2.11}$$

Proof. We refer the reader to [1], [4], [2], [9], [10]. ∎

Let $\widetilde{\rho}$ is an extension of ρ, i.e.,

$$\widetilde{\rho}(x) = \begin{cases} \rho & : \quad x \in I, \\ 0 & : \quad x \in R \setminus I. \end{cases}$$

It is obvious that $\|\widetilde{\rho}(x)\|_{L^p(R)} \leq C\|\rho\|_{L^p(I)}$ holds by Theorem 5.4.1 of [19]. Similarly, we can state the following result:

Lemma 4. *Let $\partial_x w \in X$, then one has*

$$||\Theta_\hbar\,[\rho]\,w||_X \leq C\|w\|_X(\|w\|_X + \|\partial_x w\|_X). \tag{2.12}$$

Moreover,

$$\begin{aligned} &\|\Theta_\hbar\,[\rho_1]\,w_1 - \Theta_\hbar\,[\rho_2]\,w_2\|_X \\ &\quad \leq C(\|w_1 - w_2\|_X + \|\partial_x(w_1 - w_2)\|_X)(\|w_1\|_X + \|\partial_x w_1\|_X + \|w_2\|_X). \end{aligned} \tag{2.13}$$

Proof. Using Plancherel's formula and Hölder's inequality, this implies

$$||\Theta_\hbar\,[\rho]\,w||_{L^2(\Omega)} \leq 2\|\rho\|_{L^\infty(I)}\|w\|_{L^2(\Omega)}.$$

From (2.9), we see that

$$\|v\Theta_\hbar\,[\rho]\,w\|_{L^2(\Omega)} \leq C\|\partial_\eta(\delta\rho\mathcal{F}_v w)\|_{L^2(\Omega)}.$$

where ∂_η denotes the derivation operator with respect to the variables η. Similar to Lemma 3, we have

$$\begin{aligned} \|\partial_\eta\left(\delta\rho \cdot (\mathcal{F}_v w)\right)\|_{L^2(\Omega)} &= ||\partial_\eta \delta\rho \cdot \mathcal{F}_v w + \delta\rho \cdot \partial_\eta(\mathcal{F}_v w)||_{L^2(\Omega)} \\ &\leq ||\partial_\eta \delta\rho \cdot \mathcal{F}_v w||_{L^2(\Omega)} + ||\delta\rho \cdot \partial_\eta(\mathcal{F}_v w)||_{L^2(\Omega)} \\ &\leq ||\partial_\eta \delta\rho \cdot \mathcal{F}_v w||_{L^2(\Omega)} + ||\delta\rho||_{L^\infty(I)}||\partial_\eta(\mathcal{F}_v w)||_{L^2(\Omega)}. \end{aligned}$$

On the other hand, for all functions $f \in H^1(R)$, there exists a constant C, independent of f, such that

$$\|f\|_{L^\infty(R)} \leq C\|f\|_{H^1(R)}, \tag{2.14}$$

by Sobolev embedding theorem. Then, if we apply estimate (2.14) to the function $\rho, \mathcal{F}_v w$, we get

$$\|\rho\|_{L^\infty(I)}\|w\|_{L^2(\Omega)} \leq C\|\rho\|_{H^1}\|w\|_X$$

and

$$\begin{aligned} \|\partial_\eta\left(\delta\rho \cdot (\mathcal{F}_v w)\right)\|_{L^2(\Omega)} &\leq C[\|\partial_x\rho\|_{L^2}\|\mathcal{F}_v w\|_{L^2(R;H^1)} + \|\rho\|_{H^1}\|vw\|_{L^2}] \\ &\leq C\|\rho\|_{H^1}\|w\|_X \\ &\leq C(\|w\|_X + \|\partial_x w\|_X)\|w\|_X. \end{aligned}$$

Moreover, the first assertion follows directly by combining these estimates. Subsequently, we study the Lipschitz property of the operator $\Theta_\hbar\,[\rho]\,w$. In fact,

$$\begin{aligned}
&\|\Theta_\hbar\,[\rho_1]\,w_1 - \Theta_\hbar\,[\rho_2]\,w_2\|_X \\
&\quad \le \|\Theta_\hbar\,[\rho_1]\,w_1 - \Theta_\hbar\,[\rho_1]\,w_2\|_X + \|\Theta_\hbar\,[\rho_1]\,w_2 - \Theta_\hbar\,[\rho_2]\,w_2\|_X \\
&\quad \le \|\Theta_\hbar\,[\rho_1]\,(w_1 - w_2)\|_X + \|(\Theta_\hbar\,[\rho_1] - \Theta_\hbar\,[\rho_2])w_2\|_X = \Pi_1 + \Pi_2.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\Pi_1 &\le \|\Theta_\hbar\,[\rho_1]\,(w_1 - w_2)\|_{L^2} + \|v\Theta_\hbar\,[\rho_1]\,(w_1 - w_2)\|_{L^2} \\
&\le \|\delta\rho_1\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2} + \|\partial_\eta[\delta\rho_1]\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2} \\
&\quad +\|\delta\rho_1\partial_\eta\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2} \\
&\le C(\|\rho_1\|_{L^\infty}\|w_1 - w_2\|_{L^2} + \|\partial_x\rho_1\|_{L^2}\|\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2(R;L^\infty)} \\
&\quad +\|\rho_1\|_{L^\infty}\|\partial_\eta\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2}) \\
&\le C(\|\rho_1\|_{H^1}\|w_1 - w_2\|_{L^2} + \|\partial_x\rho_1\|_{L^2}\|\mathcal{F}_{v\to\eta}(w_1 - w_2)\|_{L^2(R;H^1)}) \\
&\le C(\|w_1\|_X + \|\partial_x w_1\|_X)\|w_1 - w_2\|_X.
\end{aligned}$$

By an analogous way, we get

$$\begin{aligned}
\Pi_2 &\le \|\Theta_\hbar\,[\rho_1 - \rho_2]\,w_2\|_{L^2} + \|v\Theta_\hbar\,[\rho_1 - \rho_2]\,w_2\|_{L^2} \\
&\le C(\|w_1 - w_2\|_X + \|\partial_x(w_1 - w_2)\|_X)\|w_2\|_X.
\end{aligned}$$

This concludes the proof of result. ∎

2.2. Basic properties of the quantum Fokker-Planck

In this subsection, we will show that the quantum Fokker-Planck term is a semigroup-generator in the weighted L^2 space (2.1). An easy calculation shows that

$$\Theta_\hbar[V_{\text{ext}}]w(t,x,v) \to (2ax + b)\cdot\nabla_v w(t,x,v), \tag{2.15}$$

in $L^2(R^n_v \times R^n_x)$ with $\hbar \to 0$, see [16] and therein. Without loss of generality, we can shift this problem with respect to x and then set $b = 0$.

We will consider the linear operator $A : D(A) \to X$

$$Aw = 2axw_v + Q[w] - vw_x = 2axw_v - vw_x + \beta(vw)_v + \sigma w_{vv} + 2\gamma w_{xv} + \kappa w_{xx} \tag{2.16}$$

defined on

$$D(A) = \left\{ w \in X \,\middle|\, vw_x, vw_v, w_{vv}, w_{xv}, w_{xx} \in X, w(0,v,t) = w(1,v,t) \right\}$$

We will study the dissipation property of the operator A, which is defined on X space, i.e., whether $\langle Au, u\rangle_X \le 0, \forall u \in D(A)$ holds.

Lemma 5. *Let the coefficients of the operator A satisfy $\kappa\sigma \geq \gamma^2$, then $A - \left(\sigma + \frac{\beta}{2} + a\right) I$ and its closure are dissipative.*

Proof. Indeed, by [2], if the coefficients of the operator $Q[w] - vw_x$ satisfy $\kappa\sigma \geq \gamma^2$, then $Q[w] - vw_x - \left(\sigma + \frac{\beta}{2}\right) I$ and its closure are dissipative in X. On the other hand, since $\int_0^1 \int_R xw_v w dv dx = 0$ and

$$\int_0^1 \int_R xv^2 w_v w dv dx = -\int_0^1 \int_R (2xvw^2 + xv^2 ww_v) dv dx,$$

then we have

$$\int_0^1 \int_R xv^2 w_v w dv dx \leq \frac{1}{2} \int_0^1 \int_R (x^2 + v^2) w^2 dv dx \leq \frac{1}{2} \int_R (1 + v^2) w^2 dv dx \leq \frac{1}{2} \|w\|_X^2.$$

Moreover, $A - \left(a + \sigma + \frac{\beta}{2}\right) I$ and its closure are dissipative in X. ∎

Furthermore, we have the following theorem:

Theorem 2. *For $w_0 \in X$, the linear WFP system*

$$w_t = \overline{A} w, \;\; w(0, x, v) = w_0(x, v) \tag{2.17}$$

has a unique solution $w \in C((0, +\infty); X)$. If furthermore $w_0 \in \mathcal{D}(\overline{A})$, then the solution is a classical solution, i.e., it belongs to $w \in C^1((0, +\infty); X)$. Moreover,

$$||w(t)||_X = \leq e^{(a + \sigma + \frac{\beta}{2})t} ||w_0||_X. \tag{2.18}$$

Proof. By the Lumer-Phillips theorem (Corollary 1.4.4 of [7]), we should show that $(\overline{A})^* - (a + \sigma + \frac{\beta}{2}) I$ is also dissipative. Indeed, A^* is densely defined on $D(A^*) \supseteq D(A)$, and hence A is a closable operator (see Theorem VIII.1.b of [21]). Its closure $\overline{A}$ satisfies $(\overline{A})^* = A^*$ (see [21], Theorem VIII.1.c). Thus, by the analogous arguments in [4, 2] (see Lemma 2.2 of [2] and Lemma 2.10 of [4] for detail), we can obtain that $A^* - (a + \sigma + \frac{\beta}{2}) I$ is dissipative on all of $D(A^*)$. Applying Corollary 1.4.4 of [7] to $\overline{A} - (a + \sigma + \frac{\beta}{2}) I$ with $(\overline{A})^* = A^*$ then implies that $\overline{A} - (a + \sigma + \frac{\beta}{2})$ generates a C_0 semigroup of contractions on X. Moreover, the semigroup C_0 generated by the operator $\overline{A}$ satisfies

$$||w(t)||_X = ||e^{\overline{A}t} w_0||_X \leq e^{lt} ||w_0||_X,$$

where $l = a + \sigma + \frac{\beta}{2}$. This concludes the proof of Theorem 2. ∎

Remark 5. In order to obtain the dissipative properties of $A^* - \tau I$ (τ is a positive constant) on the corresponding $D(A^*)$, see [2], [4] for detail, the arguments are introduced on the weighted $L^2(R_x^n \times R_v^n, (1 + |v|^2)^m dx dv, m > \frac{n}{2})$ space for the WPFP equation in R and R^3, respectively. It would be very interesting to study an analogous arguments for the Fokker-Planck type operators in [8], [12].

On the other hand, The unique solution of linear equation (2.17) can be expressed explicitly with the help of a fundamental solution, i.e., a distributional solution of the linear equation

$$\begin{aligned} \partial_t G(t,x,v,x_0,v_0) &= \overline{A}G(t,x,v,x_0,v_0), \\ \lim_{t\to 0} G(t,x,v,x_0,v_0) &= \delta(x-x_0, v-v_0), \end{aligned} \tag{2.19}$$

where δ is the Dirac distribution, see also [4], [16]. The Green's function reads

$$G(t,x,v,x_0,v_0) = e^{\beta t} g(t, X(-t,x,v)-x_0, \dot{X}(-t,x,v)-v_0),$$

where $g(t,x,v) = \frac{F_1(t,x,v)}{F_2(t,x,v)}$ with

$$\begin{aligned} F_1(t,x,v) &= \exp\Big(-\frac{\nu(t)|x|^2 + \mu(t)(x\cdot v) + \lambda(t)|v|^2}{4\lambda(t)\nu(t) - \mu^2(t)}\Big), \\ F_2(t,x,v) &= 2\pi(4\lambda(t)\nu(t) - \mu^2(t))^{\frac{1}{2}}. \end{aligned}$$

The coefficients $\lambda(t), \nu(t), \mu(t)$ are defined by

$$\begin{aligned} \lambda(t) &= \int_0^t (\alpha(s)\ \beta(s)) D \begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix} ds, \\ \nu(t) &= \int_0^t (\dot{\alpha}(s)\ \dot{\beta}(s)) D \begin{pmatrix} \dot{\alpha}(s) \\ \dot{\beta}(s) \end{pmatrix} ds, \\ \mu(t) &= -2\int_0^t (\dot{\alpha}(s)\ \dot{\beta}(s)) D \begin{pmatrix} \alpha(s) \\ \beta(s) \end{pmatrix} ds \end{aligned}$$

where $\alpha(\cdot)$ and $\beta(\cdot)$ are such that

$$X(-t,x,v) = \alpha(t)x + \beta(t)v.$$

In our framework,

$$D = \begin{pmatrix} \kappa & \gamma \\ \gamma & \sigma \end{pmatrix}.$$

With these preliminaries, the following parabolic regularization result can be deduced, see Proposition 3.1 of [4] for $a = 0$.

Lemma 6. *For all $T > 0$, the derivatives of the solution f of the equation* (2.17) *can be estimated by:*

$$\|\partial_x w\|_X \le C_T t^{-\frac{1}{2}} \|w_0\|_X, \ \|\partial_v w\|_X \le C_T t^{-\frac{1}{2}} \|w_0\|_X, \ \forall t \in (0,T], \tag{2.20}$$

for all $w_0 \in X$.

3. Proof of Theorem 1

In this section, we shall use a contractive fixed point map to establish a local solution of the Wigner-Poisson-Fokker-Plank system with exchange potential in X. Without loss of generality, we can set $\varrho = 0$. Lemmas 4 and 6 motivate the definition of the Banach space

$$Y_T = \{z \in C([0,T];X) | \partial_x z \in C([0,T];X) \\ \text{with } \|\partial_x z\|_X \leq Ct^{-\frac{1}{2}} \text{ for } t \in (0,T),\ z(t=0) = z_0\} \tag{3.1}$$

endowed the norm

$$||z||_{Y_T} = \sup_{t\in[0,T]} ||z(t)||_X + \sup_{t\in[0,T]} ||t^{\frac{1}{2}} \partial_x z(t)||_X, \tag{3.2}$$

for every fixed $T > 0$. We shall obtain the (local-in-time) well-posedness result for the Wigner-Poisson–Fokker-Plank system with exchange potential by introducing a non-linear iteration in the space Y_T, with an appropriate (small enough) T. First, for a given $w \in Y_T$ we shall now consider the linear Cauchy problem for the function z,

$$z_t = \overline{A}z(t) + \Theta_\hbar[V_{\text{cou}}[z(t)]]w(t) + Q_\hbar[\rho[w(t)]]z(t), \\ \forall t \in [0,T],\ z(t=0) = w_0 \in X, \tag{3.3}$$

where, $T > 0$ is a fixed time.

Lemma 7. *For all $w_0 \in X$ and $w \in Y_T$, the initial value problem* (3.3) *has a unique mild solution $z \in C([0,T];X)$, which satisfies*

$$z(t) = e^{t\overline{A}}w_0 + \int_0^t e^{(t-s)\overline{A}}(\Theta_\hbar[V_{cou}[z(s)]]w(s) + Q_\hbar[\rho[w(s)]]z(s)),\ \forall t \in [0,T]. \tag{3.4}$$

Moreover, the solution z belongs to the space Y_T.

Proof. The first assertion follows directly by applying Theorem 6.1.2 in [7]. Moreover, the following inequalities hold

$$\begin{aligned} ||z(t)||_X &\leq e^{kt}||w_0||_X + C\int_0^t e^{k(t-s)}||z(s)||_X(||w(s)||_X + ||\partial_x w(s)||_X)ds \\ &\leq e^{kt}||w_0||_X + Ce^{kT}||w||_{Y_T}\int_0^t ||z(s)||_X\left(1+s^{-\frac{1}{2}}\right)ds, \end{aligned}$$

for all $t \in [0,T]$, $k = a + \sigma + \frac{\beta}{2}$. Then, by Gronwall's inequality,

$$||z(t)||_X \leq Ce^{kT}||w_0||_X\left[1 + \|w\|_{Y_T}(t + 2t^{\frac{1}{2}})\exp[Ce^{kT}\|w\|_{Y_T}(T + 2T^{\frac{1}{2}})]\right]. \tag{3.5}$$

By differentiating equation (3.4) in the x-direction, we obtain

$$\partial_x z(t) = \partial_x e^{t\overline{A}}w_0 + \int_0^t \partial_x e^{(t-s)\overline{A}}(\Theta_\hbar[V_{\text{cou}}[z(s)]]w(s) + Q_\hbar[\rho[w(s)]]z(s)),\ \forall t \in [0,T].$$

Using Lemmas 3, 4 and 6, we can get

$$\|\partial_x z(t)\|_X \le C\left[e^{kT}t^{-\frac{1}{2}}\|w_0\|_X+\|w\|_{Y_T}\int_0^t \|z(s)\|_X e^{k(t-s)}(t-s)^{-\frac{1}{2}}(1+s^{-\frac{1}{2}})ds\right]. \tag{3.6}$$

Hence, substituting (3.5) into (3.6), by [4],

$$\int_0^t e^{k(t-s)}(t-s)^{-\frac{1}{2}}(1+s^{-\frac{1}{2}})ds \le e^{kt}(2t^{\frac{1}{2}}+\pi)$$

and

$$\int_0^t e^{k(t-s)}(t-s)^{-\frac{1}{2}}(1+s^{-\frac{1}{2}})(s+2s^{\frac{1}{2}})ds \le 2e^{kt}(2t^{\frac{1}{2}}+3t+t^{\frac{3}{2}}),$$

we can get that the function z belongs to the space Y_T. ∎

We now define the linear map τ on Y_T (for any fixed $0 < T \le T_0$):

$$w \mapsto \tau w = z,$$

where z is the unique mild solution of the (3.3). Next, we shall show that τ is a strict contraction on a closed subset of Y_T, for T sufficiently small. This will yield the local-in-time solution of the non-linear problem (1.10)–(1.11).

Lemma 8. *For any fixed $w_0 \in X$ and $w \in Y_T$, then there exists a $l > 0$ such that the map τ,*

$$\tau w(t) = e^{\overline{A}t}w_0 + \int_0^t e^{(t-s)\overline{A}}(\Theta_\hbar[V_{cou}[\tau w(s)]]w(s) + Q_\hbar[\rho[w(s)]]\tau w(s)),\ \forall t \in [0,\tau],$$

is a strict contraction from the ball of radius $R(>0)$ of Y_l into itself.

Proof. Similar to the proof of Lemma 7 the function $z = \tau w \in Y_l$ satisfies

$$||\tau w(t)||_X \ \le\ Ce^{kT}||w_0||_X\left[1+Ce^{kT}\|w\|_{Y_l}(t+2t^{\frac{1}{2}})\exp[Ce^{kT}\|w\|_{Y_l}(T+2T^{\frac{1}{2}})]\right].$$

Under the assumption $||w||_{Y_l} \le R$, this estimate reads

$$||\tau w(t)||_X \ \le\ Ce^{kT}||w_0||_X\left[1+Ce^{kT}R(t+2t^{\frac{1}{2}})\exp[Ce^{kT}R(T+2T^{\frac{1}{2}})]\right].$$

If we assume

$$Ce^{kT}||w_0||_X\left[1+Ce^{kT}R(t+2t^{\frac{1}{2}})\exp[ce^{kT}R(T+2T^{\frac{1}{2}})]\right] \le \frac{1}{3}R, \tag{3.7}$$

then $||\tau w(t)||_X \le \frac{1}{3}R$. Similar to (3.6), for enough small l and large R, we have $t^{\frac{1}{2}}\|\partial_x \tau w(t)\|_X \le \frac{1}{3}R$. Then, the estimate $\|\tau w(t)||_{Y_l} \le \frac{2}{3}\|w(t)\|_{Y_l}$ holds, and hence the operator τ maps the ball of radius R of Y_l into itself.

To prove contractivity we shall estimate $\tau u - \tau w$ for all $w, u \in Y_l$ with $||w||_{Y_l}, ||u||_{Y_l} \leq R$. Since

$$\tau u - \tau w = \int_0^t e^{\overline{A}(t-s)} \left(\Theta_\hbar[V_{\mathrm{cou}}[\tau u(s) - \tau w(s)]]u(s) + \Theta_\hbar[V_{\mathrm{cou}}[\tau w(s)]](u-w)(s)\right) ds$$
$$+ \int_0^t e^{\overline{A}(t-s)} \left(\Theta_\hbar[\rho[\tau u(s) - \tau w(s)]]u(s) + \Theta_\hbar[\rho[u(s)]](\tau u - \tau w)(s)\right) ds,$$

by analogous estimates,

$$\begin{aligned} ||\tau u - \tau w||_X &\leq Ce^{kl} \int_0^t \|u\|_X \|\tau u - \tau w\|_X + (\|u\|_X + \|\partial_x u\|_X)\|\tau u - \tau w\|_X ds \\ &+ Ce^{k\iota} \int_0^t \|u-w\|_X \|\tau w\|_X + \|\tau w\|_X(\|u-w\|_X + \|\partial_x u - \partial_x w\|_X) ds, \end{aligned}$$

moreover,

$$\begin{aligned} ||\tau u - \tau w||_X &\leq 2CRe^{kl} \int_0^t (1 + s^{-\frac{1}{2}})(\|\tau u - \tau w\|_X + \|u - w\|_{Y_l}) ds \\ &\leq 2CRe^{kl} \|u - w\|_{Y_l}(t + 2t^{\frac{1}{2}}) + \int_0^t (1 + s^{-\frac{1}{2}})\|\tau u - \tau w\|_X ds), \end{aligned}$$

hence, by applying Gronwall's Lemma,

$$||\tau u - \tau w||_X \leq 2CRe^{k\iota}\|u - w\|_{Y_l} \left(t + 2t^{\frac{1}{2}} + \left(\frac{3}{2}t^2 + \frac{4}{3}t^{\frac{3}{2}} + \frac{2}{5}t^{\frac{5}{2}}\right) e^{l + l^{\frac{3}{2}}}\right). \tag{3.8}$$

The estimate

$$t^{\frac{1}{2}}\|\partial_x(\tau u) - \partial_x(\tau w)\|_X \leq C(R, k, \iota)\|u - w\|_{Y_l} \tag{3.9}$$

can be proved in an analogous way as those of (3.8) by Lemmas 4, 6 and [4]. When choosing $l > 0$ small enough, estimates (3.8)–(3.9) imply that

$$||\tau u - \tau w||_{Y_l} \leq C(R, k, \iota)||u - w||_{Y_l}, \tag{3.10}$$

for some $C(R, k, \iota) < 1$, and the assertion is proved. ∎

Proof of Theorem 1. By [7], there exists a $t_{\max} \leq \infty$ such that the initial boundary value problem (1.10)-(1.11) has a unique mild solution w in Y_T, for all $T < t_{\max}$, which satisfies

$$w(t) = e^{t\overline{A}}w_0 + \int_0^t e^{(t-s)\overline{A}}(\Theta_\hbar[V_{\mathrm{cou}}[w(s)]]w(s) + Q_\hbar[\rho[w(s)]]w(s)), \ \forall t \in [0, T].$$

Moreover, if $t_{\max} < \infty$, then

$$\lim_{t \to t_{\max}} ||w(t)||_{Y_T} = \infty.$$

Precisely speaking, the solution of the problem (1.10)–(1.11) is a fixed point of the map τ previously introduced. Using Lemma 8, this solution exists for a time interval of length l, depending only on $\|w_0\|_X$, and it belongs to the space Y_l. If the second assertion would not hold, there would be a sequence of times $t_n \to t_{\max}$ such that $\|w(t_n)\| \leq C$ for all n. Then, by solving a problem with the initial value $w(t_n)$, with t_n sufficiently close to $t_{\max}$, we would extend the solution up to a certain time $t'_n > t_{\max}$. This construction would contradict our definition of $t_{\max}$.

The uniqueness of the mild solution follows by arguments analogous to those in the proof of Theorem 6.1.4 in [7]. ∎

References

[1] Arnold, A., Ringhofer, C., *An operator splitting method for the Wigner-Poisson problem*, SIAM Journal on Numerical Analysis, 33 (4) 1996, 1622-1643.

[2] Arnold, A., Carrillo, J.A., Dhamo, E., *On the periodic Wigner-Poisson-Fokker-Planck system*, Journal of mathematical analysis and applications, 275 (1) 2002, 263-276.

[3] Arnold, A., Lòpez, J.L., Markowich, P.A. et al., *An analysis of quantum Fokker-Planck models: A Wigner function approach*, Revista Matemática Iberoamericana, 20 (3) 2004, 771-814.

[4] Arnold, A., Dhamo, E., Manzini, C., *The Wigner-Poisson-Fokker-Planck system: global-in-time solution and dispersive effects*, Annales de l'Institut Henri Poincaré, (C), Non Linear Analysis, 24 (4) 2007, 645-676.

[5] Arnold, A., *Mathematical properties of quantum evolution equations*, In Quantum Transport, Springer, Berlin, 2008, 45-109.

[6] Arnold, A., Gamba, I., Gualdani, M.P., et al. *The Wigner-Fokker-Planck equation: stationary states and large time behavior*, Mathematical Models and Methods in Applied Sciences, 22(11) 2012, 1250034-1250065.

[7] Pazy, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer, Berlin, 1983.

[8] Helffer, B., Nier, F., *Hypoelliptic estimates and spectral theory for Fokker-Planck operators and Witten Laplacians*, Lecture Notes in Mathematics, 1862. Springer-Verlag, Berlin, 2005.

[9] Manzini, C., Barletti, L., *An analysis of the Wigner-Poisson problem with inflow boundary conditions*, Nonlinear Analysis: Theory, Methods & Applications, 60 (1) 2005, 77-100.

[10] Manzini, C., *On the three-dimensional WignerCPoisson problem with inflow boundary conditions*, Journal of mathematical analysis and applications, 313 (1) 2006, 184-196.

[11] Sparber, C., Carrillo, J.A., Dolbeault, J. et al., *On the long-time behavior of the quantum Fokker-Planck equation*, Monatshefte für Mathematik, 141 (3) 2004, 237-257.

[12] Hérau, F., Nier, F., *Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with high degree potential*, Archive for Rational Mechanics and Analysis, 171 (2) 2004, 151-218.

[13] Stimming, H.P., *The IVP for the Schrödinger-Poisson-Xα equation in one dimension*, Mathematical Models and Methods in Applied Sciences, 15 (08) 2005, 1169-1180.

[14] Catto, I., Dolbeault, J., Sánchez, O., et al., *Existence of steady states for the Maxwell-Schrödinger-Poisson system: exploring the applicability of the concentration-compactness principle*, Mathematical Models and Methods in Applied Sciences, 23 (10) 2013, 1915-1938.

[15] Gasser, I., Lin, C.K., Markowich, P.A., *A review of dispersive limits of (non)linear Schrödinger-type equations*, Taiwanese Journal of Mathematics, 4 (4) 2000, 501-529.

[16] Cañizo, J.A., López, J.L., Nieto, J., *Global L^1 theory and regularity for the 3D nonlinear Wigner-Poisson-Fokker-Planck system*, Journal of Differential Equations, 198 (2) 2004, 356-373.

[17] Slater, J.C., *A simplification of the Hartree-Fock method*, Physical Review, 81 (3) 1951, 385-390.

[18] Barletti, L., *A mathematical introduction to the Wigner formulation of quantum mechanics*, Bollettino dell'Unione Matematica Italiana, 6 (3) 2003, 693-716.

[19] Evans, L.C., *Partial Differential Equations*, American Math Society, Providence, RI, 1998.

[20] Donarini, A., Novotný, T., Jauho, A.P., *Simple models suffice for the single-dot quantum shuttle*, New Journal of Physics, 7 (1) 2005, 237-262.

[21] Reed, M., Simon, B., *Methods of modern mathematical physics I: functional analysis*, Academic Press, 1980.

[22] Elk, M., Lambropoulos, P., *Connection between approximate Fokker-Planck equations and the Wigner function applied to the micromaser*, Quantum and Semiclassical Optics: Journal of the European Optical Society Part B, 8 (1) 1996, 23-37.

[23] Mauser, N.J., *The Schrödinger-Poisson-Xα equation*, Applied mathematics letters, 14 (6) 2001, 759-763.

[24] Sánchez, O., Soler, J., *Long-time dynamics of the Schrödinger-Poisson-Slater system*, Journal of statistical physics, 114 (1-2) 2004, 179-204.

[25] Markowich, P.A., Ringhofer, C., Schmeiser, C., *Semiconductor Equations*, Springer-Verlag, Vienna, 1990.

[26] Markowich, P.A., *On the equivalence of the Schröinger and the quantum Liouville equations*, Mathematical Methods in the Applied Sciences, 11 (1989), 459-469.

[27] Markowich, P.A., Ringhofer, C.A., *An analysis of the quantum Liouville equation*, ZAMM-Journal of Applied Mathematics and Mechanics/Zeitschrift fr̈ Angewandte Mathematik und Mechanik, 69 (3) 1989, 121-127.

[28] Bao, W., Mauser, N.J., Stimming, H.P., *Effective one particle quantum dynamics of electrons: A numerical study of the schrodinger-poisson-Xα model*, Communications in Mathematical Sciences, 1 (4) 2003, 809-828.

[29] Coffey, W.T., Kalmykov, Y.P., Titov, S.V. et al., *Wigner function approach to the quantum Brownian motion of a particle in a potential*, Physical Chemistry Chemical Physics, 9 (26) 2007, 3361-3382.

Accepted: 23.04.2015