EXTENSIONS OF GENERALIZED LIE GROUPS IN TERMS OF COHOMOLOGY

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Abstract. In this paper we introduce cohomology of generalized Lie groups, explain the problem of generalized Lie group extensions and present them in terms of cohomology of generalized Lie groups. Moreover we find a cohomological obstruction to the existence of extensions in non-Abelian case.

1. Introduction

A Lie group is, roughly speaking, an analytic manifold with a group structure such that the group operations are analytic. Lie groups arise in a natural way as transformation groups of geometric objects. For example, the group of all affine transformations of a connected manifold with an affine connection and the group of all isometries of a pseudo-Riemannian manifold are known to be Lie groups in the compact open topology. However, the group of all diffeomorphisms of a manifold is too big to form a Lie group in any reasonable topology, [12]. The problem of extending a group in terms of cohomology can be found in [1], [10] and in the non-Abelian Case in [2]. This problem can be generalized to Lie groups and their generalizations. Since the case of Lie groupoids and Lie algebroids is spelled out in the literature [3], we had to choose another generalization of Lie groups. A special generalization of Lie groups is called generalized Lie groups or top spaces which was introduced by M.R. Molaei in 1998, [6]. In this generalized field, several authors (Araujo, Molaei, Mehrabi, Oloomi, Tahmoresi, Ebrahimi, etc.) have studied different aspects of generalized groups and top spaces, [8], [4], [9], [5]. We begin by the definition of a generalized group.

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Definition 1.1. [6] A generalized group is a non-empty set \( \mathcal{G} \) admitting an operation called multiplication which satisfies the following conditions:

(i) \((g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)\), for all \(g_1, g_2, g_3 \in \mathcal{G}\).

(ii) For each \(g \in \mathcal{G}\), there exists a unique \(e(g)\) in \(\mathcal{G}\) such that
\[
g \cdot e(g) = e(g) \cdot g = g.
\]

(iii) For each \(g \in \mathcal{G}\), there exists \(h \in \mathcal{G}\) such that
\[
g \cdot h = h \cdot g = e(g).
\]

In this paper, by \(e(\mathcal{G})\) we mean \(\{e(g) : g \in \mathcal{G}\}\).

For any generalized group \(\mathcal{G}\) and any \(g \in \mathcal{G}\),
\[
e^{-1}(e(g)) = \{h \in \mathcal{G} | e(h) = e(g)\}
\]
has a canonical group structure. If \(e(g)e(h) = e(gh)\), for all \(g, h \in \mathcal{G}\), then \(e(\mathcal{G})\) is an idempotent semigroup with this product.

A top space is a smooth manifold which its points can be (smoothly) multiplied together by a generalized group operation and generally its identity is a semigroup morphism, i.e.,

Definition 1.2. [4] A top space \(T\) is a Hausdorff \(d\)-dimensional differentiable manifold which is endowed with a generalized group structure such that the generalized group operations:

(i) \(\cdot : T \times T \to T\) by \((t_1, t_2) \mapsto t_1 \cdot t_2\) which is called the multiplication map;

(ii) \(-1 : T \to T\) by \(t \mapsto t^{-1}\) which is called the inverse map; are differentiable and it holds

(iii) \(e(t_1 \cdot t_2) = e(t_1) \cdot e(t_2)\), for all \(t_1, t_2 \in T\).

Example 1.3. [4] The \(d\)-dimensional Euclidean space \(\mathbb{R}^d\) with the product:
\[
((a_1, ..., a_d), (b_1, ..., b_d)) \mapsto \left(\frac{da_1 + \sum b_i}{d}, ..., \frac{da_d + \sum b_i}{d}\right),
\]
is a top space which is not a Lie group.

Throughout this paper by \(T_a\) we mean \(T \cap e^{-1}(e(a))\).

Definition 1.4. [8] If \(T\) and \(S\) are two top spaces, then a homomorphism \(f : T \to S\) is called a morphism if it is also a \(C^\infty\) map.

By \(f_a\) we mean \(f|_{e^{-1}(e(a))}\), where \(f\) is a morphism of top spaces.

For a smooth manifold \(M\), the set of all smooth functions from \(M\) to \(M\) such that their restriction to a submanifold of \(M\) is a diffeomorphism, i.e., partial diffeomorphisms of \(M\) is denoted by \(D_P(M)\).
Definition 1.5. [11] An action of a top space \( T \) on a smooth manifold \( M \) is a map
\[
\phi : T \to D_P(M),
\]
which satisfies the following conditions:

(i) The map \( T \times M \to M \) which maps \((t, m)\) to \( \phi_t(m) \) is a smooth function;

(ii) \( \phi_{ts} = \phi_t \circ \phi_s \), for all \( t, s \in T \).

Example 1.6. [11] Let \( T \) be a top space.
\[
Ad : T \to D_P(M),
\]
where \( Ad_t(s) = tst^{-1} \) is an action of top spaces on manifolds which is called adjoint action.

Definition 1.7. [11] If \( G \) is a Lie group and \( M \) a smooth manifold, a partial action of \( G \) on \( M \) is a map
\[
\varphi : G \to D_P(M),
\]
such that the map \( G \times M \to M \) is a smooth function and \( \varphi^{gh} = \varphi^g \circ \varphi^h \), for all \( g, h \in G \).

Theorem 1.8. [11] Let \( T \) be a top space. Then \( \phi \) is an action of \( T \) on \( M \) if and only if there exists a family of partial actions \( \{ \varphi_i \}_{i \in e(T)} \) of \( e^{-1}(i) \) on \( M \), for any \( i \in e(T) \), such that \( \varphi^{ts}_{e(ts)} = \varphi^t_{e(t)} \circ \varphi^s_{e(s)} \) and \( \phi_t = \varphi^t_{e(t)} \), for all \( t, s \in T \).

Proof. Let \( \phi \) be an action of top space \( T \) on \( M \) and \( \varphi_{e(t)} = \phi \mid_{e^{-1}(e(t))} \), for all \( t \in T \). Then we know that \( \varphi_{e(t)} \) is a partial action of \( e^{-1}(e(t)) \) on \( M \). Since \( \varphi^t_{e(t)} = \phi \mid_{e^{-1}(e(t))} (t) = \phi_t \), we have \( \varphi^{ts}_{e(ts)} = \phi(st) = \phi_s \circ \phi_t = \varphi^t_{e(t)} \circ \varphi^s_{e(s)} \). Also \( \phi_t = \varphi^t_{e(t)} \).

Conversely, assume that there exists a family of partial actions \( \{ \varphi_i \}_{i \in e(T)} \) of \( e^{-1}(i) \) on \( M \), such that \( \varphi^{ts}_{e(ts)} = \varphi^t_{e(t)} \circ \varphi^s_{e(s)} \), for all \( t, s \in T \). Define \( \phi : T \to D_P(M) \) such that \( \phi_t = \varphi^t_{e(t)} \) for all \( t \in T \). Let \( t, s \in T \), \( \phi_{ts} = \varphi^{ts}_{e(ts)} = \varphi^t_{e(t)} \circ \varphi^s_{e(s)} = \phi_t \circ \phi_s \) and hence \( \phi \) is an action of \( T \) on \( M \).

By Theorem (1.8) we have:

Example 1.9. [11] Let \( T \) be a top space. Then
\[
L : T \to D_P(T),
\]
where \( L_t(s) = ts \) is an action of \( T \) on \( T \), which is called left action of \( T \).

A vector field \( X \) on \( T \) is called partially left invariant if \( (L_t(X))_s = X_{L_t(s)} \).

Example 1.10. [11] Any partially left invariant vector field on \( T \) is a left invariant vector field on \( e^{-1}(e(t)) \), for all \( t \in T \).
Theorem 1.11. [11] Let $T$ be a generalized group such that $e(t)e(s) = e(ts)$, for any $t, s \in T$. Then there is a generalized group isomorphism between $T$ and

$$e(T) \rtimes \{G_i\}_{i \in e(T)},$$

where $G_i = e^{-1}(i)$, for all $i \in e(T)$ and

$$e(T) \rtimes \{G_i\}_{i \in e(T)} = \{(i, g) | g \in G_i\}.$$

Proof. Suppose $G_i = e^{-1}(i)$ for all $i \in e(T)$ and

$$f : T \to e(T) \rtimes \{G_i\}_{i \in S},$$

such that $f(t) = (e(t), t)$. It is easy to show that $f$ is a generalized group isomorphism.

2. Extensions of top spaces and cohomology

We begin this section by introducing the notation of an exact sequence. Let $\{f_i\}_{i \in Z}$, where $f_i : T_i \to T_{i+1}$ be a collection of top spaces homomorphisms, i.e.

$$\ldots \to T_i \xrightarrow{f_i} T_{i+1} \xrightarrow{f_{i+1}} T_{i+2} \to \ldots \, .$$

The sequence $f_i$ is called exact if

$$\text{range}(f_i)_a = \ker(f_{i+1})_a, \text{ for all } a \in e(T_{i+1}).$$

For an exact sequence $(f_i)_a \circ (f_{i+1})_a = e(a)$, i.e. the composition of two adjacent homomorphisms is the trivial homomorphism on each $e(T_{i+1})$. The canonical surjective homomorphism $f_a : T_a \to \frac{T_a}{S_a}$ maps $t \in T_a$ onto the element of $T_a/S_a$ defined by the class to which $t$ belongs in $T_a$. Clearly, $\ker f_a = S_a$ since the Lie subgroup $S_a$, viewed as an element of $T_a/S_a$, is the identity of $T_a/S_a$. Thus, if $S_a \prec T_a$, the sequence of homomorphisms

$$e(a) \to S_a \to T_a \xrightarrow{f_a} \frac{T_a}{S_a} \to e(a),$$

where $f_a$ is the mentioned canonical homomorphism is exact, since $S_a \to T_a$ is an injection and $T_a/S_a \to e(a)$ is the trivial homomorphism. Moreover, the exact sequence

$$e(a) \to S_a \to T_a \to B_a \to e(a)$$

implies that $B_a \approx T_a/S_a$.

Definition 2.1. Let $T, K$ be top spaces. A top space $\tilde{T}$ is said to be an extension of $T$ by $K$ if $K$ is a top generalized normal subgroup of $\tilde{T}$, i.e $K \prec \tilde{T}$, and $\tilde{T}/K = T$. 
In terms of exact sequences, according to [8] and Theorem 1.11, Definition 2.1 is equivalent to saying that

\[ e(a) \rightarrow K_a \rightarrow \tilde{T}_a \rightarrow T_a \rightarrow e(a) \]

is exact for all \( a \in e(T) \); thus \( K_a \) is injected into \( \tilde{T}_a \) and \( \tilde{T}_a \) projected onto \( T_a \) by the canonical homomorphism so that \( T_a = \tilde{T}_a/K_a \). Since \( K \) is a top generalized normal subgroup of \( \tilde{T} \), the elements of \( T_a \) act on \( K_a \) by conjugation. Let \( \text{Aut}K \) be the group of all automorphisms of \( K \). Then there exist functions \( f_a : \tilde{T}_a \rightarrow \text{Aut}K_a \), such that \( t \mapsto [\tilde{t}] \), where \( [\tilde{t}] \) is defined by

\[ [\tilde{t}] : k \in K_a \mapsto \tilde{t}k\tilde{t}^{-1} \in K_a. \]

The kernel of \( f_a \) is the centeralizer \( C_{\tilde{T}_a}(K_a) \) of \( K_a \) in \( \tilde{T}_a \). Thus, we have the following exact sequence of top space homomorphisms:

\[ e(a) \rightarrow C_{\tilde{T}_a}(K_a) \rightarrow \tilde{T}_a \rightarrow \text{Aut}K_a. \]

The center \( C_{K_a} \) of \( K_a \) is top generalized normal subgroup in \( C_{\tilde{T}_a}(K_a) \); if \( H_a \) denotes the quotient group \( H_a = C_{\tilde{T}_a}(K_a)/C_{K_a} \),

\[ e(a) \rightarrow C_{K_a} \rightarrow C_{\tilde{T}_a}(K_a) \rightarrow H_a \rightarrow e(a) \]

is exact.

Let \( T \) be a top space, and \( A \) an Abelian top space, on which \( T \) operates through the homomorphism \( \sigma : T \rightarrow \text{Aut}A = \text{Out}A \) (since \( A \) is Abelian, \( \text{Int}A \) is reduced to the trivial automorphism). As a consequence of Theorem 1.8 there exist partial actions

\[ \sigma_a : T_a \rightarrow \text{Aut}A_a, \]

for all \( a \in e(T) \).

To construct a cohomology for top spaces, we need to introduce the n-cochain maps. A mapping \( \alpha_n : T \times \ldots \times T \rightarrow A \) is a n-cochain, i.e.,

\[ \alpha_n : (t_1, \ldots, t_n) \mapsto \alpha_n(t_1, \ldots, t_n) \in A. \]

The n-cochains form an Abelian top space, i.e. a Lie group, which will be denoted \( C^n(T, A) \), the addition being defined by the pointwise addition of functions and obtaining the usual topology of functions. \( C^0(H, A) \) is defined by the zero-dimensional cochains, i.e., the constant mappings \( \alpha_0 \); thus \( C^0(T, A) = A \). Note that \( \alpha_a \) for all \( a \in e(T) \) is defined on \( T_a \times \ldots \times T_a \) with values in \( A_a \). So we may consider \( C^n(T_a, A_a) \). The operator \( \delta_a : C^n \rightarrow C^{n+1} \) (the coboundary operator) can be defined according to the way that the action \( \sigma(t) \in \text{Aut}A \) of the elements \( t \) of \( T \) is defined on \( A \).
Definition 2.2. For the left action $\sigma$, the operator $\delta_a$ is given by

\[
(\delta_a \alpha_n)(t_1, \ldots, t_{n+1}) := \sigma_a(t_1)\alpha_n(t_2, \ldots, t_{n+1}) + \sum_{i=1}^{n} (-1)^i \alpha_n(t_1, \ldots, t_{i-1}, t_it_{i+1}, t_{i+2}, \ldots, t_{n+1}) + (-1)^{n+1} \alpha_n(t_1, \ldots, t_n).
\]

And when the action $\sigma$ is a right action, $\delta_a$ is given by

\[
(\delta_a \alpha_n)(t_1, \ldots, t_{n+1}) := (-1)^{n+1} \alpha_n(t_2, \ldots, t_{n+1}) + \sum_{i=1}^{n} (-1)^{i+n+1} \alpha_n(t_1, \ldots, t_{i-1}, t_it_{i+1}, t_{i+2}, \ldots, t_{n+1}) + \alpha_n(t_1, \ldots, t_n)\sigma_a(t_{n+1}).
\]

Theorem 2.3. The operator $\delta_a$ satisfies $\delta_a \circ \delta_a = 0$.

Proof. It's a straightforward but cumbersome calculations from the above definitions.

Consider the following sequence of Abelian top spaces:

\[
\begin{array}{c}
C_a^0 \xrightarrow{\delta_a^0} C_a^1 \xrightarrow{\delta_a^1} C_a^2 \xrightarrow{\delta_a^2} \cdots \xrightarrow{\delta_a^n} C_a^{n+1} \xrightarrow{\delta_a^{n+1}} \cdots
\end{array}
\]

Since $\delta_a^2 = 0$,

\[
\text{range}(\delta_a^{n+1} \circ \delta_a^n) = 0, \quad \text{range}\delta_a^n \subset \text{ker}\delta_a^{n+1},
\]

for every $a \in e(T)$, which motivates the definitions

\[
Z_{\sigma_a}^n := \text{ker}\delta_a^n \equiv \{\text{cocycles}\},
\]

\[
B_{\sigma_a}^n := \text{range}\delta_a^{n-1} \equiv \{\text{coboundaries}\}.
\]

Both $Z_{\sigma_a}^n$ and $B_{\sigma_a}^n$ are Lie subgroups of $C^n(T_a, A_a)$.

The quotient group

\[
H_{\sigma_a}^n(T_a, A_a) := Z_{\sigma_a}^n(T_a, A_a) / B_{\sigma_a}^n(T_a, A_a)
\]

is called the $n$-th cohomology of $T_a$ with values on $A_a$ for every $a \in e(T)$.

As in the case of abstract groups, the elements of the second cohomology group, characterize the extensions $\tilde{T}$ of the top space $T$ by the Abelian top space $A$ for the given action $\sigma$ of $T$ on $A$.

We are here concerning about extensions of the top space $T$ by $K$ in the case where $K$ is not Abelian. The main difference from the Abelian case is that
not every top space is associated with one or more extensions, i.e., not every top space is extendible. In fact, one of the aims of this chapter is to show that the top space $K$ determines an obstruction to the extension in the form of a certain three-cocycle; the top space $T$ is extendible if this cocycle is, by an abuse of language, trivial.

For every $a \in e(T)$, consider the following exact sequence

$$e(a) \longrightarrow \text{Int}K_a \longrightarrow \text{Aut}K_a \longrightarrow \text{Out}K_a \longrightarrow e(a),$$

which makes $\text{Aut}K_a$ as an extension of $\text{Out}K_a$ by $\text{Int}K_a$. Let $g_a$ be a trivializing section and let $\alpha_a = g_a \circ \sigma_a$ be defined by

$$e(a) \longrightarrow \text{Int}K_a \longrightarrow \text{Aut}K_a \longrightarrow \text{Out}K_a \longrightarrow e(a).$$

It is clear that there exists an element $h_a(t', t) \in K_a$ such that

$$\alpha_a(t')\alpha_a(t) = [h_a(t', t)]\alpha_a(t'),$$

since $\alpha_a(t')\alpha_a(t)$ and $\alpha_a(t', t)$ differ by an internal automorphism. Consequently, (2.1) defines a mapping

$$[h]_a : T_a \times T_a \rightarrow \text{Int}K_a, \quad [h]_a : (t', t) \mapsto [h_a(t', t)],$$

$$[h_a(t', t)]k := h_a(t', t)kh_a^{-1}(t', t).$$

The associative property in $\text{Aut}K_a$,

$$\alpha_a(t'')\alpha_a(t')\alpha_a(t) = (\alpha_a(t'')\alpha_a(t'))\alpha_a(t),$$

leads to the two-cocycle property for $[h_a(t', t)] \in Z^2_{\alpha_a}(T_a, \text{Int}K_a)$, where

$$[\alpha_a(t'')h_a(t', t)] = \alpha_a(t'')[h_a(t', t)]\alpha_a(t'')^{-1},$$

or

$$[\alpha_a(t''h_a(t', t))] = [h_a(t', t)h_a(t''t', t)].$$

The above equation implies that the elements

$$(\alpha_a(t'')h_a(t', t))h_a(t'', t', t), \quad h_a(t'', t')h_a(t''t', t)$$

of $K_a$ determine the same element of $\text{Int}K_a$. Thus they differ by an element of the center $C_{K_a}$. Therefore the equality (2.2) in $\text{Int}K_a$ leads to an equality in $K_a$,

$$\alpha_a(t'')h_a(t', t) = f_a(t'', t', t)h_a(t'', t')h_a(t''t', t);$$

note that $h_a(t', t)$ would itself be a two-cocycle for $f_a = e(a)$. Equation (2.3) determines a mapping $f_a : T_a \times T_a \times T_a \rightarrow C_{K_a}$, i.e. a three-cochain on $T_a$ with values in the Abelian top space $C_{K_a}$.
Theorem 2.4. The map \( f_a \in Z^3_{(\sigma_0)_a}(T_a, C_{K_a}) \) for \( (\sigma_0)_a(t) = \sigma_a(t) \) acting on \( C_{K_a} \), where it coincides with \( \alpha_a(t) \) for all \( a \in e(T) \).

Proof. To prove the theorem it is sufficient to write \( f_a \) for every \( a \in e(T) \) as
\[
(2.4) \quad f_a(t'', t', t) = (\alpha_a(t'')h_a(t', t))h_a^{-1}(t''t', t)h_a^{-1}(t'', t')
\]
and use this expression to check that \( f_a \) satisfies the cocycle condition,
\[
(\delta_a f_a)(t'', t', t, t_1) = (\alpha_a(t'')f_a(t', t, t_1))f_a^{-1}(t''t', t, t_1) \cdot f_a(t'', t't, t_1)f_a^{-1}(t'', t', t_1) \cdot f_a(t'', t', t) = e(a),
\]
which is obtained from the definition (2.2) written in multiplicative notation.  

Our next step is to prove the following theorem.

Theorem 2.5. Non-Abelian top space \( K \) together with the action \( \sigma \) characterize an element of the third cohomology group \( H^3_{(\sigma_0)_a}(T_a, C_{K_a}) \) for every \( a \in e(T) \).

Proof. We begin the proof by noting that for each action \( \sigma \) on \( K \) there exist partial actions \( \sigma_a \) on \( K_a \) for each \( a \in e(T) \). We have to show that, if the choice of \( g_a \) (which modifies \( \alpha_a \)) or that of \( h_a \) in (2.3) \( [h_a(t', t)] \) is uniquely determined by (2.1), but this fixes \( h_a \) only up to an element of \( C_{K_a} \) is modified, the new \( f_a \) differ from the previous one by a three-coboundary. The proof is done in two steps.

a) Let \( h'_a \) be another element of the class \( h_a \ast C_{K_a} \). Then \( [h'_a(t', t)] = [h_a(t', t)] \) and it is related to \( h_a \) by
\[
(2.5) \quad h'_a(t', t) = c_a(t', t)h_a(t', t), c_a(t', t) \in C_{K_a}.
\]
Let us now consider (2.4) written for \( f'_a \) and \( h'_a \) and replace \( h'_a \) in it by its expression (2.5),
\[
(2.6) \quad f'_a(t'', t', t) = (\alpha_a(t'')(c_a(t', t)h_a(t', t)))c_a(t'', t')h_a(t'', t')
\]
\[
= (\alpha_a(t'')h_a(t', t))c_a(t', t')h_a^{-1}(t''t', t)h_a^{-1}(t'', t')c_a^{-1}(t'', t').
\]
Since \( h_a \) takes values in \( K_a \) and \( c_a \) in \( C_{K_a} \), the \( c_a \)'s locations are irrelevant, and (2.6) may be written as
\[
(2.7) \quad f'_a(t'', t', t) = (\alpha_a(t'')c_a(t'', t)c_a(t', t'))c_a^{-1}(t''t', t)c_a^{-1}(t'', t')
\]
\[
= (\alpha_a(t'')h_a(t', t))h_a(t'', t')h_a^{-1}(t''t', t)h_a^{-1}(t'', t')c_a^{-1}(t'', t').
\]

b) Now let \( \alpha'_a \) be the mapping associated to a different trivializing section \( g'_a \), and let \( [k(t)] \in Int K_a \) be the element which satisfies
\[
(2.8) \quad \alpha'_a(t) = [k(t)]\alpha_a(t), t \in T.
\]
A way of proving that the change of $\alpha_a$ is unimportant is to take advantage of the freedom in $h_a$ to choose an $h'_a$ in such a way that $f_a$ remains unaltered under the combined changes $\alpha_a \rightarrow \alpha'_a$, $h_a \rightarrow h'_a$. First we note that

\begin{equation}
(2.9) \quad \alpha'_a(t') \alpha_a(t) = [k(t')]\alpha_a(t')[k(t)]\alpha_a(t)
\end{equation}

\begin{align*}
&= [k(t')]\alpha_a(t')k(t)\alpha_a(t) \\
&= [k(t')]\alpha_a(t')k(t)[h_a(t', t)]\alpha_a(t') \\
&= [k(t')]\alpha_a(t')k(t)[h_a(t', t)][k^{-1}(t't)]\alpha'_a(t') \\
&\quad [k(t')(\alpha_a(t')k(t))h_a(t', t)k^{-1}(t't)]\alpha'_a(t').
\end{align*}

Now, looking at (2.9), we select an $h'_a(t', t)$ such that

\[ h'_a(t', t) = k(t')(\alpha_a(t')k(t))h_a(t', t)k^{-1}(t't). \]

Then we write (2.4) for for $\alpha'_a$ and $h'_a$ above,

\begin{equation}
(2.10) \quad f'_a(t'', t', t) = \alpha'_a(t'')(k(t')(\alpha_a(t')k(t))h_a(t', t)k^{-1}(t't)) \\
\quad k(t'')(\alpha_a(t'')(k(t't))h_a(t'', t't)k^{-1}(t''t't) \\
\quad k(t''t't)h_a^{-1}(t''t', t)(\alpha_a(t''t')k^{-1}(t))k^{-1}(t''t') \\
\quad k(t''t')h_a^{-1}(t'', t')(\alpha_a(t'')(k(t'')k^{-1}(t'))k^{-1}(t'').
\end{equation}

Using the relation implied by (2.8),

\begin{equation}
(2.11) \quad k(t'') \cdot \alpha_a(t'')() = \alpha'_a(t'')(t') \cdot k(t''),
\end{equation}

equation (2.10) becomes equal to

\begin{equation}
(2.12) \quad \alpha'_a(t'')(k(t')(\alpha_a(t')k(t))h_a(t, t)k^{-1}(t't)) \\
\quad (\alpha_a(t'')k(t't))h_a(t'', t't) \\
\quad h_a^{-1}(t''t', t)(\alpha_a(t''t')k^{-1}(t)) \\
\quad h_a^{-1}(t'', t')(\alpha_a(t'')k^{-1}(t'))k^{-1}(t'').
\end{equation}

Using (2.11) again, (2.12) is seen to be equal to

\begin{equation}
(2.13) \quad \alpha'_a(t'')(k(t')(\alpha_a(t')k(t))k(t'') \\
\quad \{(\alpha_a(t''))h_a^{-1}(t'', t')h_a^{-1}(t''t', t)h_a^{-1} - 1(t'', t')\} \\
\quad (\alpha_a(t''))\alpha_a(t'')k^{-1}(t')(\alpha_a(t'')k^{-1}(t'))k^{-1}(t'') \\
\quad = f_a(t'', t', t)k(t'')\alpha_a(t'')(k(t')\alpha_a(t')k(t)) \\
\quad \alpha_a(t'')(\alpha(a(t')k^{-1}(t'))k^{-1}(t'))k^{-1}(t'') = f_a(t'', t', t).
\end{equation}
If a top space $T_a$ is extendible, then we have the diagram

\[
\begin{array}{cccccc}
  e(a) & \rightarrow & K_a & \rightarrow & \tilde{T}_a & \rightarrow & e(a) \\
  \downarrow \psi_a & & \downarrow \pi & & \downarrow \sigma_a & & \\
  e(a) & \rightarrow & IntK_a & \rightarrow & AutK_a & \rightarrow & OutK_a & \rightarrow & e(a),
\end{array}
\]

where $\psi_a : T_a \rightarrow AutK_a$ is the natural homomorphism defined by the action of the elements $t \in T_a$ on its top generalized normal subgroup $K_a$.

(2.14)

\[\psi_a : \tilde{t} \mapsto [\tilde{t}], [\tilde{t}]k = tkt^{-1}.\]

If $\tilde{T}_a$ is an extension, it is clear that the homomorphisms $\psi_a$ and $\sigma_a$ are compatible.

**Theorem 2.6.** A top space $T$ is extendible if and only if the cocycle $f_a$ which it determines for every $a \in e(T)$ is a three-coboundary for such $a$.

**Proof.** Let $s_a$ be a trivializing section of the principal bundle $\tilde{T}_a(K_a, T_a)$. Now define

\[\alpha_a : T_a \rightarrow AutK_a,\]

by

\[\alpha_a = \psi_a \circ s_a, \alpha_a(t) = \psi_a(s_a(t)) = [s_a(t)].\]

We have

\[
\begin{array}{cccccc}
  e(a) & \rightarrow & K_a & \rightarrow & \tilde{T}_a & \rightarrow & e(a) \\
  \downarrow \psi_a & & \downarrow \pi & & \downarrow \sigma_a & & \\
  e(a) & \rightarrow & IntK_a & \rightarrow & AutK_a & \rightarrow & OutK_a & \rightarrow & e(a),
\end{array}
\]

($T_a$ is supposed here to be known, and so is $\psi_a$ through (2.14)). Then

(2.15)

\[\alpha_a(t')\alpha_a(t) \equiv (\psi_a \circ s_a)(t')(\psi_a \circ s_a)(t)\]

\[\equiv [s_a(t')]s_a(t) \equiv [\omega(t', t)]s_a(t') \equiv [s_a(t')][s_a(t)] \equiv \omega_a(t', t)[s_a(t')] \equiv [\omega_a(t', t)]s_a(t') \equiv \omega_a(t', t)[s_a(t')]\]

Then from (2.1) we obtain

\[\omega_a(t', t) = [\omega_a(t', t)]\]

and an $h_a(t', t)$ in its class by $C_{K_a}$ may be chosen such that

(2.16)

\[\omega_a(t', t) = h_a(t', t).\]

It is now easy to check that $f_a = e(a)$; from its definition we get

\[f_a(t'', t', t) = ([s_a(t'')]\omega_a(t', t)\omega_a(t'', t'')\omega_a^{-1}(t''t', t)\omega_a^{-1}(t'', t')) = e(a),\]
where the last equality follows from the two-cocycle condition for $\omega_a$.

Conversely, we follow the lower part of the diagram to define

\begin{equation}
\alpha_a = g \circ \sigma_a, 
\end{equation}

and choose $h_a$ (by theorem (2.5)) so that $f_a = e(a)$. Let us now label the elements of $T_a$ by $\tilde{t} = (k, t)$ and define the multiplication law by

\begin{equation}
\tilde{t}'' = (k', t')(k, t) = (k' (\alpha_a(t) k) h_a(t', t), t't).
\end{equation}

Since $f_a = e(a)$, $h_a(t', t)$ is a two-cocycle. Then the two equations

\begin{equation}
(\alpha_a(t') h_a(t', t)) h_a(t'', t') = h_a(t''', t') h_a(t'''', t),
\end{equation}

show immediately that the group law is associative, that $\tilde{e} = (1, e)$ is the identity element and that

\begin{equation}
\tilde{t}^{-1} \equiv (k, t)^{-1} = (\alpha(t)^{-1} (k^{-1} h^{-1}(t, t^{-1})), t^{-1}).
\end{equation}

Thus, $g \circ \sigma_a = \psi_a \circ s_a$, and our diagram is well defined. Also $\tilde{T}_a$ has a topology which makes it a top space according to [8].

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\section*{References}


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