EXTENSIONS OF GENERALIZED LIE GROUPS IN TERMS OF COHOMOLOGY

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Abstract. In this paper we introduce cohomology of generalized Lie groups, explain the problem of generalized Lie group extensions and present them in terms of cohomology of generalized Lie groups. Moreover we find a cohomological obstruction to the existence of extensions in non-Abelian case.

1. Introduction

A Lie group is, roughly speaking, an analytic manifold with a group structure such that the group operations are analytic. Lie groups arise in a natural way as transformation groups of geometric objects. For example, the group of all affine transformations of a connected manifold with an affine connection and the group of all isometries of a pseudo-Riemannian manifold are known to be Lie groups in the compact open topology. However, the group of all diffeomorphisms of a manifold is too big to form a Lie group in any reasonable topology, [12]. The problem of extending a group in terms of cohomology can be found in [1], [10] and in the non-Abelian Case in [2]. This problem can be generalized to Lie groups and their generalizations. Since the case of Lie groupoids and Lie algebroids is spelled out in the literature [3], we had to choose another generalization of Lie groups. A special generalization of Lie groups is called generalized Lie groups or top spaces which was introduced by M.R. Molaei in 1998, [6]. In this generalized field, several authors (Araujo, Molaei, Mehrabi, Oloomi, Tahmoresi, Ebrahimi, etc.) have studied different aspects of generalized groups and top spaces, [8], [4], [9], [5]. We begin by the definition of a generalized group.

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Definition 1.1. [6] A generalized group is a non-empty set $\mathcal{G}$ admitting an operation called multiplication which satisfies the following conditions:

(i) $(g_1 \cdot g_2) \cdot g_3 = g_1 \cdot (g_2 \cdot g_3)$, for all $g_1, g_2, g_3 \in \mathcal{G}$.

(ii) For each $g \in \mathcal{G}$, there exists a unique $e(g)$ in $\mathcal{G}$ such that

$$g \cdot e(g) = e(g) \cdot g = g.$$

(iii) For each $g \in \mathcal{G}$, there exists $h \in \mathcal{G}$ such that $g \cdot h = h \cdot g = e(g)$.

In this paper, by $e(\mathcal{G})$ we mean

$$\{e(g) : g \in \mathcal{G}\}.$$

For any generalized group $\mathcal{G}$ and any $g \in \mathcal{G}$,

$$e^{-1}(e(g)) = \{h \in \mathcal{G} | e(h) = e(g)\}$$

has a canonical group structure. If $e(g)e(h) = e(gh)$, for all $g, h \in \mathcal{G}$, then $e(\mathcal{G})$ is an idempotent semigroup with this product.

A top space is a smooth manifold which its points can be (smoothly) multiplied together by a generalized group operation and generally its identity is a semigroup morphism, i.e.,

Definition 1.2. [4] A top space T is a Hausdorff d-dimensional differentiable manifold which is endowed with a generalized group structure such that the generalized group operations:

(i) $\cdot : T \times T \to T$ by $(t_1, t_2) \mapsto t_1 \cdot t_2$ which is called the multiplication map;

(ii) ${}^{-1} : T \to T$ by $t \mapsto t^{-1}$ which is called the inverse map; are differentiable and it holds

(iii) $e(t_1 \cdot t_2) = e(t_1) \cdot e(t_2)$, for all $t_1, t_2 \in T$.

Example 1.3. [4] The d-dimensional Euclidean space $\mathbb{R}^d$ with the product:

$$((a_1, ..., a_d), (b_1, ..., b_d)) \mapsto \left(\frac{da_1 + \sum b_i}{d}, ..., \frac{da_d + \sum b_i}{d} \right),$$

is a top space which is not a Lie group.

Throughout this paper by T_a we mean $T \cap e^{-1}(e(a))$.

Definition 1.4. [8] If T and S are two top spaces, then a homomorphism $f : T \to S$ is called a morphism if it is also a C^∞ map.

By f_a we mean $f|_{e^{-1}(e(a))}$, where f is a morphism of top spaces.

For a smooth manifold M, the set of all smooth functions from M to M such that their restriction to a submanifold of M is a diffeomorphism, i.e., partial diffeomorphisms of M is denoted by $D_P(M)$.

Definition 1.5. [11] An action of a top space T on a smooth manifold M is a map

$$\phi : T \to D_P(M),$$

which satisfies the following conditions:

(i) The map $T \times M \to M$ which maps (t, m) to $\phi_t(m)$ is a smooth function;

(ii) $\phi_{ts} = \phi_t \circ \phi_s$, for all $t, s \in T$.

Example 1.6. [11] For a top space T,

$$Ad : T \to D_P(M),$$

where $Ad_t(s) = tst^{-1}$ is an action of top spaces on manifolds which is called adjoint action.

Definition 1.7. [11] If G is a Lie group and M a smooth manifold, a partial action of G on M is a map

$$\varphi : G \to D_P(M),$$

such that the map $G \times M \to M$ is a smooth function and $\varphi^{gh} = \varphi^g \circ \varphi^h$, for all $g, h \in G$.

Theorem 1.8. [11] *Let T be a top space. Then ϕ is an action of T on M if and only if there exists a family of partial actions $\{\varphi_i\}_{i\in e(T)}$ of $e^{-1}(i)$ on M, for any $i \in e(T)$, such that $\varphi^{ts}_{e(ts)} = \varphi^t_{e(t)} \circ \varphi^s_{e(s)}$ and $\phi_t = \varphi^t_{e(t)}$, for all $t, s \in T$.*

Proof. Let ϕ be an action of top space T on M and $\varphi_{e(t)} = \phi \mid_{e^{-1}(e(t))}$, for all $t \in T$. Then we know that $\varphi_{e(t)}$ is a partial action of $e^{-1}(e(t))$ on M. Since $\varphi^t_{e(t)} = \phi \mid_{e^{-1}(e(t))} (t) = \phi_t$, we have $\varphi^{ts}_{e(ts)} = \phi(st) = \phi_s \circ \phi_t = \varphi^t_{e(t)} \circ \varphi^s_{e(s)}$. Also $\phi_t = \varphi^t_{e(t)}$.

Conversely, assume that there exists a family of partial actions $\{\varphi_i\}_{i\in e(T)}$ of $e^{-1}(i)$ on M, such that $\varphi^{ts}_{e(ts)} = \varphi^t_{e(t)} \circ \varphi^s_{e(s)}$, for all $t, s \in T$. Define $\phi : T \to D_P(M)$ such that $\phi_t = \varphi^t_{e(t)}$ for all $t \in T$. Let $t, s \in T$, $\phi_{ts} = \varphi^{ts}_{e(ts)} = \varphi^t_{e(t)} \circ \varphi^s_{e(s)} = \phi_t \circ \phi_s$ and hence ϕ is an action of T on M. ∎

By Theorem (1.8) we have:

Example 1.9. [11] Let T be a top space. Then

$$L : T \to D_P(T),$$

where $L_t(s) = ts$ is an action of T on T, which is called left action of T.

A vector field X on T is called partially left invariant if $(L_{*t}(X))_s = X_{L_t(s)}$.

Example 1.10. [11] Any partially left invariant vector field on T is a left invariant vector field on $e^{-1}(e(t))$, for all $t \in T$.

Theorem 1.11. [11] *Let T be a generalized group such that $e(t)e(s) = e(ts)$, for any $t, s \in T$. Then there is a generalized group isomorphism between T and*

$$e(T) \ltimes \{G_i\}_{i\in e(T)},$$

where $G_i = e^{-1}(i)$, for all $i \in e(T)$ and

$$e(T) \ltimes \{G_i\}_{i\in e(T)} = \{(i, g)|g \in G_i\}.$$

Proof. Suppose $G_i = e^{-1}(i)$ for all $i \in e(T)$ and

$$f : T \to e(T) \ltimes \{G_i\}_{i\in S},$$

such that $f(t) = (e(t), t)$. It is easy to show that f is a generalized group isomorphism. ∎

2. Extensions of top spaces and cohomology

We begin this section by introducing the notation of an exact sequence. Let $\{f_i\}_{i\in\mathbb{Z}}$, where $f_i : T_i \to T_{i+1}$ be a collection of top spaces homomorphisms, i.e.

$$\cdots \longrightarrow T_i \xrightarrow{f_i} T_{i+1} \xrightarrow{f_{i+1}} T_{i+2} \longrightarrow \cdots .$$

The sequence f_i is called exact if

$$range(f_i)_a = ker(f_{i+1})_a, \ for \ all \ a \in e(T_{i+1}).$$

For an exact sequence $(f_i)_a \circ (f_{i+1})_a = e(a)$, i.e. the composition of two adjacent homomorphisms is the trivial homomorphism on each $e(T_{i+1})$. The canonical surjective homomorphism $f_a : T_a \to \frac{T_a}{S_a}$ maps $t \in T_a$ onto the element of T_a/S_a defined by the class to which t belongs in T_a. Clearly, $\ker f_a = S_a$ since the Lie subgroup S_a, viewed as an element of T_a/S_a, is the identity of T_a/S_a. Thus, if $S_a \prec T_a$, the sequence of homomorphisms

$$e(a) \longrightarrow S_a \longrightarrow T_a \xrightarrow{f_a} T_a/S_a \longrightarrow e(a),$$

where f_a is the mentioned canonical homomorphism is exact, since $S_a \to T_a$ is an injection and $T_a/S_a \to e(a)$ is the trivial homomorphism. Moreover, the exact sequence

$$e(a) \longrightarrow S_a \longrightarrow T_a \longrightarrow B_a \longrightarrow e(a)$$

implies that $B_a \approx T_a/S_a$.

Definition 2.1. Let T, K be top spaces. A top space $\tilde{T}$ is said to be an extension of T by K if K is a top generalized normal subgroup of $\tilde{T}$, i.e $K \prec \tilde{T}$, and $\tilde{T}/K = T$.

In terms of exact sequences, according to [8] and Theorem 1.11, Definition 2.1 is equivalent to saying that

$$e(a) \longrightarrow K_a \longrightarrow \tilde{T}_a \longrightarrow T_a \longrightarrow e(a)$$

is exact for all $a \in e(T)$; thus K_a is injected into $\tilde{T}_a$ and $\tilde{T}_a$ projected onto T_a by the canonical homomorphism so that $T_a = \tilde{T}_a/K_a$. Since K is a top generalized normal subgroup of $\tilde{T}$, the elements of $\tilde{T}_a$ act on K_a by conjugation. Let Aut$K$ be the group of all automorphisms of K. Then there exist functions $f_a : \tilde{T}_a \to AutK_a$, such that $t \mapsto [\tilde{t}]$, where $[\tilde{t}]$ is defined by

$$[\tilde{t}] : k \in K_a \mapsto \tilde{t}k\tilde{t}^{-1} \in K_a \ .$$

The kernel of f_a is the centeralizer $C_{\tilde{T}_a}(K_a)$ of K_a in $\tilde{T}_a$. Thus, we have the following exact sequence of top space homomorphisms:

$$e(a) \longrightarrow C_{\tilde{T}_a}(K_a) \longrightarrow \tilde{T}_a \xrightarrow{f_a} AutK_a \ .$$

The center C_{K_a} of K_a is top generalized normal subgroup in $C_{\tilde{T}}(K)$; if H_a denotes the quotient group $H_a = C_{\tilde{T}_a}(K_a)/C_{K_a}$,

$$e(a) \longrightarrow C_{K_a} \longrightarrow C_{\tilde{T}_a}(K_a) \longrightarrow H_a \longrightarrow e(a)$$

is exact.

Let T be a top space, and A an Abelian top space, on which T operates through the homomorphism $\sigma : T \to \mathrm{Aut}A = \mathrm{Out}\ A$ (since A is Abelian, Int A is reduced to the trivial automorphism). As a consequence of Theorem 1.8 there exist partial actions

$$\sigma_a : T_a \to AutA_a,$$

for all $a \in e(T)$.

To construct a cohomology for top spaces, we need to introduce the n-cochain maps. A mapping $\alpha_n : T \times \ldots \times T \to A$ is a n-cochain, i.e.,

$$\alpha_n : (t_1, \ldots, t_n) \mapsto \alpha_n(t_1, \ldots, t_n) \in A \ .$$

The n-cochains form an Abelian top space, i.e. a Lie group, which will be denoted $C^n(T, A)$, the addition being defined by the pointwise addition of functions and obtaining the usual topology of functions. $C^0(H, A)$ is defined by the zero-dimensional cochains, i.e., the constant mappings α_0; thus $C^0(T, A) = A$. Note that α_a for all $a \in e(T)$ is defined on $T_a \times \ldots \times T_a$ with values in A_a. So we may consider $C^n(T_a, A_a)$. The operator $\delta_a : C^n :\to C^{n+1}$ (the coboundary operator) can be defined according to the way that the action $\sigma(t) \in \mathrm{Aut}A$ of the elements t of T is defined on A.

Definition 2.2. For the left action σ, the operator δ_a is given by

$$\begin{aligned}(\delta_a\alpha_n)(t_1,\ldots,t_{n+1}) :=& \sigma_a(t_1)\alpha_n(t_2,\ldots,t_{n+1})\\ &+\sum_{i=1}^{n}(-1)^i\alpha_n(t_1,\ldots,t_{i-1},t_it_{i+1},t_{i+2},\ldots,t_{n+1})\\ &+(-1)^{n+1}\alpha_n(t_1,\ldots,t_n).\end{aligned}$$

And when the action σ is a right action, δ_a is given by

$$\begin{aligned}(\delta_a\alpha_n)(t_1,\ldots,t_{n+1}) :=& (-1)^{n+1}\alpha_n(t_2,\ldots,t_{n+1})\\ &+\sum_{i=1}^{n}(-1)^{i+n+1}\alpha_n(t_1,\ldots,t_{i-1},t_it_{i+1},t_{i+2},\ldots,t_{n+1})\\ &+\alpha_n(t_1,\ldots,t_n)\sigma_a(t_{n+1}).\end{aligned}$$

Theorem 2.3. *The operator δ_a satisfies $\delta_a\circ\delta_a = 0$.*

Proof. It's a straightforward but cumbersome calculations from the above definitions. ∎

Consider the following sequence of Abelian top spaces:

$$C_a^0 \xrightarrow{\delta_a^0} C_a^1 \xrightarrow{\delta_a^1} C_a^2 \longrightarrow \cdots \xrightarrow{\delta_a^n} C_a^{n+1} \longrightarrow \cdots .$$

Since $\delta_a^2 = 0$,

$$range(\delta_a^{n+1}\circ\delta_a^n) = 0,\ range\delta_a^n \subset ker\delta_a^{n+1},$$

for every $a \in e(T)$, which motivates the definitions

$$Z_{\sigma_a}^n := ker\delta_a^n \equiv \{cocycles\},$$

$$B_{\sigma_a}^n := range\delta_a^{n-1} \equiv \{coboundaries\}.$$

Both $Z_{\sigma_a}^n$ and $B_{\sigma_a}^n$ are Lie subgroups of $C^n(T_a, A_a)$.

The quotient group

$$H_{\sigma_a}^n(T_a, A_a) := Z_{\sigma_a}^n(T_a, A_a)/B_{\sigma_a}^n(T_a, A_a)$$

is called the n-th cohomology of T_a with values on A_a for every $a \in e(T)$.

As in the case of abstract groups, the elements of the second cohomology group, characterize the extensions $\tilde{T}$ of the top space T by the Abelian top space A for the given action σ of T on A.

We are here concerning about extensions of the top space T by K in the case where K is not Abelian. The main difference from the Abelian case is that not every top space is associated with one or more extensions, i.e., not every top space is extendible. In fact, one of the aims of this chapter is to show that the top space K determines an obstruction to the extension in the form of a certain three-cocycle; the top space T is extendible if this cocycle is, by an abuse of language, trivial.

For every $a \in e(T)$, consider the following exact sequence

$$e(a) \longrightarrow IntK_a \longrightarrow AutK_a \longrightarrow OutK_a \longrightarrow e(a),$$

which makes $AutK_a$ as an extension of $OutK_a$ by $IntK_a$. Let g_a be a trivializing section and let $\alpha_a = g_a \circ \sigma_a$ be defined by

$$\begin{array}{ccccccccc} & & & & & & T_a & & \\ & & & & & \swarrow & \downarrow \sigma & & \\ e(a) & \longrightarrow & IntK_a & \longrightarrow & AutK_a & \underset{g_a}{\rightleftarrows} & OutK_a & \longrightarrow & e(a). \end{array}$$

It is clear that there exists an element $h_a(t', t) \in K_a$ such that

$$\alpha_a(t')\alpha_a(t) = [h_a(t', t)]\alpha_a(t't), \tag{2.1}$$

since $\alpha_a(t')\alpha_a(t)$ and $\alpha_a(t', t)$ differ by an internal automorphism. Consequently, (2.1) defines a mapping

$$[h]_a : T_a \times T_a \to IntK_a,\ [h]_a : (t', t) \mapsto [h_a(t', t)],$$

$$[h_a(t', t)]k := h_a(t', t)kh_a^{-1}(t', t).$$

The associative property in $AutK_a$,

$$\alpha_a(t'')(\alpha_a(t')\alpha_a(t)) = (\alpha_a(t'')\alpha_a(t'))\alpha_a(t),$$

leads to the two-cocycle property for $[h_a(t', t)] \in Z^2_{\alpha_a}(T_a, IntK_a)$, where

$$[\alpha_a(t'')h_a(t', t)] = \alpha_a(t'')[h_a(t', t)]\alpha_a(t'')^{-1},$$

or

$$[(\alpha_a(t'')h_a(t', t))h_a(t'', t't)] = [h_a(t'', t')h_a(t''t', t)]. \tag{2.2}$$

The above equation implies that the elements

$$(\alpha_a(t'')h_a(t', t))h_a(t'', t', t),\ h_a(t'', t')h_a(t''t', t)$$

of K_a determine the same element of $IntK_a$. Thus they differ by an element of the center C_{K_a}. Therefore the equality (2.2) in $IntK_a$ leads to an equality in K_a,

$$(\alpha_a(t'')h_a(t', t))h_a(t'', t't) = f_a(t'', t', t)h_a(t'', t')h_a(t''t', t); \tag{2.3}$$

note that $h_a(t', t)$ would itself be a two-cocycle for $f_a = e(a)$. Equation (2.3) determines a mapping $f_a : T_a \times T_a \times T_a \to C_{K_a}$, i.e. a three-cochain on T_a with values in the Abelian top space C_{K_a}.

Theorem 2.4. *The map $f_a \in Z^3_{(\sigma_0)_a}(T_a, C_{K_a})$ for $(\sigma_0)_a(t) = \sigma_a(t)$ acting on C_{K_a}, where it coincides with $\alpha_a(t)$ for all $a \in e(T)$.*

Proof. To prove the theorem it is sufficient to write f_a for every $a \in e(T)$ as

$$f_a(t'', t', t) = (\alpha_a(t'')h_a(t', t))h_a^{-1}(t''t', t)h_a^{-1}(t'', t') \tag{2.4}$$

and use this expression to check that f_a satisfies the cocycle condition,

$$\begin{aligned}(\delta_a f_a)(t'', t', t, t_1) \\ &= (\alpha_a(t'')f_a(t', t, t_1))f_a^{-1}(t''t', t, t_1) \cdot f_a(t'', t't, t_1)f_a^{-1}(t'', t', tt_1) \cdot f_a(t'', t', t) \\ &= e(a),\end{aligned}$$

which is obtained from the definition (2.2) written in multiplicative notation. ∎

Our next step is to prove the following theorem.

Theorem 2.5. *Non-Abelian top space K together with the action σ characterize an element of the third cohomology group $H^3_{(\sigma_0)_a}(T_a, C_{K_a})$ for every $a \in e(T)$.*

Proof. We begin the proof by noting that for each action σ on K there exist partial actions σ_a on K_a for each $a \in e(T)$. We have to show that, if the choice of g_a (which modifies α_a) or that of h_a in (2.3) ($[h_a(t', t)]$ is uniquely determined by (2.1), but this fixes h_a only up to an element of C_{K_a}) is modified, the new f_a differ from the previous one by a three-coboundary. The proof is done in two steps.

a) Let h'_a be another element of the class $h_a * C_{K_a}$. Then $[h'_a(t', t)] = [h_a(t', t)]$ and it is related to h_a by

$$h'_a(t', t) = c_a(t', t)h_a(t', t), c_a(t', t) \in C_{K_a}. \tag{2.5}$$

Let us now consider (2.4) written for f'_a and h'_a and replace h'_a in it by its expression (2.5),

$$\begin{aligned}f'_a(t'', t', t) =& (\alpha_a(t'')(c_a(t', t)h_a(t', t)))c_a(t'', t't)h_a(t'', t't) \\ & h_a^{-1}(t''t', t)c_a^{-1}(t''t', t)h_a^{-1}(t'', t')c_a^{-1}(t'', t').\end{aligned} \tag{2.6}$$

Since h_a takes values in K_a and c_a in C_{K_a}, the c_a's locations are irrelevant, and (2.6) may be written as

$$\begin{aligned}f'_a(t'', t', t) =& (\alpha_a(t'')c_a(t', t))c_a(t'', t't)c_a^{-1}(t''t', t)c_a^{-1}(t'', t') \\ & (\alpha_a(t'')h_a(t', t))h_a(t'', t't)h_a^{-1}(t''t', t)h_a^{-1}(t'', t') \\ & \equiv (\delta_a c_a)(t'', t', t)f_a(t'', t', t).\end{aligned} \tag{2.7}$$

b) Now let α'_a be the mapping associated to a different trivializing section g'_a, and let $[k(t)] \in IntK_a$ be the element which satisfies

$$\alpha'_a(t) = [k(t)]\alpha_a(t), t \in T. \tag{2.8}$$

A way of proving that the change of α_a is unimportant is to take advantage of the freedom in h_a to choose an h'_a in such a way that f_a remains unaltered under the combined changes $\alpha_a \to \alpha'_a$, $h_a \to h'_a$. First we note that

$$\begin{aligned}\alpha'_a(t')\alpha_a(t) =& [k(t')]\alpha_a(t')[k(t)]\alpha_a(t) \\ &= [k(t')][\alpha_a(t')k(t)]\alpha_a(t')\alpha_a(t) \\ &= [k(t')][\alpha_a(t')k(t)][h_a(t',t)]\alpha_a(t't) \\ &= [k(t')][\alpha_a(t')k(t)][h_a(t',t)][k^{-1}(t't)]\alpha'_a(t't) \\ &\quad [k(t')(\alpha_a(t')k(t))h_a(t',t)k^{-1}(t't)]\alpha'_a(t't).\end{aligned} \tag{2.9}$$

Now, looking at (2.9), we select an $h'_a(t',t)$ such that

$$h'_a(t',t) = k(t')(\alpha_a(t')k(t))h_a(t',t)k^{-1}(t't).$$

Then we write (2.4) for for α'_a and h'_a above,

$$\begin{aligned}f'_a(t",t',t) =& \alpha'_a(t'')(k(t')(\alpha_a(t')k(t))h_a(t',t)k^{-1}(t't)) \\ & k(t'')(\alpha_a(t'')k(t't))h_a(t'',t't)k^{-1}(t''t't) \\ & k(t''t't)h_a^{-1}(t''t',t)(\alpha_a(t''t')k^{-1}(t))k^{-1}(t''t') \\ & k(t''t')h_a^{-1}(t'',t')(\alpha_a(t'')k^{-1}(t'))k^{-1}(t'').\end{aligned} \tag{2.10}$$

Using the relation implied by (2.8),

$$k(t'')\cdot\alpha_a(t'')() = \alpha'_a(t'')()\cdot k(t''), \tag{2.11}$$

equation (2.10) becomes equal to

$$\begin{aligned}& \alpha'_a(t'')(k(t')(\alpha_a(t')k(t))h_a(t,t)k^{-1}(t't)) \\ & (\alpha_a(t'')k(t't))k(t'')h_a(t'',t't) \\ & h_a^{-1}(t''t',t)(\alpha_a(t''t')k^{-1}(t)) \\ & h_a^{-1}(t",t')(\alpha_a(t")k^{-1}(t'))k^{-1}(t").\end{aligned} \tag{2.12}$$

Using (2.11) again, (2.12) is seen to be equal to

$$\begin{aligned}& \alpha'_a(t'')(k(t')\alpha_a(t')k(t))k(t'') \\ & \{(\alpha_a(t'')h_a(t',t))h_a(t'',t't)h_a^{-1}(t''t',t)h_a^{-1} - 1(t'',t')\} \\ & (\alpha_a(t'')\alpha_a(t')k^{-1}(t))(\alpha_a(t'')k^{-1}(t'))k^{-1}(t'') \\ & = f_a(t'',t',t)k(t'')\alpha_a(t'')(k(t')\alpha_a(t')k(t)) \\ & \alpha_a(t'')((a(t')k^{-1}(t))k^{-1}(t'))k^{-1}(t'') = f_a(t'',t',t).\end{aligned} \tag{2.13}$$

■

If a top space T_a is extendible, then we have the diagram

$$\begin{array}{ccccccccc}
e(a) & \longrightarrow & K_a & \xrightarrow{i} & \tilde{T}_a & \xrightarrow{\pi} & T_a & \longrightarrow & e(a) \\
 & & & & \big\downarrow{\scriptstyle \psi_a} & & \big\downarrow{\scriptstyle \sigma_a} & & \\
e(a) & \longrightarrow & IntK_a & \longrightarrow & AutK_a & \longrightarrow & OutK_a & \longrightarrow & e(a),
\end{array}$$

where $\psi_a : T_a \to AutK_a$ is the natural homomorphism defined by the action of the elements $t \in T_a$ on its top generalized normal subgroup K_a,

$$\psi_a : \tilde{t} \mapsto [\tilde{t}], [\tilde{t}]k = tkt^{-1}. \tag{2.14}$$

If $\tilde{T_a}$ is an extension, it is clear that the homomorphisms ψ_a and σ_a are compatible.

Theorem 2.6. *A top space T is extendible if and only if the cocycle f_a which it determines for every $a \in e(T)$ is a three-coboundary for such a.*

Proof. Let s_a be a trivializing section of the principal bundle $\tilde{T_a}(K_a, T_a)$. Now define

$$\alpha_a : T_a \to AutK_a,$$

by

$$\alpha_a = \psi_a \circ s_a, \alpha_a(t) = \psi_a(s_a(t)) = [s_a(t)].$$

We have

$$\begin{array}{ccccccccc}
e(a) & \longrightarrow & K_a & \longrightarrow & \tilde{T}_a & \underset{s_a}{\overset{\pi}{\rightleftarrows}} & T_a & \longrightarrow & e(a) \\
 & & & & \big\downarrow{\scriptstyle \psi_a} & \swarrow{\scriptstyle \alpha_a} & \big\downarrow{\scriptstyle \sigma_a} & & \\
e(a) & \longrightarrow & IntK_a & \longrightarrow & AutK_a & \underset{g}{\rightleftarrows} & OutK_a & \longrightarrow & e(a),
\end{array}$$

(T_a is supposed here to be known, and so is ψ_a through (2.14)). Then

$$\begin{aligned}
\alpha_a(t')\alpha_a(t) &\equiv (\psi_a \circ s_a)(t')(\psi_a o s_a)(t) \\
&\equiv [s_a(t')][s_a(t)] \equiv [\omega_a(t', t)][s_a(t't)] \\
&\equiv [\omega_a(t', t)](\psi_a \circ s_a)(t't) = [\omega_a(t', t)]\alpha_a(t't).
\end{aligned} \tag{2.15}$$

Then from (2.1) we obtain

$$[\omega_a(t', t)] = [h_a(t', t)]$$

and an $h_a(t', t)$ in its class by C_{K_a} may be chosen such that

$$\omega_a(t', t) = h_a(t', t). \tag{2.16}$$

It is now easy to check that $f_a = e(a)$; from its definition we get

$$f_a(t'', t', t) = ([s_a(t'')]\omega_a(t', t))\omega_a(t'', t't)\omega_a^{-1}(t''t', t)\omega_a^{-1}(t'', t') = e(a),$$

where the last equality follows from the two-cocycle condition for ω_a.

Conversely, we follow the lower part of the diagram to define

$$\alpha_a = g \circ \sigma_a, \tag{2.17}$$

and choose h_a (by theorem (2.5)) so that $f_a = e(a)$. Let us now label the elements of $\tilde{T}_a$ by $\tilde{t} = (k, t)$ and define the multiplication law by

$$\tilde{t}'' = (k', t')(k, t) = (k'(\alpha_a(t)k)h_a(t', t), t't). \tag{2.18}$$

Since $f_a = e(a)$, $h_a(t', t)$ is a two-cocycle. Then the two equations

$$\alpha_a(t')\alpha_a(t) = [h_a(t', t)]\alpha_a(t't), \tag{2.19}$$

$$(\alpha_a(t'')h_a(t', t))h_a(t'', t't) = h_a(t'', t')h_a(t''t', t), \tag{2.20}$$

show immediately that the group law is associative, that $\tilde{e} = (1, e)$ is the identity element and that

$$\tilde{t}^{-1} \equiv (k, t)^{-1} = (\alpha(t)^{-1}(k^{-1}h^{-1}(t, t^{-1})), t^{-1}).$$

Thus, $g \circ \sigma_a = \psi_a \circ s_a$, and our diagram is well defined. Also $\tilde{T}_a$ has a topology which makes it a top space according to [8]. ∎

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