ON STRONG AND WEAK HYPER KS-IDEALS OF DECOMPOSABLE HYPER KS-SEMIGROUPS

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Abstract. This paper extends the study of hyper KS-semigroups, an algebraic hyperstructure which is a combination of hyper BCK-algebra and semihypergroups with respect to some hyperoperations to the concept of decomposable hyper KS-semigroups. We introduce the concept of strong and weak hyper KS-ideals and present some of their properties. Furthermore, we establish the idea of decomposable hyper KS-semigroups in the context of strong, weak and hyper KS-ideals, give conditions for decomposable hyper KS-semigroups, investigate its properties and structure and provide some characterizations.

Keywords: strong hyper KS-ideals, weak hyper KS-ideal, decomposable hyper KS-semigroups.

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1. Introduction

In 1966, Y. Imai and K. Iséki [4] initiated the study of BCK-algebra as a generalization of the concept of set-theoretic difference and propositional calculi. In 2006, K.H. Kim [8] introduced a new class of algebra called KS-semigroups, which is a combination of BCK-algebra and semigroup. He characterized the KS-semigroups from its ideals up to the first isomorphism theorem while M.P. Cawi and J.P. Vilela [3], proved the second and third isomorphism theorems for KS-semigroups and gave other characterizations parallel to ring theory.

The hyperstructure theory (or multialgebras) was introduced by F. Marty [9] at the 8th Congress of Scandinavian Mathematicians in 1934. In a classical algebraic structure, the composition of two elements is an element, while in an algebraic hyperstructure, the composition of two elements is a set. In 2000, Y.B. Jun et al. applied hyperstructure theory to BCK-algebras and introduced the notion of hyper BCK-algebras as a generalization of BCK-algebra and investigated some of its properties. In 2014, A.L. Vicedo et al. [10] applied hyperstructure theory in KS-semigroups and introduced the notion of hyper KS-semigroups, a combination of concepts of hyper BCK-algebra and semihypergroups and investigated its structural properties.

This paper extends the study of hyper KS-semigroups to the concept of decomposable hyper KS-semigroups. We establish the idea of decomposable hyper KS-semigroups in the context of strong, weak and hyper KS-ideals, investigate its properties and structure and provide some characterizations. Moreover, we provide conditions for decomposable hyper KS-semigroups.

2. Preliminaries

Let H be a nonempty set endowed with a hyperoperation " $*$ ", that is, " $*$ " is a function from $H \times H$ to $P^*(H) = P(H) \setminus \{\varnothing\}$. For two nonempty subsets A and B of H, $A * B = \bigcup_{a \in A, b \in B} a * b$. We shall use $x * y$ instead of $x * \{y\}, \{x\} * y$ or $\{x\} * \{y\}$. When A is a nonempty subset of H and $x \in H$, we agree to write $A * x$ instead of $A * \{x\}$. Similarly, we write $x * A$ for $\{x\} * A$. In effect, $A * x = \bigcup_{a \in A} a * x$ and $x * A = \bigcup_{a \in A} x * a$. A set H endowed with a family Γ of hyperoperations is called a *hyperstructure*. If Γ is singleton, that is, $\Gamma = \{f\}$, then the hyperstructure is called a *hypergroupoid*. A *semihypergroup* is a hypergroupoid $(H, \cdot)$ such that for all $x, y, z \in H, (x \cdot y) \cdot z = x \cdot (y \cdot z)$, that is, $\bigcup_{u \in x \cdot y} u \cdot z = \bigcup_{v \in y \cdot z} x \cdot v$.

Definition 2.1 [5] A *hyper BCK-algebra* is a nonempty set H endowed with a hyperoperation " $*$ " and a constant 0 satisfying the following axioms: for all $x, y, z \in H$,

(H1) $(x * z) * (y * z) \ll x * y$,

(H2) $(x * y) * z = (x * z) * y$,

(H3) $x * H \ll x$,

(H4) $x \ll y$ and $y \ll x$ imply $x = y$,

where (a) $x \ll y$ is defined by $0 \in x * y$, and (b) for every $A, B \subseteq H$, $A \ll B$ is defined as follows: for all $a \in A$, there exists $b \in B$ such that $a \ll b$. In such case, we call " $\ll$ " the *hyper order* in H.

Example 2.2 [1] Let $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$ and consider the hyperoperation $*$ on $\mathbb{N}$ defined as follows: for all $x, y \in \mathbb{N}$,

$$x * y = \begin{cases} \{0, x\}, & \text{if } x \leq y, \\ \{x\}, & \text{if } x > y. \end{cases}$$

Then $(\mathbb{N}, *, 0)$ is a hyper BCK-algebra.

Theorem 2.3 [5], [6] *Let H be a hyper BCK-algebra. Then for all $x \in H$ and for all nonempty subset A of H,*

(i) $x * H \ll x$ *if and only if* $x * y \ll x$ *for all* $y \in H$.

(ii) *if* I *is a hyper BCK-ideal of* H *and* $A \ll I$, *then* $A \subseteq I$.

Definition 2.4 [7] Let I be a nonempty subset of a hyper BCK-algebra H and $0 \in I$. Then I is a strong hyper BCK-ideal of H if $(x * y) \cap I \neq \varnothing$ and $y \in I$ imply $x \in I$ for all $x, y \in H$.

Definition 2.5 [2] A hyper BCK-algebra H is called *decomposable* if there exists a nontrivial family $\{A_i\}_{i \in \Lambda}$ of hyper BCK-ideals of H such that

(i) $H \neq A_i \neq \{0\}$ for all $i \in \Lambda$,

(ii) $H = \bigcup_{i \in \Lambda} A_i$,

(iii) $A_i \cap A_j = \{0\}$ for all $i \neq j \in \Lambda$.

In this case, we say that $H = \bigcup_{i \in \Lambda} A_i$ is a decomposition of H and we write $H = \bigoplus_{i \in \Lambda} A_i$.

Theorem 2.6 [2] *Let H be a decomposable hyper BCK-algebra with decomposition $H = \bigoplus_{i \in \Lambda} A_i$. Then A_i is a strong hyper BCK-ideal of H for all $i \in \Lambda$.*

Definition 2.7 [2] Let H be a hyper BCK-algebra and $\varnothing \neq A \subset H$. Then the subset I of H is called a *weak hyper* (resp. *hyper*) *BCK-ideal of H related to A* if

(i) $0 \in I$; and

(ii) $x * y \subseteq I$ (resp. $x * y \ll I$) and $y \in I$ imply $x \in I$ for all $x \in A$.

Theorem 2.8 [2] *Let H be decomposable hyper BCK-algebra with decomposition $H = X \bigoplus Y$, where X and Y are strong hyper BCK-ideals of H and $I \subseteq X$. If I is a weak hyper (resp. hyper) BCK-ideal of H related to X, then I is a weak hyper (resp. hyper) BCK-ideal of H.*

Definition 2.9 [10] A *hyper KS-semigroup* is a nonempty set H together with two hyperoperations " $*$ " and " $\cdot$ " and a constant 0 satisfying the following conditions:

(i) $(H, *, 0)$ is a hyper BCK-algebra.

(ii) $(H, \cdot)$ is a semihypergroup having zero as a bilaterally absorbing element, that is, $x \cdot 0 = 0 \cdot x = \{0\}$ for all $x \in H$; and

(iii) " $\cdot$ " is left and right distributive over " $*$ ", that is, for any $x, y, z \in H$, $x \cdot (y * z) = (x \cdot y) * (x \cdot z)$ and $(x * y) \cdot z = (x \cdot z) * (y \cdot z)$.

Example 2.10 [10] Consider the hyper BCK-algebra $(\mathbb{N}, *, 0)$ in Example 2.2 and the semihypergroup $(\mathbb{N}, \cdot)$ defined by $x \cdot y = \{0, \min\{x, y\}\}$ for all $x, y \in \mathbb{N}$. Then $(\mathbb{N}, *, \cdot, 0)$ is a hyper KS-semigroup.

Lemma 2.11 [10] *Let H be a hyper KS-semigroup and A, B and C be nonempty subsets of H. Then $A \ll B$ and $B \subseteq C$ imply $A \ll C$.*

In what follows, H shall mean a hyper KS-semigroup $(H, *, \cdot, 0)$ and we write $x \cdot y$ as xy for all $x, y \in H$.

Definition 2.12 [10] Let I be a nonempty subset of a hyper KS-semigroup H. Then I is a *hyper subKS-semigroup* of H if for all $x, y \in I$, $x * y \subseteq I$ and $xy \subseteq I$. I is a *hyper left* (resp. *hyper right*) *stable* if $xa \subseteq I$ (resp. $ax \subseteq I$) for all $x \in H$ and for all $a \in I$. I is a *hyper stable* if I is both hyper left and right stable. I is a *hyper left* (resp. *hyper right*) *KS-ideal* if I is a hyper left (resp. hyper right) stable and for any $x, y \in H$, $x * y \ll I$ and $y \in I$ imply that $x \in I$. I is a *hyper KS-ideal* if I is both a hyper left and a hyper right KS-ideal.

Theorem 2.13 [10] *Let $\{A_i : i \in \mathscr{I}\}$ be a nonempty collection of nonempty subsets of a hyper KS-semigroup H. If A_i is a hyper KS-ideal of H for all $i \in \mathscr{I}$, then so is $\bigcap_{i \in \mathscr{I}} A_i$.*

Theorem 2.14 [10] *The hyper product of two hyper KS-semigroups is a hyper KS-semigroup.*

3. On Decomposable Hyper KS-semigroups

In this section, we introduce the notion of decomposable hyper KS-semigroups and investigate some of its properties.

Definition 3.1 A hyper KS-semigroup H is called *decomposable* if there exists a nontrivial family $\{A_i\}_{i \in \Lambda}$ of hyper KS-ideals of H such that

(i) $H \neq A_i \neq \{0\}$ for all $i \in \Lambda$,

(ii) $H = \bigcup_{i \in \Lambda} A_i$,

(iii) $A_i \cap A_j = \{0\}$ for all $i \neq j \in \Lambda$.

In this case, we say that $H = \bigcup_{i \in \Lambda} A_i$ is a decomposition of H and we write $H = \bigoplus_{i \in \Lambda} A_i$.

Example 3.2 Consider the hyper hyper KS-semigroup H with hyperoperations given in the following Cayley's tables.

$*$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0,1\}$	$\{1\}$	$\{1\}$
2	$\{2\}$	$\{2\}$	$\{0,2\}$	$\{2\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0,3\}$

$\cdot$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{0\}$	$\{0\}$
2	$\{0\}$	$\{0\}$	$\{0,2\}$	$\{0\}$
3	$\{0\}$	$\{0\}$	$\{0\}$	$\{0,3\}$

Observe that $A_1 = \{0,1\}$, $A_2 = \{0,2\}$ and $A_3 = \{0,3\}$ are nontrivial hyper KS-ideals of H such that $A_1 \cup A_2 \cup A_3 = H$ and for all $i \neq j$, $A_i \cap A_j = \{0\}$. Thus, H is a decomposable hyper KS-semigroup.

Theorem 3.3 *Let H be a decomposable hyper KS-semigroup and $N \subseteq H$. If N is a hyper KS-ideal of H, then N is decomposable.*

Proof. Let H be a decomposable hyper KS-semigroup and suppose $N \subseteq H$. Then there exist a family of nontrivial hyper KS-ideals A_i for all $i \in \Lambda$ such that $H = \bigcup_{i\in\Lambda} A_i$ and $A_i \cap A_j = \{0\}$ for $i \neq j$. By Theorem 2.13, $N \cap A_i$ is a hyper KS-ideal of H for all $i \in \Lambda$. Now, we first show that $N = \bigcup_{i\in\Lambda} (N \cap A_i)$. Clearly, $\bigcup_{i\in\Lambda} (N \cap A_i) \subseteq N$. Let $n \in N$. Then there exist $j \in \Lambda$ such that $n \in A_j$. Thus, $n \in N \cap A_j \subseteq \bigcup_{i\in\Lambda} (N \cap A_i)$. This implies that $N \subseteq \bigcup_{i\in\Lambda} (N \cap A_i)$. Next, if $i \neq j$, then $(N \cap A_i) \cap (N \cap A_j) = N \cap (A_i \cap A_j) = N \cap \{0\} = \{0\}$. Therefore, N is decomposable. ∎

Theorem 3.4 *Let H_1 and H_2 be both decomposable hyper KS-semigroups. Then $H_1 \times H_2$ is a decomposable hyper KS-semigroup.*

Proof. Let H_1 and H_2 be decomposable hyper KS-semigroups. By Theorem 2.14, $H_1 \times H_2$ is a hyper KS-semigroup. Then there exist nontrivial families $\{A_i\}_{i\in\Lambda}$ and $\{B_s\}_{s\in\Phi}$ of hyper KS-ideals of H_1 and H_2, respectively. Now, since $H_1 \neq A_i \neq \{0\}$ for all $i \in \Lambda$ and $H_2 \neq B_s \neq \{0\}$ for all $j \in \Phi$, it follows that $H_1 \times H_2 \neq A_i \times B_s \neq \{0\}$. Moreover, since $H_1 = \bigcup_{i\in\Lambda} A_i$ and $H_2 = \bigcup_{s\in\Phi} B_s$, we have

$$H_1 \times H_2 = \bigcup_{i\in\Lambda} A_i \times \bigcup_{s\in\Phi} B_s = \bigcup_{i\in\Lambda, s\in\Phi} (A_i \times B_s).$$

Finally, since $A_i \cap A_j = \{0\}$ for $i \neq j$ and $B_s \cap B_t = \{0\}$ for $s \neq t$, then we show that $(A_i \times B_s) \cap (A_j \times B_t) = \{0\}$. Let $z \in (A_i \times B_s) \cap (A_j \times B_t)$. Then $z \in (A_i \times B_s)$ and $z \in (A_j \times B_t)$. Thus, $z = (a,b)$ with $a \in A_i, A_j$ and $b \in B_s, B_t$. This implies that $a \in A_i \cap A_j = \{0\}$ and $b \in B_s \cap B_t$. Hence, $z = 0$ and so the proof is complete. ∎

Definition 3.5 Let I be a nonempty subset of a hyper KS-semigroup H and $0 \in I$. Then I is said to be a *strong hyper left* (resp. *strong hyper right*) *KS-ideal* of H if

(i) I is a hyper left (resp. hyper right) stable;

(ii) for all $x, y \in H, (x * y) \cap I \neq \varnothing$ and $y \in I$ imply that $x \in I$.

I is a *strong hyper KS-ideal* of H if I is both strong hyper left KS-ideal and strong hyper right KS-ideal of H.

Example 3.6 Consider the hyper KS-semigroup $\mathbb{N} = \{0, 1, 2, ...\}$ in Example 2.10 and take $I = \{0, 1\}$. Then I is a strong hyper KS-ideal of $\mathbb{N}$.

Remark 3.7 Every strong hyper KS-ideal of a hyper KS-semigroup H is a hyper KS-ideal of H but the converse is not true, in general.

Example 3.8 Consider the hyper KS-semigroup $H = \{0, 1, 2\}$ in [10] given by the Cayley's tables below

$*$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0, 1\}$	$\{0, 1\}$
2	$\{2\}$	$\{1, 2\}$	$\{0, 1, 2\}$

$\cdot$	0	1	2
0	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{1\}$	$\{0, 1\}$
2	$\{0\}$	$\{0, 1\}$	$\{0, 1, 2\}$

It was shown in [10] that $I = \{0, 1\}$ is a hyper KS-ideal of H. However, I is not a strong hyper KS-ideal of H since $\{1\} = (2 * 1) \cap I \neq \varnothing$ and $1 \in I$ but $2 \notin I$.

Example 3.9 Consider the decomposable hyper KS-semigroup $H = \{0, 1, 2, 3\}$ in Example 3.2 with hyper KS-ideals $A_1 = \{0, 1\}$, $A_2 = \{0, 2\}$ and $A_3 = \{0, 3\}$. Then A_1, A_2 and A_3 are strong hyper KS-ideals of H.

In Example 3.9, H is a decomposable hyper KS-semigroup and a hyper KS-ideal of H is a strong hyper KS-ideal H. This shows that the converse of Remark 3.7 holds whenever H is decomposable. We deduce this notion by the following corollary which follows from Theorem 2.6 and the fact that a hyper KS-ideal is hyper stable.

Corollary 3.10 *Let H be a decomposable hyper KS-semigroup with decomposition $H = \bigoplus_{i\in\Lambda} A_i$. Then A_i is a strong hyper KS-ideal of H for all $i \in \Lambda$.*

The following corollary is immediate from Remark 3.7 and Corollary 3.10.

Corollary 3.11 *Let H be a hyper KS-semigroup. Then H is decomposable if and only if $H = \bigoplus_{i\in\Lambda} A_i$ with A_i is a strong hyper KS-ideal of H for all $i \in \Lambda$.*

Theorem 3.12 *Let $\{A_i : i \in \mathscr{I}\}$ be a nonempty collection of nonempty strong hyper KS-ideal of a hyper KS-semigroup H. Then $\bigcap_{i\in\mathscr{I}} A_i$ is a strong hyper KS-ideal of H.*

Proof. Suppose that A_i is strong hyper KS-ideal of H for all $i \in \mathscr{I}$. Then $0 \in A_i$ for all $i \in \mathscr{I}$ by Definition 3.5. So, $0 \in \bigcap_{i\in\mathscr{I}} A_i$. Thus, $\bigcap_{i\in\mathscr{I}} A_i \neq \varnothing$. Clearly, $\bigcap_{i\in\mathscr{I}} A_i \subseteq A_i$ for all $i \in \mathscr{I}$. Let $a \in \bigcap_{i\in\mathscr{I}} A_i$ and $h \in H$. Then $a \in A_i$ for all $i \in \mathscr{I}$. Since A_i is hyper stable, it follows that $ah, ha \subseteq A_i$ for all $i \in \mathscr{I}$. Thus, $ah, ha \subseteq \bigcap_{i\in\mathscr{I}} A_i$. This means that $\bigcap_{i\in\mathscr{I}} A_i$ is hyper stable.

Suppose that $x, y \in H$ such that $(x * y) \cap \left(\bigcap_{i\in\mathscr{I}} A_i\right) \neq \varnothing$ and $y \in \bigcap_{i\in\mathscr{I}} A_i$. Then $\bigcap_{i\in\mathscr{I}} [(x * y) \cap A_i] \neq \varnothing$. Thus, $(x * y) \cap A_i \neq \varnothing$ for all $i \in \mathscr{I}$. Since A_i is a strong hyper KS-ideal of H and $y \in A_i$ for all $i \in \mathscr{I}$, we have $x \in A_i$ for all $i \in \mathscr{I}$. Hence, $x \in \bigcap_{i\in\mathscr{I}} A_i$. Therefore, $\bigcap_{i\in\mathscr{I}} A_i$ is a strong hyper KS-ideal of H. ∎

Theorem 3.13 *Let H be a decomposable hyper KS-semigroup with decomposition $H = \bigoplus_{i\in\Lambda} A_i$. Then $A_i \cup A_j$ is a strong hyper KS-ideal of H for all $i, j \in \Lambda$.*

Proof. Let $h \in H$ and $a \in A_i \cup A_j$. Then $a \in A_i$ or $a \in A_j$. Since A_i and A_j are strong hyper KS-ideals of H, it follows that $ah, ha \subseteq A_i$ or $ah, ha \subseteq A_j$. Thus, $ah, ha \subseteq A_i \cup A_j$, that is, $A_i \cup A_j$ is hyper stable. Let $i, j \in \Lambda$ and $x, y \in H$ such that $(x * y) \cap (A_i \cup A_j) \neq \varnothing$ and $y \in A_i \cup A_j$. Without loss of generality, suppose that $y \in A_i$. Since $(x * y) \cap (A_i \cup A_j) \neq \varnothing$, we have $(x * y) \cap A_i \neq \varnothing$ or $(x * y) \cap A_j \neq \varnothing$. If $(x * y) \cap A_i \neq \varnothing$, then $x \in A_i$ since A_i is a strong hyper KS-ideal of H and $y \in A_i$. Suppose $(x * y) \cap A_j \neq \varnothing$. Then there exists $t \in (x * y) \cap A_j$. Since $x \in H = \bigcup_{i\in\Lambda} A_i$, there exists $k \in \Lambda$ such that $x \in A_k$. By Definition 2.1(H3), $x * H \ll x$. By Theorem 2.3(i), $x * y \ll x \in A_k$. By Lemma 2.11, $x * y \ll A_k$ so that by Theorem 2.3(ii), $x * y \subseteq A_k$. Thus, $t \in A_k$ and so $t \in A_j \cap A_k$. If $j = k$, then $A_j = A_k$ which implies that $x \in A_j \subseteq A_i \cup A_j$. Suppose $j \neq k$. Then $t \in A_j \cap A_k = \{0\}$. Hence, $0 \in x * y$ and so $x \ll y$. Since $x \ll y \in A_i$ and A_i is a hyper BCK-ideal of H, it follows that $x \in A_i \subseteq A_i \cup A_j$. Therefore, $A_i \cup A_j$ is a strong hyper KS-ideal of H. ∎

The following corollary is immediate from Theorem 3.13.

Corollary 3.14 *Let H be a decomposable hyper KS-semigroup with decomposition $H = \bigoplus_{i\in\Lambda} A_i$. Then $\bigcup_{i\in\Omega} A_i$ is a strong hyper KS-ideal of H for a nonempty finite subset Ω of Λ.*

Theorem 3.15 *Let H be a hyper KS-semigroup. Then H is decomposable if and only if there exist nontrivial strong hyper KS-ideals $X, Y \subseteq H$ such that $H = X \cup Y$ and $X \cap Y = \{0\}$.*

Proof. Let H be a hyper KS-semigroup. The necessary condition follows directly from Corollary 3.11 and Theorem 3.13. Suppose there exist nontrivial strong hyper KS-ideals $X, Y \subseteq H$ such that $H = X \cup Y$ and $X \cap Y = \{0\}$. By Remark 3.7, X and Y are nontrivial hyper KS-ideals of H. Thus, by Definition 3.1, H is decomposable. ∎

Definition 3.16 Let H be a hyper KS-semigroup and $\varnothing \neq X \subset H$. Then a nonempty subset I of H is called

(a) a *weak hyper left (*resp. *weak hyper right) KS-ideal* of H if

  (i) $0 \in I$;

  (ii) I is a hyper left (resp. hyper right) stable; and

  (iii) for all $x, y \in H, x * y \subseteq I$ and $y \in I$ imply that $x \in I$.

  I is a *weak hyper KS-ideal* of H if I is both weak hyper left KS-ideal and weak hyper right KS-ideal of H.

(b) a *weak hyper left (*resp. *weak hyper right) KS-ideal of H related to X* if

(i) $0 \in I$;

(ii) I is a hyper left (resp. hyper right) stable related to X, that is, $xa \subseteq I$ (resp. $ax \subseteq I$) for all $x \in X$ and for all $a \in I$; and

(iii) $x * y \subseteq I$ and $y \in I$ imply that $x \in I$ for all $x \in X$.

I is a *weak hyper KS-ideal of H related to X* if I is both weak hyper left KS-ideal of H related to X and weak hyper right KS-ideal of H related to X.

Example 3.17 Consider the hyper KS-semigroup $\mathbb{N} = \{0, 1, 2, ...\}$ in Example 2.10 and take $I = \{0, 1\}$. Then I is a weak hyper KS-ideal of $\mathbb{N}$.

The following remark follows from Definition 3.16.

Remark 3.18 Let H be a hyper KS-semigroup and $\varnothing \neq X \subset H$. If I is a weak hyper KS-ideal of H, then I is a weak hyper KS-ideal of H related to X but the converse is not true, in general.

Example 3.19 Consider the hyper BCK-algebra $(H, *, 0)$ in [2] and the semi-hypergroup $(H, \cdot)$ with the following Cayley's tables:

$*$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{1\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{2\}$	$\{1\}$	$\{0\}$	$\{0\}$
3	$\{3\}$	$\{3\}$	$\{3\}$	$\{0, 3\}$

$\cdot$	0	1	2	3
0	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
1	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
2	$\{0\}$	$\{0\}$	$\{0\}$	$\{0\}$
3	$\{0\}$	$\{0\}$	$\{0\}$	$\{0, 3\}$

By routine calculations, we can see that H is a hyper KS-semigroup. Consider $I = \{0, 2\}$ and $X = \{0, 2, 3\}$. Then I is a weak hyper KS-ideal of H related to X. However, I is not a weak hyper KS-ideal of H since $\{0\} = 1 * 2 \subseteq I$ and $2 \in I$ but $1 \notin I$.

Example 3.20 Consider the decomposable hyper KS-semigroup $H{=}\{0, 1, 2, 3\}$ in Example 3.2. Take the strong hyper KS-ideal $A_2{=}\{0, 2\}$ and $X{=}\{0, 2, 3\}$. Then A_2 is a weak hyper KS-ideal of H related to X and A_2 is a weak hyper KS-ideal of H.

In Example 3.20, H is a decomposable hyper KS-semigroup. This shows that the converse of Remark 3.18 holds whenever H is decomposable. We deduce this notion by the following theorem.

Theorem 3.21 *Let H be decomposable with decomposition $H = X \bigoplus Y$, where X and Y are strong hyper KS-ideals of H and $I \subseteq X$. If I is a weak hyper KS-ideal of H related to X, then I is a weak hyper KS-ideal of H.*

Proof. Suppose $I \subseteq X$ and I is a weak hyper KS-ideal of H related to X. Let $h \in H$ and $a \in I$. If $h \in X$, then by hypothesis, $ah, ha \subseteq I$. Now, assume that $h \in Y$. Since X and Y are strong hyper KS-ideals of H, it follows that $ah, ha \subseteq Y$ and $ah, ha \subseteq X$. Thus, $ah, ha \subseteq X \cap Y = \{0\}$. So, $ah = ha = \{0\} \subseteq I$. By Theorem 2.8, I is a weak hyper BCK-ideal of H. Therefore, I is a weak hyper KS-ideal of H. ∎

The following corollary follows from Remark 3.18 and Theorem 3.21:

Corollary 3.22 *Let H be a decomposable hyper KS-semigroup with decomposition $H = X \bigoplus Y$, where X and Y are strong hyper KS-ideals of H and $I \subseteq X$. Then I is a weak hyper KS-ideal of H related to X if and only if I is a weak hyper KS-ideal of H.*

Definition 3.23 Let H be a hyper KS-semigroup and $\varnothing \neq X \subset H$. Then the subset I of H is called a *hyper left* (resp. *hyper right*) *KS-ideal of H related to X* if

(i) $0 \in I$;

(ii) I is a hyper left (resp. hyper right) stable related to X, that is, $xa \subseteq I$ (resp. $ax \subseteq I$) for all $x \in X$ and for all $a \in I$; and

(iii) $x * y \ll I$ and $y \in I$ imply that $x \in I$ for all $x \in X$.

I is a *hyper KS-ideal of H related to X* if I is both hyper left KS-ideal of H related to X and hyper right KS-ideal of H related to X.

The following remark follows from Definition 3.23.

Remark 3.24 Let H be a hyper KS-semigroup and $\varnothing \neq X \subset H$. If I is a hyper KS-ideal of H, then I is a hyper KS-ideal of H related to X but the converse is not true, in general.

Example 3.25 Consider the hyper KS-semigroup $H = \{0, 1, 2\}$ in Example 3.19. Take $J = \{0, 1\}$ and $Y = \{0, 1, 3\}$. Then J is a hyper KS-ideal of H related to Y. However, J is not a hyper KS-ideal of H since $\{1\} = 2 * 1 \ll J$ and $1 \in J$ but $2 \notin J$.

Example 3.26 Consider the decomposable hyper KS-semigroup $H = \{0, 1, 2, 3\}$ in Example 3.2. Take the strong hyper KS-ideal $A_1 = \{0, 1\}$ and $Y = \{0, 1, 3\}$. Then A_1 is a hyper KS-ideal of H related to Y.

In Example 3.26, H is a decomposable hyper KS-semigroup. This shows that the converse of Remark 3.24 holds whenever H is decomposable. We deduce this notion by the following theorem.

Theorem 3.27 *Let H be decomposable with decomposition $H = X \bigoplus Y$, where X and Y are strong hyper KS-ideals of H and $I \subseteq X$. If I is a hyper KS-ideal of H related to X, then I is a hyper KS-ideal of H.*

Proof. Suppose $I \subseteq X$ and I is a hyper KS-ideal of H related to X. Let $h \in H$ and $a \in I$. If $h \in X$, then by hypothesis, $ah, ha \subseteq I$. Suppose that $h \in Y$. Since X and Y strong hyper KS-ideals of H, it follows that $ah, ha \subseteq Y$ and $ah, ha \subseteq X$. Thus, $ah, ha \subseteq X \cap Y = \{0\}$. Hence, $ah = ha = \{0\} \subseteq I$. By Theorem 2.8, I is a hyper BCK-ideal of H. Therefore, I is a hyper KS-ideal of H. ∎

The following corollary follows from Remark 3.24 and Theorem 3.27:

Corollary 3.28 *Let H be a decomposable hyper KS-semigroup with decomposition $H = X \bigoplus Y$, where X and Y are strong hyper KS-ideals of H and $I \subseteq X$. Then I is a hyper KS-ideal of H related to X if and only if I is a hyper KS-ideal of H.*

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