NOTES ON THE ARITHMETIC-GEOMETRIC MEAN INEQUALITY FOR SINGULAR VALUES

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Abstract. In this short note, we present a generalization and a new equivalent form of the arithmetic-geometric mean inequality for singular values. Meanwhile, we also show some remarks related to generalizations of the arithmetic-geometric mean inequality for singular values.

Keywords: arithmetic-geometric mean inequality, positive semidefinite matrices, singular values.

MSC (2010) Subject Classification: 15A42, 47A63, 47B15.

1. Introduction

Let M_n be the space of $n \times n$ complex matrices. We shall always denote the singular values of A by $s_1(A) \geq ... \geq s_n(A) \geq 0$, that is, the eigenvalues of the positive semidefinite matrix $|A| = (AA^*)^{\frac{1}{2}}$, arranged in decreasing order and repeated according to multiplicity. For $A \in M_n$, let $A^+ = \dfrac{|A| + A}{2}, A^- = \dfrac{|A| - A}{2}$. Let $A, B \in M_n$ be Hermitian, the order relation $A \geq B$ means, as usual, that $A - B$ is positive semidefinite. We use the direct sum notation $A \oplus B$ for the block-diagonal operator $\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix}$ defined on $M_n \oplus M_n$.

The well-known arithmetic-geometric mean inequality for singular values due to Bhatia and Kittaneh [1] says that

$$2s_j(AB^*) \leq s_j(A^*A + B^*B)$$

for any $A,\ B \in M_n$ and $j = 1, ..., n$. We state this in another form: Let $A,\ B \in M_n$ be positive semidefinite, then

$$2s_j(AB) \leq s_j\left(A^2 + B^2\right) \tag{1.1}$$

for $j = 1, ..., n$. For more information on singular values and unitarily invariant norms inequalities the reader is referred to [2]-[9].

In section 2, we first give a generalization of inequality (1.1). After that, we give a new equivalent form of inequality (1.1). Section 3 contains some remarks.

2. Main results

To generalize inequality (1.1), we need the following result [5, Theorem 1].

Lemma 2.1. *Let $A,\ X,\ B \in M_n$ such that A and B are positive semidefinite. Then*

$$s_j(AX - XB) \leq \|X\|\, s_j(A \oplus B)$$

for $j = 1, ..., n$, where $\|\cdot\|$ denotes the usual operator norm on M_n.

Theorem 2.1. *Let $A,\ B \in M_n$ be positive semidefinite and suppose that $f(t),\ g(t)$ are polynomials. Then*

$$s_j(ABg(B) + f(A)AB) \leq \max(\|f(A)\|, \|g(B)\|)\, s_j\left(A^2+B^2\right)$$

for $j = 1, ..., n$.

Proof. Let

$$K_1 = \begin{bmatrix} A & 0 \\ B & 0 \end{bmatrix} \begin{bmatrix} A & B \\ 0 & 0 \end{bmatrix} \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} = \begin{bmatrix} A^2X_1 & ABX_2 \\ BAX_1 & B^2X_2 \end{bmatrix}$$

and

$$K_2 = \begin{bmatrix} X_1 & 0 \\ 0 & X_2 \end{bmatrix} \begin{bmatrix} A & 0 \\ -B & 0 \end{bmatrix} \begin{bmatrix} A & -B \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} X_1A^2 & -X_1AB \\ -X_2BA & X_2B^2 \end{bmatrix}.$$

It follows that

$$\begin{aligned} K_1 - K_2 &= \begin{bmatrix} A^2X_1 & ABX_2 \\ BAX_1 & B^2X_2 \end{bmatrix} - \begin{bmatrix} X_1A^2 & -X_1AB \\ -X_2BA & X_2B^2 \end{bmatrix} \\ &= \begin{bmatrix} A^2X_1 - X_1A^2 & ABX_2 + X_1AB \\ BAX_1 + X_2BA & B^2X_2 - X_2B^2 \end{bmatrix}. \end{aligned}$$

So, by Lemma 2.1, we have

$$\begin{aligned} s_j(K_1 - K_2) &\leq \|X_1 \oplus X_2\|\, s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix} \oplus \begin{bmatrix} A^2 & -AB \\ -BA & B^2 \end{bmatrix}\right) \\ &= \max(\|X_1\|, \|X_2\|)\, s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix} \oplus \begin{bmatrix} A^2 & -AB \\ -BA & B^2 \end{bmatrix}\right) \\ &= \max(\|X_1\|, \|X_2\|)\, s_j\left(\begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix} \oplus \begin{bmatrix} A^2 & AB \\ BA & B^2 \end{bmatrix}\right) \\ &= \max(\|X_1\|, \|X_2\|)\, s_j\left((A^2+B^2) \oplus (A^2+B^2)\right) \end{aligned} \tag{2.1}$$

for $j = 1, ..., n$. Let

$$X_1 = f(A), X_2 = g(B), K = ABg(B) + f(A)AB.$$

Then, we have

$$K_1 - K_2 = \begin{bmatrix} 0 & K \\ K^* & 0 \end{bmatrix}.$$

Note that

$$s_j \begin{bmatrix} 0 & K \\ K^* & 0 \end{bmatrix} = s_j \begin{bmatrix} K & 0 \\ 0 & K^* \end{bmatrix},$$

so, by inequality (2.1), we get

$$s_j \begin{bmatrix} K & 0 \\ 0 & K^* \end{bmatrix} \leq \max(\|f(A)\|, \|g(B)\|) s_j \left(\left(A^2+B^2\right) \oplus \left(A^2+B^2\right)\right)$$

for $j = 1, ..., n$. Thus, we obtain

$$s_j(ABg(B) + f(A)AB) \leq \max(\|f(A)\|, \|g(B)\|) s_j \left(A^2+B^2\right)$$

for $j = 1, ..., n$. This completes the proof. ∎

Theorem 2.2. *The following statements are equivalent:*

(i) *Let $A, B \in M_n$ be positive semidefinite. Then*

$$s_j(A - B) \leq s_j(A \oplus B) \tag{2.2}$$

for $j = 1, ..., n$.

(ii) *Let $A, B \in M_n$ be positive semidefinite. Then*

$$2s_j(AB) \leq s_j\left(A^2 + B^2\right)$$

for $j = 1, ..., n$.

(iii) *Let $A, B, X \in M_n$ such that $\begin{bmatrix} A & X \\ X^* & B \end{bmatrix} \geq 0$. Then*

$$2s_j(X) \leq s_j \begin{bmatrix} A & X \\ X^* & B \end{bmatrix}$$

for $j = 1, ..., n$. Let $A, B \in M_n$ such that A is self-adjoint, $B \geqslant 0$, and $\pm A \leqslant B$. Then

$$2s_j(A) \leq s_j((B + A) \oplus (B - A))$$

for $j = 1, ..., n$.

(iv) *Let $A, B \in M_n$ such that A and B are self-adjoint. Then*

$$s_j(A + B) \leq s_j\left(\left(A^+ + B^+\right) \oplus \left(A^- + B^-\right)\right) \tag{2.3}$$

for $j = 1, ..., n$.

Proof. Note that (i), (ii), (iii), and (iv) are equivalent by [4, Theorem 2.6]. We will prove that (i) is equivalent to (v), and this will complete the proof of this theorem.

(i) $\Rightarrow$ (v). Since A and B are Hermitian, it follows that

$$\pm (A+B) \leq |A| + |B| .$$

Let

$$Y_1 = A+B, \ Y_2 = |A| + |B| .$$

Now, applying inequality (2.2) to the matrices $Y_2 + Y_1$ and $Y_2 - Y_1$, we have

$$s_j \left((Y_2+Y_1) - (Y_2 - Y_1) \right) \leq s_j \left((Y_2+Y_1) \oplus (Y_2 - Y_1) \right),$$

which is equivalent to

$$s_j (A+B) \leq s_j \left(\left(A^+ + B^+ \right) \oplus \left(A^- + B^- \right) \right)$$

for $j = 1, ..., n$. So, we know that inequality (2.2) implies inequality (2.3).

(v) $\Rightarrow$ (i). Audeh and Kittaneh [4, p.2521] pointed out that inequality (2.3) implies inequality (2.2). So, (i) is equivalent to (v). This completes the proof. ∎

3. Remarks

Remark 3.1. Let $f(t) = g(t) = 1$ in Theorem 2.1, we obtain inequality (1.1). Let $f(t) = g(t) = t$ in Theorem 2.1, we get

$$s_j (A (A+B) B) \leq \max (\|A\|, \|B\|) \, s_j \left(A^2 + B^2 \right)$$

for $j = 1, ..., n$. This is a matrix version of the following inequality:

$$a (a+b) b \leq \max (a, b) \left(a^2 + b^2 \right), a \geq 0, b \geq 0.$$

Remark 3.2. Let $A, \ B, \ X \in M_n$ such that A and B are positive semidefinite. A natural extension of inequality (1.1) is

$$2 s_j (AXB) \leq s_j \left(A^2 X + X B^2 \right)$$

for $j = 1, ..., n$. This is not always true. For example, if we choose

$$A = \begin{bmatrix} 0.7389 & 0.6634 & 0.7481 \\ 0.6634 & 1.0265 & 0.7836 \\ 0.7481 & 0.7836 & 1.0908 \end{bmatrix},$$

$$X = \begin{bmatrix} 0.1820 & 0.2744 & 0.8627 \\ 0.8027 & 0.3597 & 0.5781 \\ 0.0095 & 0.5384 & 0.2392 \end{bmatrix},$$

$$B=\begin{bmatrix} 1.7006 & 0.9132 & 0.3608 \\ 0.9132 & 0.8945 & 0.5280 \\ 0.3608 & 0.5280 & 0.9509 \end{bmatrix},$$

then, we have

$$2s_3\left(AXB\right)=0.0289\geq 0.0106=s_3\left(A^2X+XB^2\right).$$

Remark 3.3. Let $A,\ B,\ X\in M_n$ such that A and B are positive semidefinite. Another possible extension of inequality (1.1) is

$$2s_j\left(AXB\right)\leq \left\|X\right\| s_j\left(A^2+B^2\right)$$

for $j=1,...,n$. This is refused by

$$A=\begin{bmatrix} 0.4327 & 0.6051 \\ 0.6051 & 0.9762 \end{bmatrix},\ X=\begin{bmatrix} 0.5730 & 0.2149 \\ 0.6816 & 0.1522 \end{bmatrix},\ B=\begin{bmatrix} 1.1111 & 0.1476 \\ 0.1476 & 0.0361 \end{bmatrix}.$$

In fact, we have

$$2s_1\left(AXB\right)=2.8102\geq 2.4311=\left\|X\right\| s_1\left(A^2+B^2\right).$$

It should be note that the inequality

$$2s_j\left(AXB\right)\leq \left\|X\right\| s_j\left(A^2+B^2\right)$$

holds when X is positive semidefinite [10].

Remark 3.4. Let $A,\ B,\ X,\ Y\in M_n$ such that A and B are positive semidefinite. A possible generalization of Theorem 2.1 is

$$s_j\left(ABX+YAB\right)\leq \max\left(\left\|X\right\|,\left\|Y\right\|\right) s_j\left(A^2+B^2\right)$$

for $j=1,...,n$. This is not always true. For example, if we choose

$$A=\begin{bmatrix} 0.0286 & 0.0414 & 0.0472 \\ 0.0414 & 0.7134 & 0.5265 \\ 0.0472 & 0.5265 & 0.6030 \end{bmatrix},\quad X=\begin{bmatrix} 0.5992 & 0.7265 & 0.6529 \\ 0.4480 & 0.0524 & 0.8624 \\ 0.1893 & 0.6981 & 0.8502 \end{bmatrix}$$

$$Y=\begin{bmatrix} 0.2547 & 0.9273 & 0.1140 \\ 0.7466 & 0.1075 & 0.5175 \\ 0.1717 & 0.2464 & 0.2433 \end{bmatrix},\quad B=\begin{bmatrix} 0.3508 & 0.2981 & 0.3669 \\ 0.2981 & 1.0854 & 0.9269 \\ 0.3669 & 0.9269 & 1.2541 \end{bmatrix},$$

then, we have

$$s_2\left(ABX+YAB\right)=0.3924\geq 0.2220=\max\left(\left\|X\right\|,\left\|Y\right\|\right) s_2\left(A^2+B^2\right).$$

References

[1] Bhatia, R., Kittaneh, F., *On the singular values of a product of operators*, SIAM J. Matrix Anal. Appl., 11 (1990), 272-277.

[2] Tao, Y., *More results on singular value inequalities*, Linear Algebra Appl., 416 (2006), 724-729.

[3] Bhatia, R., Kittaneh, F., *The matrix arithmetic-geometric mean inequality revisited*, Linear Algebra Appl., 428 (2008), 2177-2191.

[4] Audeh, W., Kittaneh, F., *Singular value inequalities for compact operators*, Linear Algebra Appl., 437 (2012), 2516-2522.

[5] Kittaneh, F., *Inequalities for commutators of positive operators*, J. Funct. Anal., 250 (2007), 132-143.

[6] Zou, L., He, C., *Weak majorization inequalities for singular values*, Oper. Matrices, 7 (2013), 733-737.

[7] Zou, L., He, C., *On some inequalities for unitarily invariant norms and singular values*, Linear Algebra Appl., 436 (2012), 3354-3361.

[8] Hu, X., *Some inequalities for unitarily invariant norms*, J. Math. Inequal., 6 (2012), 615-623.

[9] Fu, X., He, C., *On some inequalities for unitarily invariant norms*, J. Math. Inequal., 7 (2013), 727-737.

[10] Albadawi, H., *Singular value and arithmetic-geometric mean inequalities for operators*, Ann. Funct. Anal., 3 (2012), 10-18.

Accepted: 02.04.2015