MINIMAL INTUITIONISTIC GENERAL L-FUZZY AUTOMATA

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Abstract. In this paper we present an intuitionistic general L-fuzzy automaton (IGLFA) based on lattice valued intuitionistic fuzzy sets [2]. In this note, we define (α, β)-language, (α, β)-complete, (α, β)-accessible, (α, β)-reduced for an IGLFA over a bounded complete lattice L, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. In particular, we prove a theorem which is generalization of Myhill-Nerode theorem in ordinary deterministic automata. In other words for any recognizable (α, β)-language over a bounded complete lattice L, there exist minimal (α, β)-complete and deterministic IGLFA, which preserve (α, β)-language, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. Also, we show that for any given (α, β)-language $\mathcal{L}$, the minimal (α, β)-complete and deterministic IGLFA recognizing $\mathcal{L}$ is isomorphic with threshold (α, β) to any (α, β)-complete, (α, β)-accessible, deterministic, (α, β)-reduced IGLFA recognizing $\mathcal{L}$. Moreover, we give some examples to clarify these notions. Finally, by using these notions, we give some theorems and obtain some results.

Keywords: Max-min intuitionistic general L-fuzzy automata; (α, β)-language; (α, β)-reduced; Minimal intuitionistic general L-fuzzy automata; (α, β)-isomorphic.

1. Introduction

The theory of fuzzy sets was introduced by L.A. Zadeh in 1965 [40]. W.G. Wee [35] introduced the idea of fuzzy automata. E.T. Lee and L.A. Zadeh [21] in 1969 gave the concept of fuzzy finite state automata. Thereafter, there were a considerable number of authors, such as Mordeson and Malik [22], [23], Topencharov and Peeva [33] and others having contributed to this field. Fuzzy finite automata have many important applications such as in learning system, pattern recognition, neural networks and data base theory [14], [15], [22], [25], [26], [29], [36]. Atanassove [1] has extended the notion of fuzzy sets to the intuitionistic fuzzy sets (IFS) by adding non-membership value, which may express more accurate and flexible information as compared with fuzzy sets. Intuitionistic fuzzy set theory has many applications in several subject, see [5], [6], [9], [11], [16], [18], [19], [34], [7]. Using the notion of intuitionistic fuzzy sets, W.L. Jun [17] introduced the notion of intuitionistic fuzzy finite state machines as a generalization of fuzzy finite state machines. Based on the papers [17], [18], Zhang and Li [41] discussed intuitionistic fuzzy recognizers and intuitionistic fuzzy finite automata. K. Atanassov and S. Stoeva generalized the concept of IFS to intuitionistic L-fuzzy sets [2] where L is an appropriate lattice. A. Tepavcevic and T. Gerstenkorn gave a new definition of lattice valued intuitionistic fuzzy sets in [32]. Thus, on the basis of lattice valued intuitionistic fuzzy sets, Yang et al. [38] introduced the concepts of lattice-valued intuitionistic fuzzy finite state machines. In 2004, M. Doostfatemeh and S.C. Kremer [10] extended the notion of fuzzy automata and gave the notion of general fuzzy automata. In 2014, M. Shamsizadeh and M.M. Zahedi [31] gave the notion of max-min intuitionistic general fuzzy automata. We will use bounded complete lattices as the structures of truth values. Note that usually the real unit interval $[0, 1]$ is used in the literature (the reader not familiar with lattices may, without any harm, substitute $[0, 1]$ for bounded complete lattices throughout the paper). Our paper deals with intuitionistic general fuzzy automata over a bounded complete lattice L, where endowed with a t-norm T, a t-conorm S, the least element 0 and the greatest element 1, denoted by $L = (L, \leq_L, T, S, 0, 1)$. State minimization is a fundamental problem in automata theory. There are many papers on the minimization problem of fuzzy finite automata. For example, minimization of mealy type of fuzzy finite automata in discussed in [4], minimization of fuzzy finite automata with crisp final states without outputs in studied in [3], minimizing the deterministic finite automaton with fuzzy (final) states in [24] and minimization of fuzzy machines becomes the subject of [28], [27], [30], [33]. Myhill-Nerode's theorem has been extended to fuzzy regular language and also an algorithm is given for minimizing the deterministic finite automaton with fuzzy (final) states in [20], [24]. It is important to find the minimal intuitionistic general L-fuzzy automata that recognizes the same language as a given language. In this note, for a given complete lattice $L = (L, \leq_L, T, S, 0, 1)$, we define an (α, β)-language, where $\alpha, \beta \in L, \alpha <_L N(\beta)$. Furthermore, we show that for any max-min IGLFA $\tilde{F}^*$, there exist (α, β)-complete, (α, β)-accessible and deterministic max-min IGLFA recognizing $\mathcal{L}(\tilde{F}^*)$, where $\alpha, \beta \in L, \alpha <_L N(\beta)$. Also, we prove a theorem which is generalization of Myhill-Nerode theorem in ordinary deterministic automata. In other words, we have shown that for any (α, β)-language, there exist minimal (α, β)-complete and deterministic intuitionistic general L-fuzzy automata (IGLFA). Also, we define an (α, β)-reduced IGLFA. We show that for any given (α, β)-language $\mathcal{L}$, the minimal (α, β)-complete and deterministic IGLFA recognizing $\mathcal{L}$ is isomorphic with threshold (α, β) to any (α, β)-complete, (α, β)-accessible, deterministic, (α, β)-reduced IGLFA recognizing $\mathcal{L}$. Moreover we give some new notions and results as mentioned in the abstract and some examples to clarify these new notions.

2. Preliminaries

In this section we give some definitions that we need in the sequel. Assume that E is an universal set. A fuzzy set A on E is characterized by the same symbol A as a function $A : E \to [0,1]$ where $A(u) \in [0,1]$ is the membership degree of the element $u \in E$ [40].

Definition 2.1 [1] Let A be a given subset on E. An intuitionistic fuzzy set (IFS) A^+ on E is an object of the following form

$$A^+ = \{\langle x, \mu_A(x), \nu_A(x)\rangle \mid x \in E\},$$

where the functions $\mu_A : E \to [0,1]$ and $\nu_A : E \to [0,1]$ define the value of membership and the value of non-membership of the element $x \in E$ to the set A, respectively, and for every $x \in E$, $0 \le \mu_A(x) + \nu_A(x) \le 1$.

Obviously, every ordinary fuzzy set $\{(x, \mu_A(x)) \mid x \in E\}$ has an intuitionistic form $\{\langle x, \mu_A(x), 1 - \mu_A(x)\rangle \mid x \in E\}$. If $\pi_A(x) = 1 - \mu_A(x) - \nu_A(x)$, then $\pi_A(x)$ is the value of non-determinacy (uncertainty) of the membership of element $x \in E$ to the set A. In the case of ordinary fuzzy sets, where $\nu_A(x) = 1 - \mu_A(x)$, we have $\pi_A(x) = 0$ for every $x \in E$.

In the rest of this paper, we denote the set A^+ by A.

Let $L = (L, \le_L, 0, 1)$ be a bounded (complete) lattice. By an L-fuzzy set A on E we mean a function $A : E \to L$ [12].

Definition 2.2 [37] Let $L = (L, \le_L, 0, 1)$ be a bounded lattice. A binary operation $T : L \times L \to L$ is a lattice t-norm (Lt-norm) if it satisfies the following conditions:

1. $T(1, x) = x$,
2. $T(x, y) = T(y, x)$,
3. $T(x, T(y, z)) = T(T(x, y), z)$,
4. if $w \le_L x$ and $y \le_L z$, then $T(w, y) \le_L T(x, z)$.

Definition 2.3 [37] Let $L = (L, \le_L, 0, 1)$ be a bounded lattice. A binary operation $S : L \times L \to L$ is a lattice t-conorm (Lt-conorm) if it satisfies the following conditions:

1. $S(0, x) = x$,
2. $S(x, y) = S(y, x)$,
3. $S(x, S(y, z)) = S(S(x, y), z)$,
4. if $w \le_L x$ and $y \le_L z$, then $S(w, y) \le_L S(x, z)$.

In the rest of this paper, we denote $D(x_1, D^{n-1}(x_2, ..., x_{n+1}))$ by $D^n(x_1, x_2, ..., x_{n+1})$ where D is an Lt-norm or an Lt-conorm on lattice L, $D^0(x) = x$ and $D^1(x_1, x_2) = D(x_1, x_2)$, for any $n \geq 1$.

We recall from [8] that $L^* : \{(x, y) \in [0, 1]^2 \big| 0 \leq x+y \leq 1\}$ is a complete lattice with the order defined by

$$(x_1, x_2) \leq (y_1, y_2) \text{ if and only if } x_1 \leq y_1 \text{ and } y_2 \leq x_2.$$

Definition 2.4 [2] Let X be a nonempty set and L be a complete lattice with an involutive order reversing unary operation $N : L \to L$. An intuitionistic L-fuzzy set is an object of the form $A = \{(x, \mu(x), \nu(x)) \mid x \in E\}$, where μ and ν are functions $\mu : E \to L, \nu : E \to L$, such that for all $x \in X$, $\mu(x) \leq N(\nu(x))$. We use the abbreviation ILFS for intuitionistic L-fuzzy set.

Definition 2.5 [13] Let $L \subseteq X^*$ and consider the relation $\sim_L$ on X, where $x \sim_L y$ if and only if, for all $z \in X^*, xz \in L \Longleftrightarrow yz \in L$.

From now on, we let $L = (L, \leq_L, T, S, 0, 1)$ be a complete lattice, where endowed with an Lt-norm T, an Lt-conorm S, the least element element 0 and the greatest element 1, also with an involutive order reversing unary operation $N : L \to L$.

Note 2.6 Let $A, B \in L$. In this note we assume that $A <_L B$ if and only if $A \leq_L B$ and $A \neq B$. We also assume that $A \geq_L B$ if and only if $B \leq_L A$.

3. Intuitionistic general L-fuzzy automata

Definition 1.1 An intuitionistic general L-fuzzy automaton (IGLFA) $\tilde{F}$ is a ten-tuple machine denoted by $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$, where

- Q is a set of states,
- X is a finite set of input symbols, $X = \{a_1, a_2, ..., a_m\}$,
- $\tilde{R}$ is the ILFS of start states, $\tilde{R} = \{(q, \mu^{t_0}(q), \nu^{t_0}(q)) \mid q \in R\}$, where R is a finite subset of Q,
- Z is a finite set of output symbols, $Z = \{b_1, b_2, ..., b_l\}$,
- $\tilde{\delta} : (Q \times L \times L) \times X \times Q \to L \times L$ is the augmented transition function,
- $\tilde{\omega} : (Q \times L \times L) \times Z \to L \times L$ is the output function,
- $F_1 = (F_1^T, F_1^S)$, where $F_1^T : L \times L \to L$ is an Lt-norm and it is called the membership assignment function. $F_1^T(\mu, \delta_1)$ as is seen, is motivated by two parameters μ and δ_1, where μ is the membership value of a predecessor and δ_1 is the membership value of a transition. Moreover, $F_1^S : L \times L \to L$ is an Lt-conorm, where is the dual of F_1^T respect to the involutive negation and it is called non-membership assignment function. $F_1^S(\nu, \delta_2)$ as is seen, is motivated by two

parameters ν and δ_2, where ν is the non-membership value of a predecessor and δ_2 is the non-membership value of a transition.

In this definition, the process that takes place upon the transition from the state q_i to q_j on an input a_k is given by:

$$\mu^{t+1}(q_j) = \tilde{\delta}_1((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) = F_1^T(\mu^t(q_i), \delta_1(q_i, a_k, q_j)), \tag{1.1}$$

$$\nu^{t+1}(q_j) = \tilde{\delta}_2((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) = F_1^S(\nu^t(q_i), \delta_2(q_i, a_k, q_j)), \tag{1.2}$$

thus

$$\begin{aligned} &\tilde{\delta}((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) \\ &\quad = (\tilde{\delta}_1((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j), \tilde{\delta}_2((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j)), \end{aligned} \tag{1.3}$$

where, $\delta(q_i, a_k, q_j) = (\delta_1(q_i, a_k, q_j), \delta_2(q_i, a_k, q_j))$, which δ is an ILFS. It means that the membership value of the state q_j at time $t+1$ is computed by the function F_1^T using both the membership value of q_i at time t and the membership value of the transition. Also, the non-membership value of the state q_j at time $t+1$ is computed by function F_1^S using both the non-membership value of q_i at time t and the non-membership value of the transition.

Considering (1.1), (1.2) and Definitions 2.2, 2.3 and 2.4, $\tilde{\delta}$ is an ILFS.

• $F_2 = (F_2^T, F_2^S)$, where $F_2^T : L \times L \to L$ is an Lt-norm and it is called the membership assignment output function. $F_2^T(\mu, \omega_1)$ as is seen, is motivated by two parameters μ and ω_1, where μ is the membership value of present state and ω_1 is the membership value of an output function. Moreover, $F_2^S : L \times L \to L$ is an Lt-conorm where it is the dual of F_2^T respect to the involutive negation and is called non-membership assignment output function. $F_2^S(\nu, \omega_2)$ as is seen, is motivated by two parameters ν and ω_2, where ν is the non-membership value of present state and ω_2 is the non-membership value of an output function. Then, we have

$$\tilde{\omega}_1((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) = F_2^T(\mu^t(q_i), \omega_1(q_i, b_k)), \tag{1.4}$$

$$\tilde{\omega}_2((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) = F_2^S(\nu^t(q_i), \omega_2(q_i, b_k)), \tag{1.5}$$

thus,

$$\begin{aligned} &\tilde{\omega}((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) \\ &\quad = (\tilde{\omega}_1((q_i, \mu^t(q_i), \nu^t(q_i)), b_k), \tilde{\omega}_2((q_i, \mu^t(q_i), \nu^t(q_i)), b_k)), \end{aligned} \tag{1.6}$$

where, $\omega(q_i, b_k) = (\omega_1(q_i, b_k), \omega_2(q_i, b_k))$, which ω is an ILFS. It means that the output membership value of the state q_i at time t is computed by the function F_2^T using both the membership value of q_i at time t and the membership value of the output function, also output non-membership value of the state q_i at time t is computed by function F_2^S using both the non-membership value of q_i at time t and the non-membership value of the output function.

Considering (1.4), (1.5) and Definitions 2.2, 2.3 and 2.4, $\tilde{\omega}$ is an ILFS.

- $F_3 = (F_3^{TS}, F_3^{ST})$, where $F_3^{ST} : L^* \to L$ is an Lt-norm and it is called the multi-non-membership function. The multi- non-membership resolution function resolves the multi-non-membership active states and assigns a single non-membership value to them. Moreover, $F_3^{TS} : L^* \to L$ is an Lt-conorm, where it is the dual of F_3^{ST} respect to the involutive negation and it is called the multi-membership function. The multi-membership resolution function resolves the multi-membership active states and assigns a single membership value to them.
- $F_4 = (F_4^{TS}, F_4^{ST})$, where $F_4^{ST} : L^* \to L$, is an Lt-norm and it is called the multi-non-membership output function. The multi-non-membership resolution output function resolves the output multi-non-membership active state and assigns a single output non-membership value to it. Moreover, $F_4^{TS} : L^* \to L$, is an Lt-conorm, where it is the dual of F_4^{ST} respect to the involutive negation and is multi-membership output function. The multi-membership resolution output function resolves the output multi-membership active state and assigns a single output membership value to it.

Let $Q_{act}(t_i)$ be the set of all active states at time t_i for all $i \geq 0$. We have $Q_{act}(t_0)=\tilde{R}$ and $Q_{act}(t_i)=\{(q, \mu^{t_i}(q), \nu^{t_i}(q)) \mid \exists (q', \mu^{t_{i-1}}(q'), \nu^{t_{i-1}}(q')) \in Q_{act}(t_{i-1}), \exists a \in X, \delta(q', a, q) \in \Delta, \mu^{t_i}(q) >_L 0\}$ for all positive integer i.

Since $Q_{act}(t_i)$ is an ILFS, to say that a state q belongs to $Q_{act}(t_i)$, we write $q \in Domain(Q_{act}(t_i))$ and for simplicity of notation we denote it by $q \in Q_{act}(t_i)$. The combination of the operations of functions F_1^T and F_3^{TS} (F_1^S and F_3^{ST}) on a multi-membership (multi- non-membership) state q_j will lead to the multi-membership (multi-non-membership) resolution algorithm. Also, the set of all transition of IGLFA $\tilde{F}$ is denoted by Δ.

Algorithm: Multi-membership resolution (multi-non-membership resolution) for **transition function**.

If there are several simultaneous transitions to the active state q_j at time $t+1$, then the following algorithm will assign a unified membership value (non-membership value) to that

1. Each transition membership value (transition non-membership value) $\delta_1(q_i, a_k, q_j)$ ($\delta_2(q_i, a_k, q_j)$) together with $\mu^t(q_i)$ ($\nu^t(q_i)$), will be processed by the membership (non-membership) assignment function F_1^T (F_1^S) and will produce a new membership value (non-membership value) as follows:

$$\begin{aligned} \tilde{\delta}_1((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) &= F_1^T(\mu^t(q_i), \delta_1(q_i, a_k, q_j)), \\ \big(\tilde{\delta}_2((q_i, \mu^t(q_i), \nu^t(q_i)), a_k, q_j) &= F_1^S(\nu^t(q_i), \delta_2(q_i, a_k, q_j))\big). \end{aligned}$$

2. These new membership (non-membership) values are not necessarily equal. Hence, they will be processed by F_3^{TS} (F_3^{ST}), which is called the multi-membership (multi-non-membership) resolution function. By some manipulation on the product results by F_1^T (F_1^S), we obtain just one element, say

the instantaneous membership value (non-membership value) of the active state q_j

$$\begin{aligned} \mu^{t+1}(q_j) &= (F_3^{TS})^{n-1}(x_1, x_2, ..., x_n), \\ (\nu^{t+1}(q_j) &= (F_3^{ST})^{n-1}(x_1, x_2, ..., x_n)), \end{aligned}$$

where

- n is the number of simultaneous transitions to the active state q_j at time $t+1$ and $x_i = F_1^T(\mu^t(q_i), \delta_1(q_i, a_k, q_j))(x_i = F_1^S(\nu^t(q_i), \delta_2(q_i, a_k, q_j)))$, $1 \leq i \leq n$,
- $\delta_1(q_i, a_k, q_j)(\delta_2(q_i, a_k, q_j))$ is the membership (non-membership) value of transition from q_i to q_j upon input a_k,
- $\mu^t(q_i)$ $(\nu^t(q_i))$ is the membership (non-membership) value of q_i at time t,

Algorithm: Multi-membership resolution (Multi-non-membership resolution) for **output function**

If there are several simultaneous output to the active state q_i at time t, the following algorithm will assign a unified membership value (non-membership value) to that

1. Each output membership value (non-membership value) $\omega_1(q_i, b_k)$ $(\omega_2(q_i, b_k))$ together with $\mu^t(q_i)$ $(\nu^t(q_i))$, will be processed by the membership (non-membership) assignment function F_2^T (F_2^S) and will produce a new output membership value (non-membership value) as follows:

$$\begin{aligned} \tilde{\omega}_1((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) &= F_2^T(\mu^t(q_i), \omega_1(q_i, b_k)), \\ \big(\tilde{\omega}_2((q_i, \mu^t(q_i), \nu^t(q_i)), b_k) &= F_2^S(\nu^t(q_i), \omega_2(q_i, b_k))\big). \end{aligned}$$

2. These new output membership (non-membership) values are not necessarily equal. Hence, they will be processed by F_4^{TS} (F_4^{ST}), which is called the output multi-membership (multi-non-membership) resolution function. By some manipulation on the product results by F_2^T (F_2^S), we obtain just one element, say the instantaneous output membership value (non-membership value) of the active state q_i.

$$\begin{aligned} \omega_1^t(q_i) &= (F_4^{TS})^{n-1}(x_1, x_2, ..., x_n), \\ (\omega_2^t(q_i) &= (F_4^{ST})^{n-1}(x_1, x_2, ..., x_n)), \end{aligned}$$

where

- n is the number of simultaneous outputs to the active state q_i at time t, $x_j = F_2^T(\mu^t(q_i), \omega_1(q_i, b_j))(x_j = F_2^S(\nu^t(q_i), \omega_2(q_i, b_j)))$, $1 \leq j \leq n$, for some $a_k \in X$ and q_i is an active state at time t,
- $\omega_1(q_i, b_j)(\omega_2(q_i, b_j))$ is the membership (non-membership) value of output from q_i on b_k,

- $\mu^t(q_i)$ ($\nu^t(q_i)$) is the membership (non-membership) value of q_i at time t,
- $\omega_1^t(q_i)$ ($\omega_2^t(q_i)$) is the output membership (non-membership) value of q_i at time t.

Remark 1.2 We let for all $q \in Q$ such that $q \notin \tilde{R}$, $\mu^{t_0}(q) = 0$ and $\nu^{t_0}(q) = 1$ and for all $q \in \tilde{R}$, $\mu^{t_0}(q) >_L 0$.

Note 1.3 In this paper, we assume that the max-min IGLFA has a finite number of states.

Definition 1.4 Let $\tilde{F} = (Q, X, \tilde{R}, Z, \tilde{\delta}, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an IGLFA. We define the max-min intuitionistic general L-fuzzy automaton $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ such that $\tilde{\delta}^* : Q_{act} \times X^* \times Q \to L \times L$, where $Q_{act} = \{Q_{act}(t_0), Q_{act}(t_1), Q_{act}(t_2), ...\}$ and for every $i \geq 0$,

$$\tilde{\delta}_1^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), \Lambda, p) = \begin{cases} 1 & \text{if } p = q, \\ 0 & \text{otherwise,} \end{cases} \tag{1.7}$$

and

$$\tilde{\delta}_2^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), \Lambda, p) = \begin{cases} 0 & \text{if } p = q, \\ 1 & \text{otherwise.} \end{cases} \tag{1.8}$$

Also for every $i \geq 0$, $\tilde{\delta}_1^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p) = \tilde{\delta}_1((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p)$ and $\tilde{\delta}_2^*((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p) = \tilde{\delta}_2((q, \mu^{t_i}(q), \nu^{t_i}(q)), u_{i+1}, p)$ and recursively,

$$\begin{aligned} &\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1 u_2 ... u_n, p) = \\ &\quad \vee \{\tilde{\delta}_1((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1, p_1) \wedge \tilde{\delta}_1((p_1, \mu^{t_1}(p_1), \nu^{t_1}(p_1)), u_2, p_2) \wedge ... \\ &\qquad \wedge \tilde{\delta}_1((p_{n-1}, \mu^{t_{n-1}}(p_{n-1}), \nu^{t_{n-1}}(p_{n-1})), u_n, p) \mid \\ &\qquad\qquad p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\}, \end{aligned} \tag{1.9}$$

$$\begin{aligned} &\tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1 u_2 ... u_n, p) = \\ &\quad \wedge \{\tilde{\delta}_2((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1, p_1) \vee \tilde{\delta}_2((p_1, \mu^{t_1}(p_1), \nu^{t_1}(p_1)), u_2, p_2) \vee ... \\ &\qquad \vee \tilde{\delta}_2((p_{n-1}, \mu^{t_{n-1}}(p_{n-1}), \nu^{t_{n-1}}(p_{n-1})), u_n, p) \mid \\ &\qquad\qquad p_1 \in Q_{act}(t_1), ..., p_{n-1} \in Q_{act}(t_{n-1})\}, \end{aligned} \tag{1.10}$$

in which $u_i \in X$ for all $1 \leq i \leq n$ and assume that u_{i+1} is the entered input at time t_i, for all $0 \leq i \leq n-1$.

4. Minimal intuitionistic general L-fuzzy automata

Definition 4.1 Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Suppose that $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. Then we say that

1. $\tilde{F}^*$ is (α, β)-complete, if for any $q \in Q, a \in X$ there exists $p \in Q$ such that $\delta_1(q, a, p) >_L \alpha$ and $\delta_2(q, a, p) <_L \beta$,

2. $q \in Q$ is (α, β)-accessible if there exist $p \in \tilde{R}, x \in X^*$ such that $\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) >_L \alpha$, and $\tilde{\delta}_2^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) <_L \beta$,

3. $\tilde{F}^*$ is (α, β)-accessible if for any $q \in Q$, q is an (α, β)-accessible.

Definition 4.2 Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Then we say that $\tilde{F}^*$ is deterministic if there exists a unique $p_0 \in \tilde{R}$ such that $\mu^{t_0}(p_0) >_L 0$, and for any $q \in Q, a \in X$ there exists at most one $p \in Q$ such that $\delta_2(q, a, p) <_L 1$.

Example 4.3 Consider the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ as in Figure 1, where $L = \{0, a, b, c, d, 1\}$ and $N(0) = 1$, $N(1) = 0$, $N(a) = b$, $N(b) = a$, $N(c) = d$, $N(d) = c$.

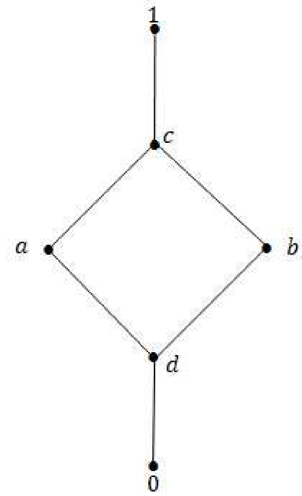

Figure 1: The complete lattice L of Example 4.3

Let the max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 2, where $Q = \{q_0, q_1, q_2, q_3\}, X = \{u, v\}, \tilde{R} = \{(q_0, 1, 0)\}, Z = \{o\}$ and $\delta : Q \times X \times Q : L \times L$ is defined as follows:

$$\begin{array}{rclcrcl} \delta(q_0, u, q_1) & = & (a, 0) & \quad & \delta(q_1, v, q_3) & = & (a, b) \\ \delta(q_2, u, q_1) & = & (1, 0) & \quad & \delta(q_2, v, q_0) & = & (c, 0) \end{array}$$

$\delta(q, x, q') = (0, 1)$ for all other $(q, x, q') \in Q \times X \times Q$ and $\omega : Q \times Z : L \times L$ is defined by: $\omega(q_1, o) = (0, 1)$ and $\omega(q, e) = (1, 0)$ for all other $(q, e) \in Q \times Z$. q_1 is $(0, \alpha)$-accessible, (d, β)-accessible, where $\alpha \in \{a, b, c, d, 1\}$ and $\beta \in \{a, b, c, d\}$, q_3 is $(0, 1)$- accessible, $(0, c)$- accessible, (d, c)- accessible but q_3 is not $(d, 1)$- accessible, since $d \nleq N(1) = 0$. $\tilde{F}^*$ is not (α, β)-accessible, not (α, β)- complete, for any $(\alpha, \beta) \in L$ in which $\alpha \leq_L N(\beta)$. Clearly, it is deterministic.

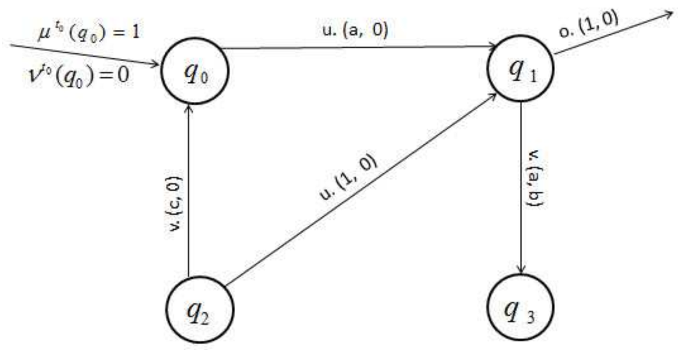

Figure 2: The IGLFA of Example 4.3

Theorem 4.4 *Let* $\tilde{F}^* = (Q, X, \{p\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ *be a deterministic max-min IGLFA. If*

$$\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \wedge \tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q') >_L \alpha,$$

and

$$\tilde{\delta}_2^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, r) \vee \tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, r') <_L \beta,$$

then $q = q' = r = r'$*, where* $q, q', r, r' \in Q, x \in X^*, \alpha, \beta \in L$ *and* $\alpha \leq_L N(\beta)$.

Proof. First, let $x = \Lambda$. Then we have

$$\tilde{\delta}_1((p, \mu^{t_0}(p), \nu^{t_0}(p)), \Lambda, q) \wedge \tilde{\delta}_1((p, \mu^{t_0}(p), \nu^{t_0}(p)), \Lambda, q') >_L \alpha, \quad \text{and}$$
$$\tilde{\delta}_2((p, \mu^{t_0}(p), \nu^{t_0}(p)), \Lambda, r) \vee \tilde{\delta}_2((p, \mu^{t_0}(p), \nu^{t_0}(p)), \Lambda, r') <_L \beta,$$

which implies that $p = q' = q = r = r'$. Thus the theorem holds for $x = \Lambda$.

Now, we continue the proof for any $x \in X^*$ and $x \notin \Lambda$ by induction on $|x|$. Suppose that $|x| = 1$. Then

$$\tilde{\delta}_1((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) = F_1^T(\mu^{t_0}(p), \delta_1(p, x, q)) >_L \alpha \quad \text{and}$$
$$\tilde{\delta}_1((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q') = F_1^T(\mu^{t_0}(p), \delta_1(p, x, q')) >_L \alpha$$

and

$$\tilde{\delta}_2((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, r) = F_1^S(\nu^{t_0}(p), \delta_1(p, x, r)) <_L \beta \quad \text{and}$$
$$\tilde{\delta}_2((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, r') = F_1^S(\nu^{t_0}(p), \delta_1(p, x, r')) <_L \beta.$$

These imply that

$$\delta_1(p, x, q) \wedge \delta_1(p, x, q') >_L \alpha \text{ and } \delta_2(p, x, r) \vee \delta_2(p, x, r') <_L \beta.$$

Since $\delta_1(p, x, q) \wedge \delta_1(p, x, q') >_L \alpha$, then $\delta_2(p, x, q) \vee \delta_2(p, x, q') <_L N(\alpha) \leq_L 1$. Then by Definition 4.2, $q = q' = r = r'$.

Now, suppose the claim holds for all $y \in X^*$ such that $|y| = n - 1$ and $n > 1$. Let $x = ya$, where $y \in X^*, a \in X$ and $|y| = n - 1$. Then

$$\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) = \vee\{\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), y, p')$$
$$\wedge\tilde{\delta}_1((p', \mu^{t_0+n-1}(p'), \nu^{t_0+n-1}(p')), a, q) | p' \in Q\} >_L \alpha,$$

and

$$\tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),x,q') = \vee\{\tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,p') \wedge\tilde{\delta}_1((p',\mu^{t_0+n-1}(p'),\nu^{t_0+n-1}(p')),a,q') \mid p'\in Q\} >_L \alpha.$$

So, there exist $d, d' \in Q$ such that

$$\vee\{\tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,p') \wedge \tilde{\delta}_1((p',\mu^{t_0+n-1}(p'),\nu^{t_0+n-1}(p')),a,q) \mid p'\in Q\} = \tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,d) \wedge \tilde{\delta}_1((d,\mu^{t_0+n-1}(d),\nu^{t_0+n-1}(d)),a,q) >_L \alpha,$$

and

$$\vee\{\tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,p') \wedge \tilde{\delta}_1((p',\mu^{t_0+n-1}(p'),\nu^{t_0+n-1}(p')),a,q') \mid p'\in Q\} = \tilde{\delta}_1^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,d') \wedge \tilde{\delta}_1((d',\mu^{t_0+n-1}(d'),\nu^{t_0+n-1}(d')),a,q') >_L \alpha.$$

Also there exist $s, s' \in Q$ such that

$$\begin{aligned}&\tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),x,r)\\ &= \wedge\{\tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,p') \vee \tilde{\delta}_2((p',\mu^{t_0+n-1}(p'),\nu^{t_0+n-1}(p')),a,r) \mid p'\in Q\}\\ &= \tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,s) \vee \tilde{\delta}_2((s,\mu^{t_0+n-1}(s),\nu^{t_0+n-1}(s)),a,r) <_L \beta,\end{aligned}$$

and

$$\begin{aligned}&\tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),x,r')\\ &= \wedge\{\tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,p') \vee \tilde{\delta}_2((p',\mu^{t_0+n-1}(p'),\nu^{t_0+n-1}(p')),a,r') \mid p'\in Q\}\\ &= \tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),y,s') \vee \tilde{\delta}_2((s',\mu^{t_0+n-1}(s'),\nu^{t_0+n-1}(s')),a,r') <_L \beta.\end{aligned}$$

Therefore, by the induction hypothesis, $s = s' = d = d'$. Hence

$$\tilde{\delta}_1((d,\mu^{t_0+n-1}(d),\nu^{t_0+n-1}(d)),a,q) \wedge \tilde{\delta}_1((d,\mu^{t_0+n-1}(d),\nu^{t_0+n-1}(d)),a,q') >_L \alpha,$$

and

$$\tilde{\delta}_2((s,\mu^{t_0+n-1}(s),\nu^{t_0+n-1}(s)),a,r) \vee \tilde{\delta}_2((s,\mu^{t_0+n-1}(s),\nu^{t_0+n-1}(s)),a,r') <_L \beta,$$

imply that $\delta_1(d,a,q) \wedge \delta_1(d,a,q') >_L \alpha$ and $\delta_2(s,a,r) \vee \delta_2(s,a,r') <_L \beta$. Then $q = q' = r = r'$. Hence the claim is hold. ∎

Corollary 4.5 *Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Notice that if $\tilde{F}^*$ is an (α,β)-complete and deterministic max-min IGLFA, then for any $p \in Q, a \in X$ there exists exactly one state $q \in Q$ such that $\delta_1(p,a,q) >_L \alpha$ and $\delta_2(p,a,q) <_L \beta$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.*

Example 4.6 Consider the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ as in Example 4.3, where $L = \{0, a, b, c, d, 1\}$. Let the max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 3, where $Q = \{q_0, q_1, q_2, q_3\}, X = \{u\}, \tilde{R} = \{(q_0, a, b)\}$, $Z = \{o\}$ and $\delta : Q \times X \times Q : L \times L$ is defined as follows:

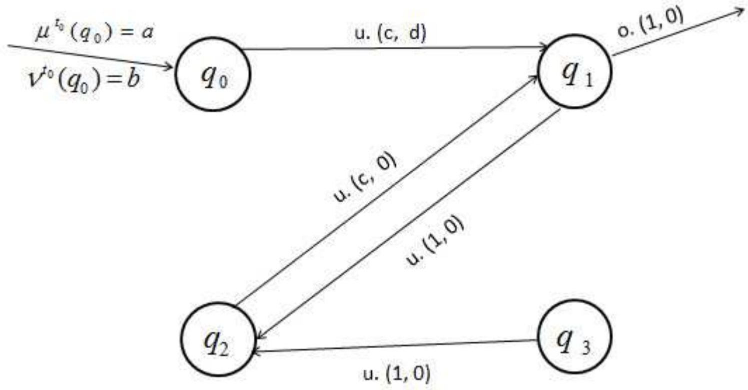

Figure 3: The IGLFA of Example 4.6

$$\delta(q_0, u, q_1) = (c, d) \qquad \delta(q_1, u, q_2) = (1, 0)$$
$$\delta(q_2, u, q_1) = (c, 0) \qquad \delta(q_3, u, q_2) = (1, 0)$$

$\delta(q, x, q') = (0, 1)$ for all other $(q, x, q') \in Q \times X \times Q$ and $\omega : Q \times Z \to L \times L$ is defined by: $\omega(q_1, o) = (1, 0)$ and $\omega(q, e) = (0, 1)$ for all other $(q, e) \in Q \times Z$. Then $\tilde{F}^*$ is (a, b)-complete and deterministic.

Definition 4.7 Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Then the (α, β)-language recognized by $\tilde{F}^*$ is a subset of X^* defined by:

$$\begin{aligned}\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \{x \in X^* \mid\ & \tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \\ & \wedge\tilde{\omega}_1((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b) >_L \alpha, \\ & \tilde{\delta}_2^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \\ & \vee\tilde{\omega}_2((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b') <_L \beta, \\ & \text{for some } p \in \tilde{R}, q \in Q, b, b' \in Z\}.\end{aligned} \tag{4.1}$$

Definition 4.8 Let X be a nonempty finite set. Then subset $\mathcal{L}$ of X^* is called recognizable (α, β)- language, if there exists a max-min IGLFA $\tilde{F}^*$ such that $\mathcal{L} = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.

Theorem 4.9 *Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA. Then there exists an (α, β)- complete IGLFA $\tilde{F}^{*c}$ such that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}^{\alpha,\beta}(\tilde{F}^{*c})$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.*

Proof. Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ does not be an (α, β)-complete max-min IGLFA. Let $Q^c = Q \cup \{t\}$, where t is an element such that $t \notin Q$. Consider $\gamma, \eta \in L$, where $\gamma >_L \alpha$, $\eta <_L \beta$ and $\gamma \leq_L N(\eta)$. We now give an algorithm in which the output is an (α, β)-complete max-min IGLFA for a given (α, β)-incomplete max-min IGLFA as input.

Algorithm: (to make (α, β)-complete)

Step 1 input incomplete $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$, where $Q = \{q_1, q_2, ..., q_n\}, X = \{a_1, a_2, ..., a_m\}$,

Step 2 $i = 1$,

Step 3 $j = 1$, if $i \leq n$ go to the next step, else go to Step 7,

Step 4 if $j \leq m$, then $T = \{q \mid \delta_1(q_i, a_j, q) >_L \alpha\}$ else $i = i + 1$ go to Step 3,

Step 5 if $T = \emptyset$, then $\delta_1^c(q_i, a_j, t) = \gamma, \delta_2^c(q_i, a_j, t) = \eta, j = j + 1$, go to Step 4, else go to Step 6,

Step 6 for $q \in T$ if $\delta_2(q_i, a_j, q) <_L \beta$, then $j = j + 1$ and go to Step 4, else $T = T - \{q\}$ and go to Step 5,

Step 7 $\delta_1^c(p, a, q) = \delta_1(p, a, q)$, and $\delta_2^c(p, a, q) = \delta_2(p, a, q)$, for all $p, q \in Q, a \in X$, $\delta_1^c(t, a, t) = \gamma$, $\delta_2^c(t, a, t) = \eta$ for all $a \in X$,

Step 8 $Q^c = Q \cup \{t\}$, $\omega_1^c(p, b) = \begin{cases} \omega_1(p, b) & if\ p \neq t, \\ 0 & if\ p = t, \end{cases}$ and

$$\omega_2^c(p, b) = \begin{cases} \omega_2(p, b) & if\ p \neq t, \\ 1 & if\ p = t, \end{cases}$$

Step 9 output $\tilde{F}^{*c} = (Q^c, X, \tilde{R}, Z, \tilde{\delta}^{*c}, \tilde{\omega}^c, F_1, F_2, F_3, F_4)$.

It is easy to see that the max-min IGLFA $\tilde{F}^{*c} = (Q^c, X, \tilde{R}, Z, \tilde{\delta}^{*c}, \tilde{\omega}^c, F_1, F_2, F_3, F_4)$ is (α, β)- complete. It is clear that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) \subseteq \mathcal{L}^{\alpha,\beta}(\tilde{F}^{*c})$. Let $x = u_1 u_2 ... u_{k+1} \in L^{\alpha,\beta}(\tilde{F}^{*c})$. Then there exist $p \in \tilde{R}, q \in Q^c, b, b' \in Z$ such that

$$\tilde{\delta}_1^{*c}((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \wedge \tilde{\omega}_1^c((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b) >_L \alpha,$$

and

$$\tilde{\delta}_2^{*c}((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \vee \tilde{\omega}_2^c((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b') <_L \beta.$$

These imply that $q \in Q$ and there exist $p_1, p_2, ..., p_k, p'_1, p'_2, ..., p'_k \in Q$ such that

$$\tilde{\delta}_1^c((p, \mu^{t_0}(p), \nu^{t_0}(p)), u_1, p_1) \wedge ... \wedge \tilde{\delta}_1^c((p_k, \mu^{t_0+|x|}(p_k), \nu^{t_0+|x|}(p_k)), u_k, q) >_L \alpha. \tag{4.2}$$

and

$$\tilde{\delta}_2^c((p, \mu^{t_0}(p), \nu^{t_0}(p)), u_1, p'_1) \vee ... \vee \tilde{\delta}_2^c((p'_k, \mu^{t_0+|x|}(p'_k), \nu^{t_0+|x|}(p'_k)), u_k, q) <_L \beta. \tag{4.3}$$

Definition 2.2 and (4.2) imply that $\mu^{t_0}(p) >_L \alpha, \delta_1^c(p, u_1, p_1) >_L \alpha, \delta_1^c(p_1, u_2, p_2) >_L \alpha, ..., \delta_1^c(p_k, u_k, q) >_L \alpha$. Now, suppose that $p_j, 1 \leq j \leq k$, be the first state that $\delta_1^c(p_j, u_j, p_{j+1}) >_L \alpha$ and $\delta_1(p_j, u_j, p_{j+1})$ was undefined. Then $p_{j+1} = t$. Therefore $p_{j+1} = p_{j+2} = ... = q = t$, which is a contradiction. Hence

$$\tilde{\delta}_1^*((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q) \wedge \tilde{\omega}_1((q, \mu^{t_0+|x|}(q), \nu^{t_0+|x|}(q)), b) >_L \alpha.$$

In a similar manner we obtain

$$\tilde{\delta}_2^*((p,\mu^{t_0}(p),\nu^{t_0}(p)),x,q)\vee\tilde{\omega}_2((q,\mu^{t_0+|x|}(q),\nu^{t_0+|x|}(q)),b')<_L\beta.$$

Then the claim is hold. ∎

Example 4.10 Consider the complete lattice $L=(L,\le_L,T,S,0,1)$ defined in Example 4.3, and max-min IGLFA $\tilde{F}^*=(Q,X,\tilde{R},Z,\tilde{\delta}^*,\tilde{\omega},F_1,F_2,F_3,F_4)$ as in Figure 4,

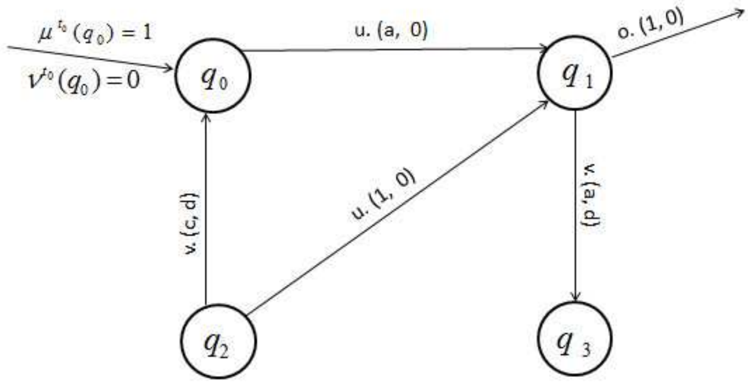

Figure 4: The IGLFA of Example 4.10

where $Q=\{q_0,q_1,q_2,q_3\},X=\{a_0=u,a_1=v\},\tilde{R}=\{(q_0,1,0)\},Z=\{o\}$ and $\delta:Q\times X\times Q\to L\times L$ is defined as follows:

$$\begin{array}{ll}\delta(q_0,u,q_1)=(a,0) & \delta(q_1,v,q_3)=(a,d)\\ \delta(q_2,u,q_1)=(1,0) & \delta(q_2,v,q_0)=(c,d)\end{array}$$

$\delta(q,x,q')=(0,1)$ for all other $(q,x,q')\in Q\times X\times Q$ and $\omega:Q\times Z\to L\times L$ is defined by: $\omega(q_1,o)=(1,0)$ and $\omega(q,e)=(0,1)$ for all other $(q,e)\in Q\times Z$.

Now, considering the complete algorithm in the proof of Theorem 4.9, and $\alpha=a$, $\beta=d$, we have

Stage 1 Let $i=0,j=0$. Then $T=\emptyset$ and $\delta_1^c(q_0,u,t)=c,\delta_2^c(q_0,u,t)=0$.

Stage 2 Let $i=0,j=1$. Then $T=\emptyset$ and $\delta_1^c(q_0,v,t)=c,\delta_2^c(q_0,v,t)=0$.

Stage 3 Let $i=1,j=0$. Then $T=\emptyset$ and $\delta_1^c(q_1,u,t)=c,\delta_2^c(q_1,u,t)=0$.

Stage 4 Let $i=1,j=1$. Then $T=\emptyset$ and $\delta_1^c(q_1,v,t)=c,\delta_2^c(q_1,u,t)=0$.

Stage 5 Let $i=2,j=0$. Then $T=\{q_1\}$ and we have $\delta_2(q_2,u,q_1)=0<_L d$.

Stage 6 Let $i=2,j=1$. Then $T=\{q_0\}$. Since $\delta_2(q_2,v,q_0)=d$,
thus $T=T-\{q_0\}=\emptyset$. Then $\delta_1^c(q_2,v,t)=c,\delta_2^c(q_2,v,t)=0$.

Stage 7 Let $i=3,j=0$. Then $T=\emptyset$ and $\delta_1^c(q_3,u,t)=c,\delta_2^c(q_3,u,t)=0$.

Stage 8 Let $i=3,j=1$. Then $T=\emptyset$ and $\delta_1^c(q_3,v,t)=c,\delta_2^c(q_3,v,t)=0$.

Therefore, we have an (a,d)-complete $\tilde{F}^{*c} = (Q^c, X, \tilde{R}, Z, \tilde{\delta}^{*c}, \tilde{\omega}^c, F_1, F_2, F_3, F_4)$ as in Figure 5.

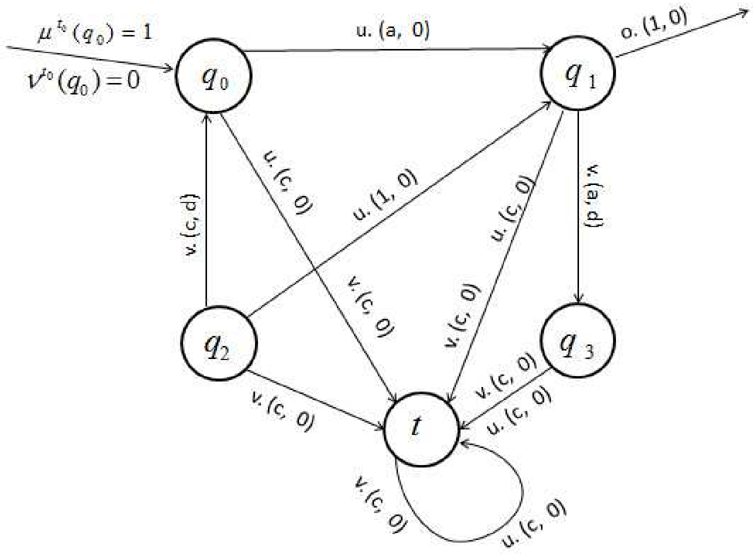

Figure 5: The (a,d)-complete $\tilde{F}^{*c}$ of Example 4.10

Theorem 4.11 *Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA and $\tilde{R} \neq \emptyset$. Then there exists a deterministic max-min IGLFA $\tilde{F}^*_d$ such that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_d)$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.*

Proof. Let

$$I_x = \{q' \in Q \mid \exists q \in \tilde{R} \text{ such that } \tilde{\delta}^*_1((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q') >_L \alpha \quad \tilde{\delta}^*_2((p, \mu^{t_0}(p), \nu^{t_0}(p)), x, q') <_L \beta\}, \tag{4.4}$$

for all $x \in X^*$. Then $I_\Lambda = \{q' \in Q \mid q' \in \tilde{R}\}$.

Let $Q_d = \{I_x \mid x \in X^*\}$. Define $\delta_d : Q_d \times X \times Q_d \to L \times L$, where

$$\delta_{d1}(I_y, a, I_x) = \begin{cases} \gamma_1 & \text{if } I_x = I_{ya}, \\ 0 & \text{otherwise}, \end{cases}$$

$$\delta_{d2}(I_y, a, I_x) = \begin{cases} \eta_1 & \text{if } I_x = I_{ya}, \\ 1 & \text{otherwise}, \end{cases}$$

and $\omega_d : Q_d \times Z_d \to L \times L$, where

$$\omega_{d1}(I_x, e) = \begin{cases} \gamma_2 & \text{if } x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*), \\ 0 & \text{otherwise}, \end{cases} \tag{4.5}$$

$$\omega_{d2}(I_x, e) = \begin{cases} \eta_2 & \text{if } x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*), \\ 1 & \text{otherwise}, \end{cases} \tag{4.6}$$

where $\gamma_1, \eta_1, \gamma_2, \eta_2 \in L, \gamma_1 \wedge \gamma_2 >_L \alpha, \eta_1 \vee \eta_2 <_L \beta$, $\gamma_1 \leq_L N(\eta_1)$ and $\gamma_2 \leq_L N(\eta_2)$. Consider $\mu^{t_0}(I_\Lambda) = \vee\{\mu^{t_0}(q) \mid q \in I_\Lambda\}, \nu^{t_0}(I_\Lambda) = \wedge\{\nu^{t_0}(q) \mid q \in I_\Lambda\}$ and $Z_d = \{e\}$. Now, suppose that $\tilde{F}_d^* = (Q_d, X, I_\Lambda, Z_d, \tilde{\delta}_d^*, \tilde{\omega}_d, F_1, F_2, F_3, F_4)$. It is clear that δ_d is well defined. Now, we show that ω_d is well defined. Let $I_x = I_y$ and $e \in Z_d$. If $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$, then there exist $q \in \tilde{R}, p \in Q, b, b' \in Z$ such that

$$\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \wedge \tilde{\omega}_1((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b) >_L \alpha,$$

and

$$\tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \vee \tilde{\omega}_2((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b') <_L \beta.$$

Thus $p \in I_x = I_y$. Therefore

$$\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), y, p) >_L \alpha \text{ and } \tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), y, p) <_L \beta$$

for some $q \in \tilde{R}$. Also $\omega_1(p, b) >_L \alpha$ and $\omega_2(p, b) <_L \beta$. Hence $y \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. In a similar way, if $y \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$, then $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. Hence $\omega_d(I_x, e) = \omega_d(I_y, e)$.

Since there exists $q \in \tilde{R}$ such that $\mu^{t_0}(q) >_L 0$, then $\mu^{t_0}(I_\Lambda) >_L 0$. It is easy to see that the max-min IGLFA $\tilde{F}_d^*$ is deterministic. We show that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}^{\alpha,\beta}(\tilde{F}_{sd}^*)$. Let $x = u_1 u_2 ... u_{k+1} \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. Then there exist $q \in \tilde{R}$, $p \in Q$, $b, b' \in Z$ such that

$$\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \wedge \tilde{\omega}_1((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b) >_L \alpha,$$

and

$$\tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \vee \tilde{\omega}_2((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b') <_L \beta.$$

Since

$$\tilde{\delta}_1^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1...u_{k+1}, p) >_L \alpha \text{ and } \tilde{\delta}_2^*((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1...u_{k+1}, p) <_L \beta,$$

then $\mu^{t_0}(q) >_L \alpha, \nu^{t_0}(q) <_L \beta$. Thus $\mu^{t_0}(I_\Lambda) >_L \alpha$ and $\nu^{t_0}(I_\Lambda) <_L \beta$. Also

$$\tilde{\delta}_{d1}^*((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), u_1, I_{u_1}) = F_1^T(\mu^{t_0}(I_\Lambda), \delta_{d1}(I_\Lambda, u_1, I_{u_1})) \geq_L \alpha,$$

thus $\mu^{t_1}(I_{u_1}) \geq_L \alpha$. Also we have

$$\tilde{\delta}_{d1}((I_{u_1}, \mu^{t_1}(I_{u_1}), \nu^{t_1}(I_{u_1})), u_2, I_{u_1 u_2}) = F_1^T(\mu^{t_1}(I_{u_1}), \delta_{d1}(I_{u_1}, u_2, I_{u_1 u_2})) \geq_L \alpha.$$

So if we continue this process, then by some manipulation we get that

$$\tilde{\delta}_{d1}^*((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), x, I_x) \geq_L \alpha.$$

Also, we have

$$\tilde{\delta}_{d2}^*((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), u_1, I_{u_1}) = F_1^S(\nu^{t_0}(I_\Lambda), \delta_{d2}(I_\Lambda, u_1, I_{u_1})) \leq_L \beta,$$

then $\nu^{t_1}(I_{u_1}) \leq_L \beta$. Therefore

$$\tilde{\delta}_{d2}((I_{u_1}, \mu^{t_1}(I_{u_1}), \nu^{t_1}(I_{u_1})), u_2, I_{u_1 u_2}) \leq_L \beta.$$

By continuing this process we obtain that

$$\tilde{\delta}^*_{d2}((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), x, I_x) \leq_L \beta.$$

Also we have $\omega_{d1}(I_x, e) = \gamma_2$ and $\omega_{d2}(I_x, e) = \eta_2$. Hence $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_d)$. Now, suppose that $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_d)$. Then

$$\tilde{\delta}^*_{d1}((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), x, I_x) \wedge \tilde{\omega}_{d1}((I_x, \mu^{t_0+|x|}(I_x), \nu^{t_0+|x|}(I_x)), e) >_L \alpha,$$

and

$$\tilde{\delta}^*_{d2}((I_\Lambda, \mu^{t_0}(I_\Lambda), \nu^{t_0}(I_\Lambda)), x, I_x) \vee \tilde{\omega}_{d2}((I_x, \mu^{t_0+|x|}(I_x), \nu^{t_0+|x|}(I_x)), e) <_L \beta.$$

Since $\omega_{d1}(I_x, e) >_L \alpha$ and $\omega_{d2}(I_x, e) <_L \beta$, then by (4.5) and (4.6) $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. ∎

Theorem 4.12 *Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a max-min IGLFA and $\tilde{R} \neq \emptyset$. Then there exists an (α, β)-accessible max-min IGLFA $\tilde{F}^*_a$ such that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_a)$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.*

Proof. By Theorem 4.11, without loss of generality we assume that $\tilde{F}^*$ is deterministic. Let S=$\{q \in Q \mid q$ be an (α, β)-accessible state$\}$, $\tilde{R}_a$=$\{(q, \mu^{t_0}(q), \nu^{t_0}(q)) \mid q \in S|_{\tilde{R}}\}$, $Z_a = Z$, $\delta_a = \delta|_{S\times X\times S}$, and $\omega_a = \omega|_{S\times Z}$, i.e., δ_a is the restriction of δ to $S \times X \times S$ and ω_a is the restriction of ω to $S \times Z$. Then the max-min IGLFA $\tilde{F}^*_a$ is (α, β)-accessible. It is clear that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*_a) \subseteq \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. Now, let $x = u_1u_2...u_{k+1} \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. Then there exist $q \in \tilde{R}, p \in Q, b \in Z$ such that

$$\tilde{\delta}^*_1((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \wedge \tilde{\omega}_1((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b) >_L \alpha,$$

and

$$\tilde{\delta}^*_2((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \vee \tilde{\omega}_2((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b') <_L \beta.$$

Therefore, there exist $p_1, p_2, ..., p_k, p'_1, p'_2, ..., p'_k \in Q$ such that

$$\tilde{\delta}^*_1((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1, p_1) \wedge ... \wedge \tilde{\delta}^*_1((p_k, \mu^{t_k}(p_k), \nu^{t_k}(p_k)), u_{k+1}, p) >_L \alpha,$$

and

$$\tilde{\delta}^*_2((q, \mu^{t_0}(q), \nu^{t_0}(q)), u_1, p'_1) \vee ... \vee \tilde{\delta}^*_2((p'_k, \mu^{t_k}(p'_k), \nu^{t_k}(p'_k)), u_{k+1}, p) <_L \beta.$$

These imply that $q \in \tilde{R}_a$ and, since $\tilde{F}^*$ is a deterministic, then

$$p_1 = p'_1, p_2 = p'_2, ..., p_k = p'_k.$$

Thus $p_1, p_2, ..., p_k$, $p \in S$. Hence

$$\tilde{\delta}^*_{a1}((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \wedge \tilde{\omega}_{a1}((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b) >_L \alpha,$$

and

$$\tilde{\delta}^*_{a2}((q, \mu^{t_0}(q), \nu^{t_0}(q)), x, p) \vee \tilde{\omega}_{a2}((p, \mu^{t_0+|x|}(p), \nu^{t_0+|x|}(p)), b') <_L \beta.$$

Then $x \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_a)$. Hence $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*) \subseteq \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_a)$. Thus the claim is hold. ∎

Example 4.13 Consider the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ defined in Example 4.3, and max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 6,

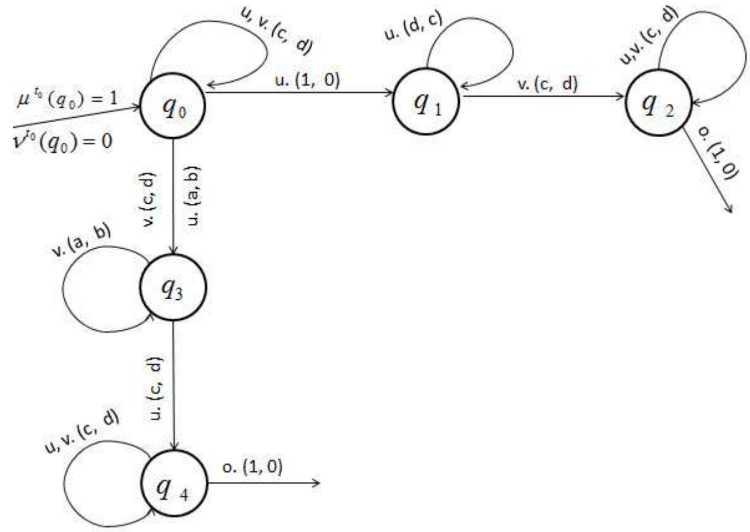

Figure 6: The IGLFA of Example 4.13

where $Q = \{q_0, q_1, q_2, q_3, q_4\}, X = \{u, v\}, \tilde{R} = \{(q_0, 1, 0)\}, Z = \{o\}$. It is clear that $\mathcal{L}^{a,b}(\tilde{F}^*) = \{u, v\}^* uv\{u, v\} \cup \{u, v\}^* vu\{u, v\}$. Considering the proof of Theorems 4.9, 4.11, 4.12, we obtain an (a, b)-complete, deterministic and (a, b)-accessible max-min IGLFA $\tilde{F}^*_{cda}$ as in Figure 7, such that $\mathcal{L}^{a,b}(\tilde{F}^*_{cda}) = \mathcal{L}^{a,b}(\tilde{F}^*)$.

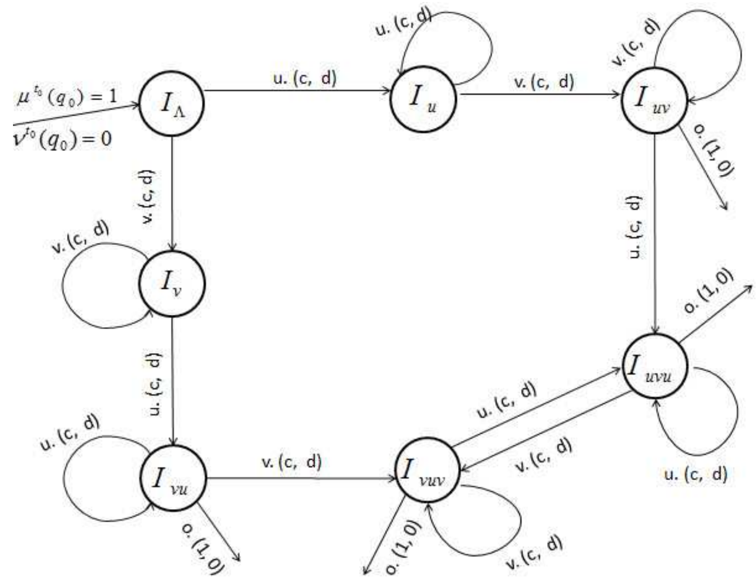

Figure 7: The (a, b)-complete, deterministic and (a, b)-accessible $\tilde{F}^*_{cda}$ of Example 4.13

Definition 4.14 Let $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an (α, β)-accessible, (α, β)-complete, deterministic max-min IGLFA, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. We define a relation on Q by $q_1 \rho^{\alpha,\beta} q_2$, if and only if the following two sets are equal

(i) $\{w \in X^* \mid \tilde{\delta}_1^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha,$
$\tilde{\delta}_2^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta,$
for some $b, b' \in Z, q \in Q\}$

and

(ii) $\{w \in X^* \mid \tilde{\delta}_1^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha,$
$\tilde{\delta}_2^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta,$
for some $b, b' \in Z, q \in Q\}$,

where $q_1 \in Q_{act}(t_i)$ $q_2 \in Q_{act}(t_j)$. It is clear that $\rho^{\alpha,\beta}$ is an equivalence relation.

Definition 4.15 We say that the (α, β)-accessible, (α, β)-complete, deterministic max-min IGLFA $\tilde{F}^*$, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$, is (α, β)-reduced if $q_1 \rho^{\alpha,\beta} q_2$ implies that $q_1 = q_2$, for any $q_1, q_2 \in Q$.

Let $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an (α, β)-accessible, (α, β)-complete, deterministic max-min IGLFA, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. Now, suppose that $\rho^{\alpha,\beta}$ be the equivalence relation defined in Definition 4.14. Consider $Q/\rho^{\alpha,\beta} = \{q\rho^{\alpha,\beta} \mid q \in Q\}$ and $\tilde{R}/\rho^{\alpha,\beta} = q_0\rho^{\alpha,\beta}$, $\mu^{t_0}(q_0\rho^{\alpha,\beta}) = \mu^{t_0}(q_0)$, $\nu^{t_0}(q_0\rho^{\alpha,\beta}) = \nu^{t_0}(q_0)$. We define $\delta_\rho : Q/\rho^{\alpha,\beta} \times X \times Q/\rho^{\alpha,\beta} \to L \times L$ by:

$$\delta_{\rho 1}(q_1\rho^{\alpha,\beta}, a, q_2\rho^{\alpha,\beta}) = \begin{cases} \gamma_1 & if \delta_1(q_1, a, q_2') >_L \alpha \ \& \ \delta_2(q_1, a, q_2') <_L \beta, \\ & \qquad \text{where } q_2'\rho^{\alpha,\beta} q_2 \\ 0 & \text{otherwise,} \end{cases}$$

$$\delta_{\rho 2}(q_1\rho^{\alpha,\beta}, a, q_2\rho^{\alpha,\beta}) = \begin{cases} \eta_1 & \text{if } \delta_1(q_1, a, q_2') >_L \alpha \ \& \ \delta_2(q_1, a, q_2') <_L \beta, \\ & \qquad \text{where } q_2'\rho^{\alpha,\beta} q_2 \\ 1 & \text{otherwise,} \end{cases}$$

where $\gamma_1, \eta_1 \in L, \gamma_1 >_L \alpha, \eta_1 <_L \beta$ and $\gamma_1 \leq_L N(\eta_1)$.

Define $\omega_\rho : Q/\rho^{\alpha,\beta} \times Z \to L \times L$ by:

$$\omega_{\rho 1}(q_1\rho^{\alpha,\beta}, b) = \begin{cases} \gamma_2 & if \omega_1(q_1, b') >_L \alpha \ \& \ \delta_2(q_1, b'') <_L \beta, \\ 0 & \text{otherwise,} \end{cases} \tag{4.7}$$

$$\omega_{\rho 2}(q_1\rho^{\alpha,\beta}, b) = \begin{cases} \eta_2 & if \omega_1(q_1, b') >_L \alpha \ \& \ \delta_2(q_1, b'') <_L \beta, \\ 1 & \text{otherwise,} \end{cases} \tag{4.8}$$

for some $b', b'' \in Z$ and $\gamma_2, \eta_2 \in L, \gamma_2 >_L \alpha, \eta_2 <_L \beta$ and $\gamma_2 \leq_L N(\eta_2)$.

Theorem 4.16 *Suppose that $\alpha, \beta \in L, \alpha \leq_L N(\beta)$. Then the following properties hold:*

1. δ_ρ *and* ω_ρ *are well defined,*
2. $\tilde{F}^*/\rho^{\alpha,\beta} = (Q/\rho^{\alpha,\beta}, X, q_0\rho^{\alpha,\beta}, Z, \tilde{\delta}^*_\rho, \tilde{\omega}_\rho, F_1, F_2, F_3, F_4)$ *is an* (α,β)*-reduced max-min IGLFA,*
3. $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*/\rho^{\alpha,\beta}) = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$.

Proof. 1. Let $q_1\rho^{\alpha,\beta}, q_2\rho^{\alpha,\beta}, p_1\rho^{\alpha,\beta}, p_2\rho^{\alpha,\beta} \in Q/\rho^{\alpha,\beta}$, $q_1\rho^{\alpha,\beta} = p_1\rho^{\alpha,\beta}$ and $q_2\rho^{\alpha,\beta} = p_2\rho^{\alpha,\beta}$. Then $q_1\rho^{\alpha,\beta}p_1$ and $q_2\rho^{\alpha,\beta}p_2$. Therefore

$$\begin{aligned} A = \{w \in X^* \mid\ & \tilde{\delta}^*_1((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha, \\ & \tilde{\delta}^*_2((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta, \\ & \text{for some } b, b' \in Z, q \in Q\} = \\ \{w \in X^* \mid\ & \tilde{\delta}^*_1((p_1, \mu^{t_j}(p_1), \nu^{t_j}(p_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha, \\ & \tilde{\delta}^*_2((p_1, \mu^{t_j}(p_1), \nu^{t_j}(p_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta, \\ & \text{for some } b, b' \in Z, q \in Q\} = B. \end{aligned}$$

Let $\delta_{\rho 1}(q_1\rho^{\alpha,\beta}, a, q_2\rho^{\alpha,\beta}) = \gamma$ and $\delta_{\rho 2}(q_1\rho^{\alpha,\beta}, a, q_2\rho^{\alpha,\beta}) = \eta$, where $\gamma, \eta \in L$ and $\gamma \leq_L N(\eta)$. Then there exists $q'_2 \in Q$ such that $q'_2\rho^{\alpha,\beta}q_2$ and $\delta_1(q_1, a, q'_2) >_L \alpha$ and $\delta_2(q_1, a, q'_2) <_L \beta$. We have $w \in A$ if and only $w \in B$, for all $w \in X^*$. This holds in particular for all $w \in aX^*$. Then

$$\begin{aligned} \{aw \in aX^* \mid\ & \tilde{\delta}_1((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), a, q'_2) \wedge \tilde{\delta}^*_1((q'_2, \mu^{t_{i+1}}(q'_2), \nu^{t_{i+1}}(q'_2)), w, q) \\ & \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha, \\ & \tilde{\delta}_2((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), a, q'_2) \vee \tilde{\delta}^*_2((q'_2, \mu^{t_{i+1}}(q'_2), \nu^{t_{i+1}}(q'_2)), w, q) \\ & \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta, \\ & \text{for some } b, b' \in Z, q, q'_2 \in Q\} = \\ \{aw \in aX^* \mid\ & \tilde{\delta}^*_1((p_1, \mu^{t_j}(p_1), \nu^{t_j}(p_1)), a, p') \wedge \tilde{\delta}^*_1((p', \mu^{t_{j+1}}(p'), \nu^{t_{j+1}}(p')), w, q) \\ & \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha, \\ & \tilde{\delta}_2((p_1, \mu^{t_j}(p_1), \nu^{t_j}(p_1)), a, p') \vee \tilde{\delta}^*_2((p', \mu^{t_{j+1}}(p'), \nu^{t_{j+1}}(p')), w, q) \\ & \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta, \\ & \text{for some } b, b' \in Z, q, p' \in Q\}. \end{aligned}$$

These imply that

$$\begin{aligned} \{w \in X^* \mid\ & \tilde{\delta}^*_1((q'_2, \mu^{t_{i+1}}(q'_2), \nu^{t_{i+1}}(q'_2)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha, \\ & \tilde{\delta}^*_2((q'_2, \mu^{t_{i+1}}(q'_2), \nu^{t_{i+1}}(q'_2)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta, \\ & \text{for some } b, b' \in Z, q \in Q\} = \\ \{w \in X^* \mid\ & \tilde{\delta}^*_1((p', \mu^{t_{j+1}}(p'), \nu^{t_{j+1}}(p')), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha, \\ & \tilde{\delta}^*_2((p', \mu^{t_{j+1}}(p'), \nu^{t_{j+1}}(p')), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta, \\ & \text{for some } b, b' \in Z, q \in Q\}. \end{aligned}$$

Thus $p'\rho^{\alpha,\beta}q'_2\rho^{\alpha,\beta}q_2\rho^{\alpha,\beta}p_2$ i.e., $p'\rho^{\alpha,\beta}p_2$. Also, $\tilde{F}^*$ is (α,β)-complete and deterministic, therefore $\delta_1(p_1,a,p') >_L \alpha$ and $\delta_2(p_1,a,p') <_L \beta$. Hence

$$\delta_{\rho 1}(p_1\rho^{\alpha,\beta},a,p_2\rho^{\alpha,\beta}) = \gamma \text{ and } \delta_{\rho 2}(p_1\rho^{\alpha,\beta},a,p_2\rho^{\alpha,\beta}) = \eta.$$

So δ_ρ is well defined.

Now, let $q_1\rho^{\alpha,\beta} = p_1\rho^{\alpha,\beta}$. If $\omega_{\rho 1}(q_1\rho^{\alpha,\beta},b) = \gamma'$ and $\omega_{\rho 2}(q_1\rho^{\alpha,\beta},b) = \eta'$, where $\gamma',\eta' \in L$ and $\gamma' \leq_L N(\eta')$, then $\omega_1(q_1,b') >_L \alpha$ and $\omega_2(q_1,b'') <_L \beta$, for some $b \in Z_\rho, b', b" \in Z$. Since $\tilde{F}^*$ is (α,β)-accessible, then there exists a positive integer j such that $\mu^{t_j}(q_1) >_L \alpha$ and $\nu^{t_j}(q_1) <_L \beta$. These imply that $\Lambda \in A$. Therefore $\Lambda \in B$. Then $\omega_1(q_2,b') >_L \alpha$ and $\omega_2(q_2,b'') <_L \beta$, for some $b', b" \in Z$. Hence

$$\omega_{\rho 1}(q_2\rho^{\alpha,\beta},b) = \gamma' \text{ and } \omega_{\rho 2}(q_2\rho^{\alpha,\beta},b) = \eta'.$$

Then ω_ρ is well defined.

2. Let $(q_1\rho^{\alpha,\beta})\rho^{\alpha,\beta}(q_2\rho^{\alpha,\beta})$. Now, we have to show that $q_1\rho^{\alpha,\beta} = q_2\rho^{\alpha,\beta}$. It suffices to show that $q_1\rho^{\alpha,\beta}q_2$.

Let for $w \in X^*$

$$\begin{aligned}
&\tilde{\delta}_1^*((q_1,\mu^{t_i}(q_1),\nu^{t_i}(q_1)),w,p_1) \wedge \tilde{\omega}_1((p_1,\mu^{t_i+|w|}(p_1),\nu^{t_i+|w|}(p_1)),b) >_L \alpha,\\
&\tilde{\delta}_2^*((q_1,\mu^{t_i}(q_1),\nu^{t_i}(q_1)),w,p_1) \vee \tilde{\omega}_2((p_1,\mu^{t_i+|w|}(p_1),\nu^{t_i+|w|}(p_1)),b') <_L \beta,
\end{aligned}$$

for some $b, b' \in Z$. Then

$$\begin{aligned}
&\tilde{\delta}_{\rho 1}^*((q_1\rho^{\alpha,\beta},\mu^{t_i}(q_1\rho^{\alpha,\beta}),\nu^{t_i}(q_1\rho^{\alpha,\beta})),w,p_1\rho^{\alpha,\beta})\\
&\qquad\qquad \wedge \tilde{\omega}_{\rho 1}((p_1\rho^{\alpha,\beta},\mu^{t_i+|w|}(p_1\rho^{\alpha,\beta}),\nu^{t_i+|w|}(p_1\rho^{\alpha,\beta})),b) >_L \alpha,\\
&\tilde{\delta}_{\rho 2}^*((q_1\rho^{\alpha,\beta},\mu^{t_i}(q_1\rho^{\alpha,\beta}),\nu^{t_i}(q_1\rho^{\alpha,\beta})),w,p_1\rho^{\alpha,\beta})\\
&\qquad\qquad \vee \tilde{\omega}_{\rho 2}((p_1\rho^{\alpha,\beta},\mu^{t_i+|w|}(p_1\rho^{\alpha,\beta}),\nu^{t_i+|w|}(p_1\rho^{\alpha,\beta})),b') <_L \beta,
\end{aligned}$$

for some $b, b' \in Z$. Therefore

$$\begin{aligned}
&\tilde{\delta}_{\rho 1}^*((q_2\rho^{\alpha,\beta},\mu^{t_j}(q_2\rho^{\alpha,\beta}),\nu^{t_j}(q_2\rho^{\alpha,\beta})),w,p'_1\rho^{\alpha,\beta})\\
&\qquad\qquad \wedge \tilde{\omega}_{\rho 1}((p'_1\rho^{\alpha,\beta},\mu^{t_j+|w|}(p'_1\rho^{\alpha,\beta}),\nu^{t_j+|w|}(p'_1\rho^{\alpha,\beta})),b) >_L \alpha,\\
&\tilde{\delta}_{\rho 2}^*((q_2\rho^{\alpha,\beta},\mu^{t_j}(q_2\rho^{\alpha,\beta}),\nu^{t_j}(q_2\rho^{\alpha,\beta})),w,p'_1\rho^{\alpha,\beta})\\
&\qquad\qquad \vee \tilde{\omega}_{\rho 2}((p'_1\rho^{\alpha,\beta},\mu^{t_j+|w|}(p'_1\rho^{\alpha,\beta}),\nu^{t_j+|w|}(p'_1\rho^{\alpha,\beta})),b') <_L \beta,
\end{aligned}$$

$p'_1\rho^{\alpha,\beta} \in Q/\rho^{\alpha,\beta}$ for some $b, b' \in Z$. Since $\tilde{F}^*$ is (α,β)-accessible, then there exists a positive integer j' such that $\mu^{t_{j'}}(q_2) >_L \alpha$ and $\nu^{t_{j'}}(q_2) <_L \beta$. Thus

$$\begin{aligned}
&\tilde{\delta}_1^*((q_2,\mu^{t_{j'}}(q_2),\nu^{t_{j'}}(q_2)),w,p) \wedge \tilde{\omega}_1((p,\mu^{t_{j'}+|w|}(p),\nu^{t_{j'}+|w|}(p)),b) >_L \alpha,\\
&\tilde{\delta}_2^*((q_2,\mu^{t_{j'}}(q_2),\nu^{t_{j'}}(q_2)),w,p) \vee \tilde{\omega}_2((p,\mu^{t_{j'}+|w|}(p),\nu^{t_{j'}+|w|}(p)),b') <_L \beta,
\end{aligned}$$

where $p\rho^{\alpha,\beta}p_1'$ for some $b, b' \in Z$. The converse follows in a similar manner. Hence

$$\{w \in X^* \mid \tilde{\delta}_1^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta,$$
$$\text{for some } b, b' \in Z, q \in Q\} =$$
$$\{w \in X^* \mid \tilde{\delta}_1^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta,$$
$$\text{for some } b, b' \in Z, q \in Q\}.$$

Hence the claim is hold.

3. Considering the proof of 2, it is trivial. ■

Example 4.17 Consider the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ defined in Example 4.3, and the max-min IGLFA $\tilde{F}^*_{cda} = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 7. By the Definition 4.14, $I_{uv}\rho^{a,b}I_{uvu}\rho^{a,b}I_{vuv}\rho^{a,b}I_{vu}$. Then the max-min IGLFA $\tilde{F}^*/\rho^{a,b}$ has the state diagram as in Figure 8.

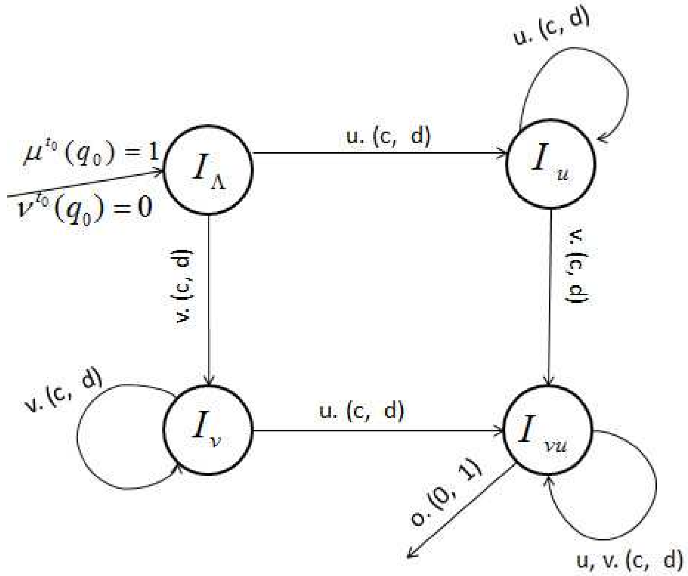

Figure 8: The reduced $\tilde{F}^*/\rho^{a,b}$ of Example 4.17

Definition 4.18 Let

$$\tilde{F}_1^* = (Q_1, X, \tilde{R}_1, Z_1, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4) \quad \text{and}$$
$$\tilde{F}_2^* = (Q_2, X, \tilde{R}_2, Z_2, \tilde{\delta}'^*, \tilde{\omega}', F_1, F_2, F_3, F_4)$$

be two max-min IGLFA. A homomorphism from $\tilde{F}_1^*$ onto $\tilde{F}_2^*$ with threshold (α, β), where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$, is a function ξ from Q_1 onto Q_2 such that for every $q', q'' \in Q_1$, $u \in X$ and $b_1, b_2 \in Z$ the following conditions hold:

1. $(\mu^{t_0}_{Q_1}(q') >_L \alpha \ \& \ \nu^{t_0}_{Q_1}(q') <_L \beta)$
   if and only if $(\mu^{t_0}_{Q_2}(\xi(q')) >_L \alpha \ \& \ \nu^{t_0}_{Q_2}(\xi(q')) <_L \beta)$,
2. $(\delta_1(q', u, q") >_L \alpha \ \& \ \delta_2(q', u, q") <_L \beta)$
   if and only if $(\delta'_1(\xi(q'), u, \xi(q")) >_L \alpha \ \& \ \delta'_2(\xi(q'), u, \xi(q")) <_L \beta)$,
3. $(\omega_1(q', b_1) >_L \alpha \ \& \ \omega_2(q', b_2) <_L \beta)$
   implies $(\omega'_1(\xi(q'), b) >_L \alpha \ \& \ \omega_2(\xi(q'), b_2) <_L \beta)$ for some $b, b' \in Z'$.

We say that ξ is an isomorphism with threshold (α, β) if and only if ξ is a homomorphism with threshold (α, β) that is one-one and $(\omega_1(q', b_1) >_L \alpha$ and $\omega_2(q', b_2) <_L \beta)$ if and only if $(\omega'_1(\xi(q'), b) >_L \alpha$ and $\omega'_2(\xi(q'), b_2) <_L \beta)$, for some $b, b' \in Z'$.

Definition 4.19 Let $\mathcal{L}$ be an (α, β)-language, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. A relation $R_\mathcal{L}$ on X^* is defined by:

> for any two strings x and y in X^*, $xR_\mathcal{L}y$ if, for all $z \in X^*$,
> either $xz, yz \in \mathcal{L}$ or $xz, yz \notin \mathcal{L}$.

Definition 4.20 Let $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be a deterministic max-min IGLFA. Then for any $\alpha, \beta \in L$, where $\alpha \leq_L N(\beta)$, define a relation $R^{\alpha,\beta}_F$ on X^*. For strings x and y in X^*, $xR^{\alpha,\beta}_F y$ if and only if there exists $q \in Q$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) <_L \beta,$$
$$\text{iff}$$
$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), y, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), y, q) <_L \beta.$$

Now, we show that $R^{\alpha,\beta}_F$ is an equivalence relation. It is clear that $xR^{\alpha,\beta}_F x$ and if $xR^{\alpha,\beta}_F y$, then $yR^{\alpha,\beta}_F x$.

Now, let $xR^{\alpha,\beta}_F y$ and $yR^{\alpha,\beta}_F z$. Suppose that there exists $q \in Q$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) <_L \beta.$$

Then

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), y, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), y, q) <_L \beta,$$

and, since $yR^{\alpha,\beta}_F z$, then

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), z, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), z, q) <_L \beta.$$

Similarity we can obtain the converse, i.e.,

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), z, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), z, q) <_L \beta,$$

implies that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) <_L \beta.$$

Then $xR^{\alpha,\beta}_F z$. Hence the claim is hold.

Note 4.21 By Definition 4.20, for any (α, β)-complete and deterministic max-min IGLFA $\tilde{F}^*$, the number of classes of equivalence relation $R_F^{\alpha,\beta}$ is less than or equal to the number of states of $\tilde{F}^*$.

Theorem 4.22 *Let $\alpha, \beta \in L$, where $\alpha \leq_L N(\beta)$. Suppose that $\mathcal{L}$ be a recognizable (α, β)-language where $\mathcal{L} = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$, for some (α, β)-complete and deterministic max-min IGLFA $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$. Then for a given equivalence class $[w]_{R_F^{\alpha,\beta}}$ of $R_F^{\alpha,\beta}$ there is an equivalence class $[w]_{R_\mathcal{L}}$ of $R_\mathcal{L}$ such that $[w]_{R_F^{\alpha,\beta}} \subseteq [w]_{R_\mathcal{L}}$. Each equivalence class $[w]_{R_\mathcal{L}}$ of the relation $R_\mathcal{L}$ is a finite union of equivalence classes of $R_F^{\alpha,\beta}$.*

Proof. Let $[w]_{R_F^{\alpha,\beta}}$ be an equivalence class of $R_F^{\alpha,\beta}$. Now, suppose that $x \in [w]_{R_F^{\alpha,\beta}}$. Since $\tilde{F}^*$ is an (α, β)-complete, then there exists $q \in Q$ such that

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) <_L \beta.$$

Therefore

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), w, q) >_L \alpha \quad \text{and} \quad \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), w, q) <_L \beta.$$

By the (α, β)-complete property of $\tilde{F}^*$, for any $y \in X^*$, there exists $q' \in Q$ such that

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), xy, q') \wedge \tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), wy, q') >_L \alpha,$$
$$\text{and}$$
$$\tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), xy, q') \vee \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), wy, q') <_L \beta.$$

If there exist $b, b' \in Z$ such that $\omega_1(q', b) >_L \alpha$ and $\omega_2(q', b) <_L \beta$, then $xy, wy \in \mathcal{L}$ otherwise $xy, wy \notin \mathcal{L}$. Then $xR_\mathcal{L}w$. Thus $x \in [w]_{R_\mathcal{L}}$ and $[w]_{R_F^{\alpha,\beta}} \subseteq [w]_{R_\mathcal{L}}$. Suppose that $[w]_{R_\mathcal{L}}$ is an equivalence class of $R_\mathcal{L}$. It is obvious that if $w \notin \mathcal{L}$, then for any $x \in [w]_{R_\mathcal{L}}$, $x \notin \mathcal{L}$ and if $w \in \mathcal{L}$, then for any $x \in [w]_{R_\mathcal{L}}$, $x \in \mathcal{L}$. If $w \notin \mathcal{L}$, then for any $q \in Q, b, b' \in Z$ we have

$$\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_0+|w|}(q), \nu^{t_0+|w|}(q)), b) \leq_L \alpha,$$

or

$$\tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_0+|w|}(q), \nu^{t_0+|w|}(q)), b') \geq_L \beta.$$

If $\tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), w, q) >_L \alpha$ and $\tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), w, q) <_L \beta$, then

$$\omega_1(q, b) \leq_L \alpha \quad \text{or} \quad \omega_2(q, b) \geq_L \beta.$$

The set

$$S = \{q \in Q \mid \tilde{\delta}_1^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) >_L \alpha, \text{ and } \tilde{\delta}_2^*((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), x, q) <_L \beta, x \in [\omega]_{R_L}\}, \tag{4.9}$$

is finite. Then we have $\omega_1(q, b) \leq_L \alpha$ or $\omega_2(q, b) \geq_L \beta$ for all $q \in S$ and $b \in Z$, or for all $q \in S$ there exist $b, b' \in Z$ such that $\omega_1(q, b) >_L \alpha$ and $\omega_2(q, b') <_L \beta$. Therefore the equivalence class $[w]_{R_\mathcal{L}}$ of $R_\mathcal{L}$ is a finite union of the equivalence classes $[w]_{R_F^{\alpha,\beta}}$ of $R_F^{\alpha,\beta}$, where $q \in S$. ∎

Theorem 4.23 *Let $\mathcal{L}$ be a recognizable (α,β)-language. Then there is an (α,β)-complete and deterministic IGLFA $\tilde{F}^*_m$ such that $\mathcal{L}^{\alpha,\beta}(\tilde{F}^*_m) = \mathcal{L}$ and $\tilde{F}^*_m$ is a minimal automaton, where $\alpha,\beta \in L$ and $\alpha \leq_L N(\beta)$.*

Proof. Let $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ be an (α,β)-complete and deterministic max-min IGLFA such that $\mathcal{L} = \mathcal{L}^{\alpha,\beta}(\tilde{F}^*)$. By Theorem 4.22, we have that the number of equivalence classes of $R_{\mathcal{L}}$ is finite. Let Q_m be the set of equivalence classes of $R_{\mathcal{L}}$ i.e., $Q_m = \{[w] \mid [w] \text{ is an equivalence class of } R_{\mathcal{L}}\}$. Let $\tilde{R}_m = \{([\Lambda], \gamma_1, \gamma_2)\}$, where $\gamma_1, \gamma_2 \in L, \gamma_1 >_L \alpha, \gamma_2 <_L \beta$ and $\gamma_1 \leq_L N(\gamma_2)$. Define $\delta_m : Q_m \times X \times Q_m \to L \times L$ by

$$\delta_{m1}([z], a, [x]) = \begin{cases} \gamma_1 & \text{if } [za] = [x], \\ 0 & \text{otherwise,} \end{cases} \tag{4.10}$$

and

$$\delta_{m2}([z], a, [x]) = \begin{cases} \gamma_2 & \text{if } [za] = [x], \\ 1 & \text{otherwise,} \end{cases} \tag{4.11}$$

Also define $\omega_m : Q_m \times \{b\} \to L \times L$ by

$$\omega_{m1}([w], b) = \begin{cases} \gamma_1 & \text{if } w \in \mathcal{L}, \\ 0 & \text{otherwise,} \end{cases} \tag{4.12}$$

and

$$\omega_{m2}([w], b) = \begin{cases} \gamma_2 & \text{if } w \in \mathcal{L}, \\ 1 & \text{otherwise.} \end{cases} \tag{4.13}$$

It is clear that $\tilde{F}^*_m = (Q_m, X, \tilde{R}_m, Z' = \{b\}, \tilde{\delta}^*_m, \tilde{\omega}_m, F_1, F_2, F_3, F_4)$ is an (α,β)-complete and deterministic IGLFA. Clearly, δ_m and ω_m are well defined.

Let $w \in \mathcal{L}$. Then, we have

$$\delta^*_{m1}(([\Lambda], \mu^{t_0}([\Lambda]), \nu^{t_0}([\Lambda])), w, [w]) >_L \alpha \quad \text{and}$$
$$\delta^*_{m2}(([\Lambda], \mu^{t_0}([\Lambda]), \nu^{t_0}([\Lambda])), w, [w]) <_L \beta,$$

and

$$\omega_{m1}([w], b) = \gamma_1 >_L \alpha, \ \omega_{m2}([w], b) = \gamma_2 <_L \beta.$$

These imply that $w \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_m)$.

Now, suppose that $w \in \mathcal{L}^{\alpha,\beta}(\tilde{F}^*_m)$. Then there exist $[z] \in Q_m, b \in Z$ such that

$$\tilde{\delta}^*_{m1}(([\Lambda], \mu^{t_0}([\Lambda]), \nu^{t_0}([\Lambda])), w, [z]) \wedge \tilde{\omega}_{m1}(([z], \mu^{t_0+|w|}([z]), \nu^{t_0+|w|}([z])), b) >_L \alpha, \tag{4.14}$$

and

$$\tilde{\delta}^*_{m2}(([\Lambda], \mu^{t_0}([\Lambda]), \nu^{t_0}([\Lambda])), w, [z]) \vee \tilde{\omega}_{m2}(([z], \mu^{t_0+|w|}([z]), \nu^{t_0+|w|}([z])), b) <_L \beta. \tag{4.15}$$

Considering (4.10), (4.11), (4.14) and (4.15) we have $[z] = [w]$. Then $\omega_{m1}([w], b) >_L \alpha$ and $\omega_{m2}([w], b) <_L \beta$. Then $w \in \mathcal{L}$. Hence $\mathcal{L}^{\alpha}(\tilde{F}^*_m) = \mathcal{L}$. By Theorem 4.22, we have that the number of equivalence classes of $R_{\mathcal{L}}$ is less than or equal to that of equivalence classes of $R_F^{\alpha,\beta}$, and by Note 4.21, the number of classes of equivalence relation $R_F^{\alpha,\beta}$ is less than or equal to the number of states of $\tilde{F}^*$. Hence the claim holds. ∎

Example 4.24 Let the complete lattice $L = (L, \leq_L, T, S, 0, 1)$ defined in Example 4.3, and the max-min IGLFA $\tilde{F}^* = (Q, X, \tilde{R}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$ as in Figure 6. Considering Definition 4.19, we find $[u]$, $[v]$, $[uv] = [vu]$, $[u^2] = [u]$, $[v^2] = [v]$, $[uv^2] = [uv]$ and $[uvu] = [uv]$. Then we have $\tilde{F}^*_m$ as in Figure 9. It is

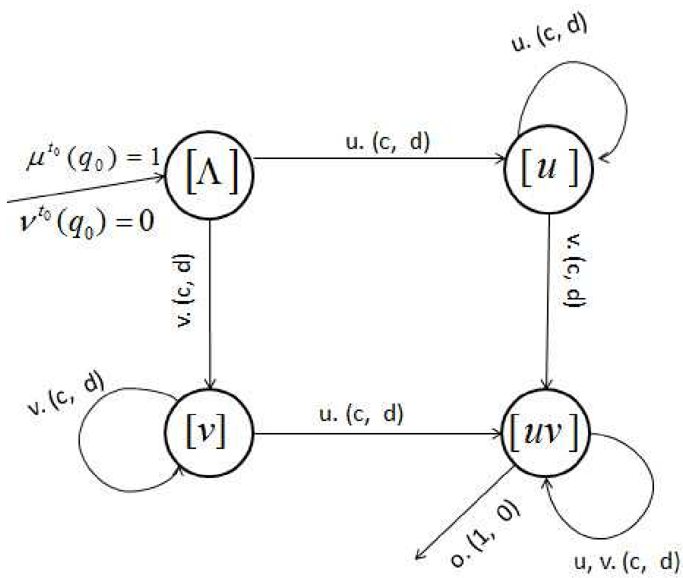

Figure 9: The minimal $\tilde{F}^*_m$ of Example 4.24

obvious that $\tilde{F}^*/\rho^{a,b}$ as in Figure 8, and $\tilde{F}^*_m$ as in Figure 9, have the same number of states.

Theorem 4.25 *For every recognizable $(\alpha, \beta)-$language $\mathcal{L}$, the minimal max-min IGLFA $\tilde{F}^*$ defined in proof of Theorem 4.23, is (α, β)-reduced, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$.*

Proof. Let $\tilde{F}^*$ be an (α, β)-accessible, (α, β)-complete, deterministic max-min IGLFA. Suppose that $q_1 = [u_1], q_2 = [u_2]$ be equivalence classes of $R_{\mathcal{L}}$. Now, let $q_1 \rho^{\alpha,\beta} q_2$. Then

$$\begin{aligned}
A=\{w \in X^* | & \tilde{\delta}_1^*(([u_1], \mu^{t_i}([u_1]), \nu^{t_i}([u_1])), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha, \\
& \tilde{\delta}_2^*(([u_1], \mu^{t_i}([u_1]), \nu^{t_i}([u_1])), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta, \\
& \text{for some } b, b' \in Z, q \text{ is an equivalence classes of } R_{\mathcal{L}}\} \\
=\{w \in X^* | & \tilde{\delta}_1^*(([u_2], \mu^{t_j}([u_2]), \nu^{t_j}([u_2])), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha, \\
& \tilde{\delta}_2^*(([u_2], \mu^{t_j}([u_2]), \nu^{t_j}([u_2])), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta,
\end{aligned}$$

for some $b, b' \in Z, q$ is an equivalence classes of $R_{\mathcal{L}}\} = B$.

Therefore $w \in A \iff w \in B$, for all $w \in X^*$, this implies that $u_1 w \in \mathcal{L} \iff u_2 w \in \mathcal{L}$, for all $w \in X^*$. Then $[u_1] = [u_2]$. Hence $\tilde{F}^*$ is an (α, β)-reduced. ∎

Theorem 4.26 *Let $\mathcal{L}$ be a recognizable (α, β)-language, where $\alpha, \beta \in L$ and $\alpha \leq_L N(\beta)$. Let $\tilde{F}^*_m$ be the max-min IGLFA defined in the proof of Theorem 4.23, and $\tilde{F}^*$ be an (α, β)-complete, (α, β)-accessible, deterministic and (α, β)-reduced max-min IGLFA. Then $\tilde{F}^*_m$ and $\tilde{F}^*$ are isomorphism with threshold (α, β).*

Proof. Let $\tilde{F}^*_m = (Q_,, X, \{[\Lambda]\}, \{b\}, \tilde{\delta}^*_m, \tilde{\omega}_m, F_1, F_2, F_3, F_4)$ and $\tilde{F}^* = (Q, X, \{q_0\}, Z, \tilde{\delta}^*, \tilde{\omega}, F_1, F_2, F_3, F_4)$. Define $\xi : Q \to Q_m$ by $\xi(q) = [u]$, where

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Since $\tilde{F}^*$ is (α, β)-accessible, then $\mu^{t_0}_Q(q_0) >_L \alpha$ and $\nu^{t_0}_Q(q_0) <_L \beta$.
Also, $\mu^{t_0}_{Q_m}([\Lambda]) >_L \alpha$ and $\nu^{t_0}_{Q_m}([\Lambda]) <_L \beta$.
Let $q_1, q_2 \in Q$ and $q_1 = q_2$. Then $q_1 \rho^{\alpha,\beta} q_2$. Therefore

$$\begin{aligned}\{w \in X^* \mid\, & \tilde{\delta}^*_1((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b) >_L \alpha,\\ & \tilde{\delta}^*_2((q_1, \mu^{t_i}(q_1), \nu^{t_i}(q_1)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_i+|w|}(q), \nu^{t_i+|w|}(q)), b') <_L \beta,\\ & \text{for some } b, b' \in Z, q \in Q\} =\\ \{w \in X^* \mid\, & \tilde{\delta}^*_1((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \wedge \tilde{\omega}_1((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b) >_L \alpha,\\ & \tilde{\delta}^*_2((q_2, \mu^{t_j}(q_2), \nu^{t_j}(q_2)), w, q) \vee \tilde{\omega}_2((q, \mu^{t_j+|w|}(q), \nu^{t_j+|w|}(q)), b') <_L \beta,\\ & \text{for some } b, b' \in Z, q \in Q\}.\end{aligned}$$

Since $\mu^{t_i}(q_1) >_L \alpha, \nu^{t_i}(q_1) <_L \beta$ and $\mu^{t_j}(q_2) >_L \alpha, \nu^{t_j}(q_2) <_L \beta$, then there exist $u, v \in X^*$, where $|u| = i, |v| = j$, and

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q_1) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q_1) <_L \beta,$$

and

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), v, q_2) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), v, q_2) <_L \beta.$$

Thus $uz \in \mathcal{L}$ if and only if $vz \in \mathcal{L}$ for all $z \in X^*$. Then $[u] = [v]$, i.e., $\xi(q_1) = \xi(q_2)$. Hence ξ is well defined.

Let $[u] \in Q_m$. By the (α, β)-complete property of $\tilde{F}^*$, there exists $q \in Q$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Then $\xi(q) = [u]$. Therefore ξ is surjective.

Now, let $\delta_1(q, a, q') >_L \alpha$ and $\delta_2(q, a, q') <_L \beta$. Since $\tilde{F}^*$ is (α, β)- accessible, then there exists $u \in X^*$ such that

$$\tilde{\delta}^*_1((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) >_L \alpha \text{ and } \tilde{\delta}^*_2((q_0, \mu^{t_0}(q_0), \nu^{t_0}(q_0)), u, q) <_L \beta.$$

Therefore

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),ua,q') >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),ua,q') <_L \beta.$$

Hence $\xi(q)=[u]$ and $\xi(q')=[ua]$ and $\delta_{m1}(\xi(q),a,\xi(q')) >_L \alpha$, $\delta_{m2}(\xi(q),a,\xi(q')) <_L \beta$.

Let $\delta_{m1}(\xi(q),a,\xi(q')) >_L \alpha$ and $\delta_{m2}(\xi(q),a,\xi(q')) <_L \beta$, where $\xi(q)=[u]$ and $\xi(q')=[v]$. Then $[ua]=[v]$. Thus

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) <_L \beta,$$

and

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),ua,q') >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),ua,q') <_L \beta.$$

Since $\tilde{F}^*$ is deterministic, then $\delta_1(q,a,q') >_L \alpha$ and $\delta_2(q,a,q') <_L \beta$.

Let $q \in Q$ and $\omega_1(q,b) >_L \alpha$ and $\omega_2(q,b') <_L \beta$ for some $b,b' \in Z$. By the (α,β)-accessibility property

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) <_L \beta.$$

Then $u \in \mathcal{L}$. Therefore $\omega_{m1}([u],b) >_L \alpha$ and $\omega_{m2}([u],b') <_L \beta$ for some $b,b' \in Z$. Hence $\omega_{m1}(\xi(q),b) >_L \alpha$ and $\omega_{m2}(\xi(q),b') <_L \beta$ for some $b,b' \in Z$.

Then $\tilde{F}_m^*$ and $\tilde{F}^*$ are homomorphism with threshold (α,β). Now, let $q_1,q_2 \in Q$ and $\xi(q_1)=\xi(q_2)$. Then there exist $x,y \in X^*$ such that $[x]=\xi(q_1)=\xi(q_2)=[y]$. Therefore $q_1\rho^{\alpha,\beta}q_2$. Since $\tilde{F}^*$ is (α,β)-reduced, then $q_1=q_2$. Thus ξ is one-one.

Now, let $\omega_{m1}(\xi(q),b) >_L \alpha$ and $\omega_{m2}(\xi(q),b') <_L \beta$ for some $b,b' \in Z$, where $\xi(q)=[u]$. On the other hand, $\omega_{m1}([u],b) >_L \alpha$ and $\omega_{m2}([u],b') <_L \beta$ for some $b,b' \in Z$. These imply that $u \in \mathcal{L}$. Then

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q') \wedge \tilde{\omega}_1((q',\mu^{t_0+|u|}(q'),\nu^{t_0+|u|}(q')),b) >_L \alpha,$$
$$\tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q') \vee \tilde{\omega}_2((q',\mu^{t_0+|u|}(q'),\nu^{t_0+|u|}(q')),b') <_L \beta,$$

for some $b,b' \in Z$. Also, we have

$$\tilde{\delta}_1^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) >_L \alpha \text{ and } \tilde{\delta}_2^*((q_0,\mu^{t_0}(q_0),\nu^{t_0}(q_0)),u,q) <_L \beta.$$

Since $\tilde{F}^*$ is deterministic, then $q=q'$. Therefore $\omega_1(q,b) >_L \alpha$ and $\omega_2(q,b') <_L \beta$ for some $b,b' \in Z$.

Hence $\tilde{F}_m^*$ and $\tilde{F}^*$ are isomorphism with threshold (α,β). ∎

5. Conclusion

In this paper, we present the notions of intuitionistic general L-fuzzy automata based on lattice valued intuitionistic fuzzy sets, (α,β)-language, (α,β)-accessible, (α,β)-reduced, (α,β)-complete, deterministic and isomorphic with threshold (α,β). Afterthat, it is shown that for any recognizable (α,β)-language over a bounded lattice L, there exists a minimal (α,β)-complete and deterministic IGLFA, which preserve (α, β)-language. Finally, we show that for any given (α, β)-language $\mathcal{L}$, the minimal (α, β)-complete and deterministic IGLFA recognizing $\mathcal{L}$ is isomorphic with threshold (α, β) to any (α, β)-complete, (α, β)-accessible, deterministic, (α, β)-reduced IGLFA recognizing $\mathcal{L}$.

Now, there is an important question:

> Suppose that there is a max-min IGLFA, say $\tilde{F}^*$. Is there a minimal intuitionistic general L-fuzzy automata with more that one initial state, in which it has the same (α, β)-language as the (α, β)-language of $\tilde{F}^*$?

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References

[1] Atanassov, K., *Intuitionistic fuzzy sets*, Fuzzy Sets and Systems, 20 (1986), 87-96.

[2] Atanassov, K., Stoeva, S., *Intuitionistic L-fuzzy sets.* In: R. Trappl (ed.) Cybernetics and Systems Research 2. Elsevier, North-Holland, (1984), 539-540.

[3] Basak, N.C., Gupta, A., *On quotient machines of a fuzzy automata and the minimal machine*, Fuzzy Sets and Systems, 125 (2002), 223-229.

[4] Cheng, W., Mo, Z.W., *Minimization algorithm of fuzzy finite automata*, Fuzzy Sets and Systems, 141 (2004), 439-448.

[5] Coskun, E., *Systems on intuitionistic fuzzy special sets and intuitionistic fuzzy special measures*, Information Sciences, 128 (2000), 105-118.

[6] Davvaz, B., Dudek, W.A., Jun, Y.B., *Intuitionistic fuzzy Hv-submodules*, Information Sciences, 176 (2006), 285-300.

[7] De, S.K., Biswas, R., Roy, A.R., *An application of intuitionistic fuzzy sets in medical diagnosis*, Fuzzy Sets and Systems 117 (2001), 209-213.

[8] Deschrijver, G., Cornelis, C., Kerre, E.E., *Intuitionistic fuzzy connectives revisited*, Proceedings of 9th International Conference on Information Processing and Management of Uncertainty in Knowledge-Based Systems, July 1-5, 2002.

[9] Deschrijver, G., Kerre, E.E., *On the position of intuitionistic fuzzy set theory in the framework of theories modelling imprecision*, Information Sciences, 177 (2007), 1860-1866.

[10] Doostfatemeh, M., Kremer, S.C., *New directions in fuzzy automata*, International Journal of Approximate Reasoning, 38 (2005), 175-214.

[11] Dudek, W.A., Davvaz, B., Jun, Y.B., *On intuitionistic fuzzy sub-hyperquasigroups of hyperquasigroups*, Information Sciences, 170 (2005), 251-262.

[12] Goguen, J.A., *L-Fuzzy sets, Journal of mathematical analysis and applications*, 18 (1967), 145-173.

[13] Fellows, M.R., Langston, M.A., *An analogue of the Myhill-Nerode theorem and its use in computing finite-basis characterizations*, IEEE, (1989), 520-525.

[14] Giles, C.L., Omlin, C.W., Thornber, K.K., *Equivalence in knowledge representation: automata, recurrent neural networks, and dynamical fuzzy systems*, Proceedings of IEEE, 87 (1999), 1623-1640.

[15] Gupta, M.M., Saridis, G.N., Gaines, B.R., *Fuzzy Automata and Decision Processes*, North Holland, New York, 1977, 111-175.

[16] Hung, W.L., Wu, J.W., *Correlation of intuitionistic fuzzy sets by centroid method*, Information Sciences, 144 (2002), 219-225.

[17] Jun, Y.B., *Intuitionistic fuzzy finite state machines*, Journal of Applied Mathematics and Computing, 17 (2005), 109-120.

[18] Jun, Y.B., *Quotient structures of intuitionistic fuzzy finite state machines*, Information Sciences, 177 (2007), 4977-4986.

[19] Jun, Y.B., Ozturk, M.A., Park, C.H., *Intuitionistic nil radicals of intuitionistic fuzzy ideals and Euclidean intuitionistic fuzzy ideals in rings*, Information Sciences, 177 (2007), 4662-4677.

[20] Lee, H.S., *Minimizing fuzzy finite automata*, 2000. Fuzzy Systems, FUZZ IEEE 2000. The Ninth IEEE International Conference on, 1 (2000), 65-70.

[21] Lee, E.T., Zadeh, L.A., *Note on fuzzy languages*, Information Sciences, 1 (1969), 421-434.

[22] Malik , D.S., Mordeson, J.N., *Fuzzy Automata and Languages: Theory and Applications*, Chapman Hall, CRC Boca Raton, London, New York, Washington DC, 2002.

[23] Malik , D.S., Mordeson, J.N., *Fuzzy Discrete Structures*, Physica-Verlag, New York, 2000, London, 2002.

[24] Mateescu, A., Salomaa, A., Salomaa, K., Yu, S., *Lexical Analysis with a Simple Finite Fuzzy Automaton Model*, Journal of Universal Computer Science, 1 (1995), 292-311.

[25] Omlin, C.W., Thornber, K.K., Giles, C.L., *Fuzzy finite-state automata can be deterministically encoded in recurrent neural networks*, IEEE Transactions on Fuzzy Systems, 5 (1998), 76-89.

[26] Pedrycz, W., Gacek, A., *Learning of fuzzy automata*, International Journal of Computational Intelligence and Applications, 1 (2001), 19-33.

[27] Peeva, K., *Behavior, reduction and minimization of finite L-automata*, Fuzzy Sets and Systems, 28 (1988), 171-181.

[28] Peeva, K., *Equivalence, reduction and minimization of finite automata over semirings*, Theoretical Computer Science, 88 (1991), 269-285.

[29] Qiu, D., *Supervisory control of fuzzy discrete event systems: a formal approach*, IEEE Transactions on Systems, Man and CyberneticsPart B, 35 (2005), 72-88.

[30] Ray, A.K., Chatterjee, B., Majumdar, A.K., *A formal power series approach to the construction of minimal fuzzy automata*, Information Sciences, 55 (1991), 189-207.

[31] Shamsizadeh, M., Zahedi, M.M., *Intuitionistic General Fuzzy Automata*, submitted.

[32] Tepavcevic, A., Gerstenkorn, T., *Lattice valued intuitionistic fuzzy sets*, Central European Journal of Mathematics, 2 (2004), 388-398.

[33] Topencharov, V., Peeva, K., *Equivalence, reduction and minimization of finite fuzzy automata*, Journal of Mathematical Analysis and Applications, 84 (1981), 270-281.

[34] Uma, A., Rajasekar, M., *Intuitionistic fuzzy automata and the minimal machine*, Engineering Mathematics Letters, (2014), 1-11.

[35] Wee, W.G., *On generalization of adaptive algorithm and application of the fuzzy sets concept to pattern classification*, Ph.D. Thesis, Purdue University, Lafayette, IN, 1967.

[36] Wee, W.G., Fu, K.S., *A formulation of fuzzy automata and its application as a model of learning systems*, IEEE Transactions on Systems, Man and Cybernetics, 5 (1969), 215-223.

[37] Wu, W., *Commutative implications on complete lattices*, International Journal of Uncertainty, Fuzziness and Knowledge-based System, 2 (1994), 333-341.

[38] Yang, L., Mo, CZ.W., *ascade and Wreath Products of Lattice-Valued Intuitionistic Fuzzy Finite State Machines and Coverings*, Fuzzy Information & Engineering and Operations Research & Management Advances in Intelligent Systems and Computing, 211 (2014), 97-106.

[39] Zadeh, L.A., *Fuzzy languages and their relation to human and machine intelligence*, Electrn. Research Laboratory University California, Berkeley, CA,Technical Report ERL-M 302, 1971.

[40] Zadeh, L.A., *Fuzzy sets*, Information and Control, 8 (1965,) 338-353.

[41] Zhang, X.W., Li, Y.M., *Intuitionistic fuzzy recognizers and intuitionistic fuzzy finite automata*, Soft Computing, 13 (2009), 611-616.

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