# EQUITABLE COLORINGS OF CARTESIAN PRODUCTS OF FANS WITH BIPARTITE GRAPHS[1]

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**Abstract.** In this paper, by the sorting method of vertices, it is obtained that the equitable chromatic number and the equitable chromatic threshold of the Cartesian products of fans with bipartite graphs.

**Keywords:** Cartesian product; equitable coloring; equitable chromatic number; equitable chromatic threshold.

**MR (2000) Subject Classification:** 05C70, 05C15.

## 2. Introduction

All graphs considered in this paper are finite, undirected, loopless and without multiple edges. For a positive integer $k$ and a real number $x$, let $[k] = \{1, 2, ..., k\}$, $\lceil x \rceil$ and $\lfloor x \rfloor$ denote the smallest integer not less than $x$ and the largest integer not greater than $x$, respectively.

A graph $G$ is said to be $k$-colorable if there is a map $c : V(G) \to [k]$ such that adjacent vertices are mapped to distinct numbers. The map $c$ is called a proper $k$-coloring of $G$, and all pre-images of a fixed number form a so-called color class. No two vertices are adjacent in each color class. The chromatic number of $G$, denoted by $\chi(G)$, is the smallest number $k$ such that $G$ is $k$-colorable.

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A graph $G = (V, E)$ is said to be equitably $k$-colorable if $V(G)$ can be divided into $k$ independent sets $V_1, V_2, ..., V_k$ such that $||V_i| - |V_j|| \leq 1$ for all $i, j \in [k]$. The smallest integer $k$ for which $G$ is equitably $k$-colorable is called equitable chromatic number of a graph $G$, and denoted by $\chi_=(G)$. The equitable chromatic threshold of a graph $G$, denoted by $\chi_=^*(G)$, is the minimum $t$ such that $G$ is equitably $k$-colorable for all $k \geq t$. It is evident from the definition that

$$\chi(G) \leq \chi_=(G) \leq \chi_=^*(G)$$

for any graph $G$.

Equitable coloring as a special vertex coloring on graphs was first introduced by Meyer[1]. His motivation came from the problem of assigning one of the six days of the work week to each garbage collection route. Here, the vertices represent garbage collection routes and two such vertices are joined by an edge when the corresponding routes should not be run on the same day. The problem of assigning one of the six days of the work week to each route becomes the problem of 6-coloring of $G$. On practical grounds it might also be desirable to have an approximately equal number of routes run on each of the six days, so we have to color the graph in the equitable way.

Another application of equitable coloring is in scheduling and timetabling. Consider, for example, a problem of constructing university timetables. It is known that this problem can be modeled as coloring the vertices of a graph $G$ whose nodes correspond to classes, edges correspond to time conflicts between classes, and colors to hours. If the set of available rooms is restricted, then we may be forced to partition the vertex set into independent subsets of as near equal size as possible, since then the room usage is the highest. For applications of equitable coloring such as scheduling and constructing timetables, please see [1], [5], [11], [12], [13].

In [3], by Lin and Chang, it is obtained that the exact values or upper bounds of the equitable chromatic number on Kronecker products of $G$ and $H$, when $G$ and $H$ are complete graphs, bipartite graphs, paths or cycles, and in [4], it is studied that the equitable colorings of Cartesian product of paths and cycles, respectively, with bipartite graphs. In [16], Lih and Wu studied the equitable colorings of bipartite graphs, and in [17], Lih gave a good survey for this coloring. In [23], Zhu gave a survey for Hedetniemi's conjecture about the tensor product of graphs. The general problem of deciding if $\chi_=(G) \leq 3$ is NP-complete [10]. If, however, $G$ has a regular or simplified structure we are sometimes able to provide a polynomial algorithm coloring it in the equitable way. For more details about this coloring, please see [1], [2], [6], [7], [8], [14], [20], [21], [22].

The Cartesian product of graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$ is the graph $G \square H$ with vertex set $\{(u, x) \, | u \in V_1, x \in V_2\}$ and edge set

$$\{(u, x) \; (v, y) \, | u = v \text{ } with \text{ } xy \in E_2 \text{ } or \text{ } x = y \text{ } with \text{ } uv \in E_1\}.$$

Graph products are interesting and useful in many situations. For example, Sabidussi [19] showed that any graph has the unique decomposition into prime factors under the Cartesian product. The complexity of many problems, also equitable coloring, that deal with very large and complicated graphs is reduced greatly if one is able to fully characterize the properties of less complicated prime factors.

In the present paper, we study the equitable colorings of Cartesian products of fans with complete bipartite graphs.

## 2. Main results

In the following, let $s, l, m, n, n'$ be all nonnegative integers, $F_{n'+1}$ represent the Fan with vertex set $V(F_{n'+1}) = \{x, x_1, x_2, \ldots, x_{n'}\}$, and $H$ represent a complete bipartite graph with two parts $Y = \{y_1, y_2, ..., y_m\}$ and $Z = \{z_1, z_2, ..., z_n\}$ where $m \geq n$. We will study the equitable chromatic number and the equitable chromatic threshold of the Cartesian product $F_{n'+1} \square H$ according to the parity of $n'$ and $m$.

On the other hand, if $n = 1$, then $H$ is a star and denoted by $K_{m,1}$. If $n'+1 = 2$ or $n'+1 = 3$, then $F_{n'+1}$ is a path or a cycle. In this paper, we always suppose that $n'+1 > 3$. Clearly, $\chi_=(F_{n'+1} \square H) \geq \chi(F_{n'+1}) = 3$.

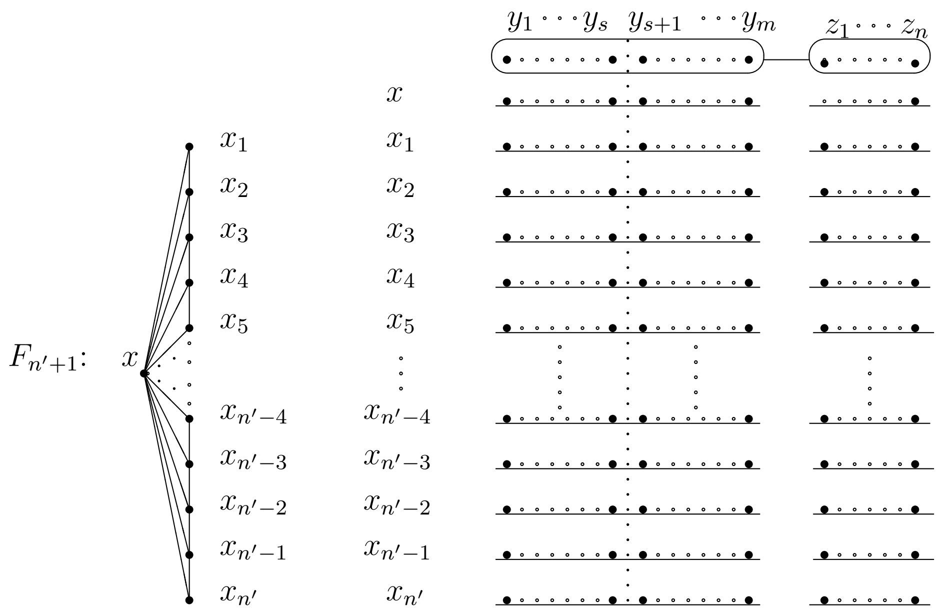


Figure 1. The Cartesian product $F_{n'+1} \square H$

**Theorem 2.1.** *Suppose that $m \geq n \geq 1$, $l \geq 2$ and $k \geq 4$. If $n' = 2l$, then $F_{n'+1} \square H$ is equitably $k$-colorable.*

**Proof.** The structure of the Cartesian product graph $F_{n'+1} \square H$ is represented in Figure 1.

For $m = 2p + 1$, $p \geq 0$ and $s = p + 1$ ( or for $m = 2p$, $p \geq 1$ and $s = p$), we will sort the vertices of $F_{n'+1} \Box H$ as following.

$(x, y_1), (x, y_2), \ldots, (x, y_s), (x_1, z_1), (x_3, z_1), \ldots, (x_{2h-1}, z_1), \ldots, (x_{n'-1}, z_1),$
$(x_1, z_2), (x_3, z_2), \ldots, (x_{2h-1}, z_2), \ldots, (x_{n'-1}, z_2), \ldots, (x_1, z_n), (x_3, z_n), \ldots,$
$(x_{2h-1}, z_n), \ldots, (x_{n'-1}, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), \ldots, (x_{2h}, y_{s+1}), \ldots, (x_{n'}, y_{s+1}),$
$(x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_{2h}, y_{s+2}), \ldots, (x'_n, y_{s+2}), \ldots, (x_2, y_m), (x_4, y_m), \ldots,$
$(x_{2h}, y_m), \ldots, (x_{n'}, y_m), (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s),$
$\ldots, (x_{2h}, y_1), (x_{2h}, y_2) \ldots, (x_{2h}, y_s), \ldots, (x_{n'}, y_1), (x_{n'}, y_2), \ldots, (x_{n'}, y_s),$
$(x, z_1), (x, z_2), \ldots, (x, z_n), (x_1, y_{s+1}), (x_3, y_{s+1}), \ldots,$
$(x_{2h-1}, y_{s+1}), \ldots, (x_{n'-1}, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots,$
$(x_{2h-1}, y_{s+2}), \ldots, (x_{n'-1}, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m), \ldots,$
$(x_{2h-1}, y_m), \ldots, (x_{n'-1}, y_m), (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s),$
$(x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), \ldots, (x_{2h-1}, y_1), (x_{2h-1}, y_2) \ldots, (x_{2h-1}, y_s),$
$\ldots, (x_{n'-1}, y_1), (x_{n'-1}, y_2), \ldots, (x_{n'-1}, y_s), (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n),$
$(x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), \ldots, (x_{2h}, z_1), (x_{2h}, z_2), \ldots, (x_{2h}, z_n), \ldots,$
$(x_{n'}, z_1), (x_{n'}, z_2), \ldots, (x_{n'}, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m),$

where $h$ is a positive integer and $1 \leq h \leq l$. It is not difficult to verify that the smallest cardinality of independent set consisting of consecutive vertices is at least $\min\{lp + nl + p + 1, 2pl + l + n\}$ for $m = 2p + 1$ and $\min\{2lp + n, nl + pl + p\}$ for $m = 2p$.

Let

$$\sigma_t = \left\lfloor \frac{(2l + 1)(m + n) + t - 1}{k} \right\rfloor,$$

where $t \in [k]$. By $l \geq 2$ and $k \geq 4$, we have

$$\begin{aligned} \sigma_1 &= \left\lfloor \frac{(2l + 1)(m + n)}{k} \right\rfloor \leq \sigma_k = \left\lfloor \frac{(2l + 1)(m + n) + k - 1}{k} \right\rfloor \\ &= \left\lceil \frac{(2l + 1)(m + n)}{k} \right\rceil \leq \left\lceil \frac{(2l + 1)(m + n)}{4} \right\rceil. \end{aligned}$$

If $m = 2p + 1$, then

$$\sigma_t \leq \left\lceil \frac{(2l + 1)(m + n)}{4} \right\rceil \leq \min\{lp + nl + p + 1, 2pl + l + n\}.$$

If $m = 2p$, then

$$\sigma_t \leq \left\lceil \frac{(2l + 1)(m + n)}{4} \right\rceil < \min\{2lp + n, nl + pl + p\}.$$

Hence, according to the vertex sorting above, the vertex set of $F_{n'+1} \Box H$ can be partitioned into $k$ independent sets with cardinality $\sigma_1, \sigma_2, \ldots, \sigma_k$, respectively. Therefore $F_{n'+1} \Box H$ is equitably $k$-colorable. ∎

**Theorem 2.2.** *Suppose that $m \geq n \geq 1$.*

(1) *For* $n' = 4$ *and* $2n - m \in \{0, \pm 1, \pm 2\}$*, or*

(2) *for* $n' = 2l$*,* $l \geq 3$ *and* $m = 2n$*,*

*we have that* $F_{n'+1} \square H$ *is equitably* 3*-colorable, and then* $\chi^*_{=}(F_{n'+1} \square H) = 3$*.*

**Proof.** (1) Assume that $s$ is a nonnegative integer, then we can partition the vertex set $V(F_{n'+1} \square H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x, y_1), (x, y_2), \ldots, (x, y_s), (x_1, z_1), (x_3, z_1), (x_1, z_2), (x_3, z_2), \ldots, (x_1, z_n), \\ (x_3, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), (x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_2, y_m), (x_4, y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s), (x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), (x_2, z_1), (x_2, z_2), \\ \ldots, (x_2, z_n), (x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s), (x, z_1), (x, z_2), \ldots, \\ (x, z_n), (x_1, y_{s+1}), (x_3, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m) \end{array} \right\},$$

where $|V_1| = 2n + 2m - s$, $|V_2| = 2n + m + s$, and $|V_3| = n + 2m$.

If $m = 2n + 1$ and $s = n + 1$, then $|V_1| = 2n + 2m - s = 5n + 1$, $|V_2| = 2n + m + s = 5n + 2$, and $|V_3| = n + 2m = 5n + 2$.If $m = 2n - 1$ and $s = n$, then $|V_1| = 5n - 2$, $|V_2| = 5n - 1$, and $|V_3| = 5n - 2$.

If $m = 2n$ and $s = n$, then $|V_1| = 2n + 2m - s = 5n$, $|V_2| = 2n + m + s = 5n$, and $|V_3| = n + 2m = 5n$.

If $m = 2n + 2$ and $s = n + 1$, then $|V_1| = 2n + 2m - s = 5n + 3$, $|V_2| = 2n + m + s = 5n + 3$, and $|V_3| = n + 2m = 5n + 4$.

If $m = 2n - 2$ and $s = n - 1$, then $|V_1| = 2n + 2m - s = 5n - 3$, $|V_2| = 2n + m + s = 5n - 3$, and $|V_3| = n + 2m = 5n - 4$.

Therefore, $F_{n'+1} \square H$ is equitably 3-colorable. By Theorem 2.1, we have $\chi^*_{=}(F_{n'+1} \square H) = 3$.

(2) Assume that $s = n$ and $1 \leq h \leq l$, then we will partition the vertex set $V(F_{n'+1} \square H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x, y_1), (x, y_2), \ldots, (x, y_s), (x_1, z_1), (x_3, z_1), \ldots, (x_{2h-1}, z_1), \ldots, \\ (x_{n'-1}, z_1), (x_1, z_2), (x_3, z_2), \ldots, (x_{2h-1}, z_2), \ldots, (x_{n'-1}, z_2), \ldots, \\ (x_1, z_n), (x_3, z_n), \ldots, (x_{2h-1}, z_n), \ldots, (x_{n'-1}, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), \\ \ldots, (x_{2h}, y_{s+1}), \ldots, (x_{n'}, y_{s+1}), (x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_{2h}, y_{s+2}), \\ \ldots, (x_{n'}, y_{s+2}), \ldots, (x_2, y_m), (x_4, y_m), \ldots, (x_{2h}, y_m), \ldots, (x_{n'}, y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s), (x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), \\ \ldots, (x_{2h-1}, y_1), (x_{2h-1}, y_2), \ldots, (x_{2h-1}, y_s), \ldots, (x_{n'-1}, y_1), \\ (x_{n'-1}, y_2) \ldots, (x_{n'-1}, y_s), (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n), \\ (x_4, z_1), (x_4, z_2), \ldots, (x_4, z_n), \ldots, (x_{2h}, z_1), (x_{2h}, z_2), \ldots, (x_{2h}, z_n), \\ \ldots, (x_{n'}, z_1), (x_{n'}, z_2), \ldots, (x_{n'}, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s), \ldots, (x_{2h}, y_1), \\ (x_{2h}, y_2), \ldots, (x_{2h}, y_s), \ldots, (x_{n'}, y_1), (x_{n'}, y_2), \ldots, (x_{n'}, y_s), (x, z_1), \\ (x, z_2), \ldots, (x, z_n), (x_1, y_{s+1}), (x_3, y_{s+1}), \ldots, (x_{2h-1}, y_{s+1}), \\ \ldots, (x_{n'-1}, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots, (x_{2h-1}, y_{s+2}), \ldots, \\ (x_{n'-1}, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m), \ldots, (x_{2h-1}, y_m), \ldots, (x_{n'-1}, y_m) \end{array} \right\}.$$

It is easy to see that $|V_1| = (2l+1)n$, $|V_2| = (2l+1)n$, and $|V_3| = (2l+1)n$. Hence $F_{n'+1} \Box H$ is equitably 3-colorable. By Theorem 2.1, we have $\chi^*_=(F_{n'+1} \Box H) = 3$. ∎

**Theorem 2.3.** *Suppose that $m \geq n \geq 1$, $l \geq 1$ and $k \geq 4$. If $n' = 2l+1$, then $F_{n'+1} \Box H$ is equitably $k$-colorable.*

*Proof.* For $m = 2p+1$, $p \geq 0$ and $s = p+1$ (or $m = 2p$, $p \geq 1$ and $s = p$), we sort the vertex set of $F_{n'+1} \Box H$ as following:

$$\begin{array}{l} (x, y_1), (x, y_2), \ldots, (x, y_s), (x_2, z_1), (x_4, z_1), \ldots, (x_{2h+2}, z_1), \ldots, (x_{n'-1}, z_1), \\ (x_2, z_2), (x_4, z_2), \ldots, (x_{2h+2}, z_2), \ldots, (x_{n'-1}, z_2), \ldots, (x_2, z_n), (x_4, z_n), \\ \ldots, (x_{2h+2}, z_n), \ldots, (x_{n'-1}, z_n), (x_1, y_{s+1}), (x_3, y_{s+1}), \ldots, (x_{2h'+1}, y_{s+1}), \\ \ldots, (x_{n'}, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots, (x_{2h'+1}, y_{s+2}), \ldots, \\ (x_{n'}, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m), \ldots, (x_{2h'+1}, y_m), \ldots, (x_{n'}, y_m), \\ (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s), (x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), \ldots, \\ (x_{2h'+1}, y_1), (x_{2h'+1}, y_2) \ldots, (x_{2h'+1}, y_s), \ldots, (x_{n'}, y_1), (x_{n'}, y_2), \\ \ldots, (x_{n'}, y_s), (x, z_1), (x, z_2), \ldots, (x, z_n), (x_2, y_{s+1}), (x_4, y_{s+1}), \ldots, \\ (x_{2h+2}, y_{s+1}), \ldots, (x_{n'-1}, y_{s+1}), (x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_{2h+2}, y_{s+2}), \\ \ldots, (x_{n'-1}, y_{s+2}), \ldots, (x_2, y_m), (x_4, y_m), \ldots, (x_{2h+2}, y_m), \ldots, (x_{n'-1}, y_m), \\ (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s), \ldots, (x_{2h+2}, y_1), \\ (x_{2h+2}, y_2) \ldots, (x_{2h+2}, y_s), \ldots, (x_{n'-1}, y_1), (x_{n'-1}, y_2) \ldots, (x_{n'-1}, y_s), \\ (x_1, z_1), (x_1, z_2), \ldots, (x_1, z_n), (x_3, z_1), (x_3, z_2), \ldots, (x_3, z_n), \ldots, \\ (x_{2h'+1}, z_1), (x_{2h'+1}, z_2), \ldots, (x_{2h'+1}, z_n), \ldots, (x_{n'}, z_1), (x_{n'}, z_2), \ldots, \\ (x_{n'}, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m), \end{array}$$

where $h, h'$ are all nonnegative integers, $0 \leq h \leq l-1$, and $0 \leq h' \leq l$. It is obvious that the smallest cardinality of independent set consisting of consecutive vertices in the order above is at least

$$\min\{lp + nl + 2p + 1, 2lp + l + n, pl + nl + p + l + n\}$$

when $m = 2p+1$ and

$$\min\{2lp + n, nl + p(l+2), (l+1)p + nl + n\}$$

when $m = 2p$.

Let

$$\sigma_t = \left\lfloor \frac{(2l+2)(m+n) + t - 1}{k} \right\rfloor,$$

where $t \in [k]$. By $l \geq 1$ and $k \geq 4$, we can obtain that

$$\sigma_1 = \left\lfloor \frac{(2l+2)(m+n)}{k} \right\rfloor$$

$$\leq \sigma_k = \left\lfloor \frac{(2l+2)(m+n)+k-1}{k} \right\rfloor = \left\lceil \frac{(2l+2)(m+n)}{k} \right\rceil \leq \left\lceil \frac{(2l+2)(m+n)}{4} \right\rceil .$$

If $m = 2p+1$, then

$$\sigma_t \leq \left\lceil \frac{(2l+2)(m+n)}{4} \right\rceil \leq \min\{lp+nl+2p+1, 2lp+l+n, pl+nl+p+l+n\}.$$

If $m = 2p$, then

$$\sigma_t \leq \left\lceil \frac{(2l+2)(m+n)}{4} \right\rceil \leq \min\{2lp+n, nl+p(l+2), (l+1)p+nl+n\}.$$

Therefore, according to the vertex ordering above, the vertex set of $F_{n'+1}\square H$ can be partitioned into $k$ independent sets with cardinality $\sigma_1, \sigma_2, \ldots, \sigma_k$, respectively. Hence $F_{n'+1}\square H$ is equitably $k$-colorable for $k \geq 4$. $\square$

**Theorem 2.4.** *Suppose that $m \geq n \geq 1$.*
*(1) If $n' = 3$, and $2n - m \in \{0, \pm 1, \pm 2\}$, or*
*(2) if $l \geq 2$, $n' = 2l+1$ and $m = 2n$,*
*then $F_{n'+1}\square H$ is equitably 3-colorable, and $\chi^*_=(F_{n'+1}\square H) = 3$.*

**Proof.** (1) Assume that $s$ is a nonnegative integer, then we can partition the vertex set $V(F_{n'+1}\square H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x, y_1), (x, y_2), \ldots, (x, y_s), (x_2, z_1), (x_2, z_2), \ldots, (x_2, z_n), \\ (x_1, y_{s+1}), (x_3, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_1, z_1), (x_1, z_2), \ldots, (x_1, z_n), \\ (x_3, z_1), (x_3, z_2), \ldots, (x_3, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_1, y_1), (x_1, y_2) \ldots, (x_1, y_s), (x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), \\ (x, z_1), (x, z_2), \ldots, (x, z_n), (x_2, y_{s+1}), (x_2, y_{s+2}), \ldots, (x_2, y_m) \end{array} \right\},$$

where $|V_1| = n + 2m - s$, $|V_2| = 2n + m$, and $|V_3| = n + m + s$.

If $m = 2n+1$ and $s = n+1$, then $|V_1| = n+2m-s = 4n+1$, $|V_2| = 2n+m = 4n+1$, and $|V_3| = n+m+s = 4n+2$. If $m = 2n-1$ and $s = n$, then $|V_1| = 4n-2$, $|V_2| = 4n-1$, and $|V_3| = 4n-1$.

If $m = 2n$ and $s = n$, then $|V_1| = n + 2m - s = 4n$, $|V_2| = 2n + m = 4n$, and $|V_3| = n + m + s = 4n$. If $m = 2n - 2$ and $s = n - 1$, then $|V_1| = 4n - 3$, $|V_2| = 4n - 2$, and $|V_3| = 4n - 3$. If $m = 2n + 2$ and $s = n + 1$, then $|V_1| = 4n + 3$, $|V_2| = 4n + 2$, and $|V_3| = 4n + 3$.

Hence $F_{n'+1}\Box H$ is equitably 3-colorable. Applying Theorem 2.3, we have $\chi^*_{=}(F_{n'+1}\Box H) = 3$.

(2) Assume that $s = n$, $0 \le h \le l-1$, and $0 \le h' \le l$. Then we can partition the vertex set $V(F_{n'+1}\Box H)$ into the following three parts:

$$V_1 = \left\{ \begin{array}{l} (x, y_1), (x, y_2), \ldots, (x, y_s), (x_2, z_1), (x_4, z_1), \ldots, (x_{2h+2}, z_1), \ldots, \\ (x_{n'-1}, z_1), (x_2, z_2), (x_4, z_2), \ldots, (x_{2h+2}, z_2), \ldots, (x_{n'-1}, z_2), \ldots, \\ (x_2, z_n), (x_4, z_n), \ldots, (x_{2h+2}, z_n), \ldots, (x_{n'-1}, z_n), (x_1, y_{s+1}), \\ (x_3, y_{s+1}), \ldots, (x_{2h'+1}, y_{s+1}), \ldots, (x_{n'}, y_{s+1}), (x_1, y_{s+2}), (x_3, y_{s+2}), \\ \ldots, (x_{2h'+1}, y_{s+2}), \ldots, (x_{n'}, y_{s+2}), \ldots, (x_1, y_m), (x_3, y_m), \\ \ldots, (x_{2h'+1}, y_m), \ldots, (x_{n'}, y_m) \end{array} \right\},$$

$$V_2 = \left\{ \begin{array}{l} (x_2, y_1), (x_2, y_2) \ldots, (x_2, y_s), (x_4, y_1), (x_4, y_2) \ldots, (x_4, y_s), \ldots, \\ (x_{2h+2}, y_1), (x_{2h+2}, y_2) \ldots, (x_{2h+2}, y_s), \ldots, (x_{n'-1}, y_1), (x_{n'-1}, y_2), \\ \ldots, (x_{n'-1}, y_s), (x_1, z_1), (x_1, z_2), \ldots, (x_1, z_n), (x_3, z_1), (x_3, z_2), \\ \ldots, (x_3, z_n), \ldots, (x_{2h'+1}, z_1), (x_{2h'+1}, z_2), \ldots, (x_{2h'+1}, z_n), \ldots, \\ (x_{n'}, z_1), (x_{n'}, z_2), \ldots, (x_{n'}, z_n), (x, y_{s+1}), (x, y_{s+2}), \ldots, (x, y_m) \end{array} \right\},$$

and

$$V_3 = \left\{ \begin{array}{l} (x_1, y_1), (x_1, y_2), \ldots, (x_1, y_s), (x_3, y_1), (x_3, y_2) \ldots, (x_3, y_s), \\ \ldots, (x_{2h'+1}, y_1), (x_{2h'+1}, y_2), \ldots, (x_{2h'+1}, y_s), \ldots, \\ (x_{n'}, y_1), (x_{n'}, y_2), \ldots, (x_{n'}, y_s), (x, z_1), (x, z_2), \ldots, (x, z_n), \\ (x_2, y_{s+1}), (x_4, y_{s+1}), \ldots, (x_{2h+2}, y_{s+1}), \ldots, (x_{n'-1}, y_{s+1}), \\ (x_2, y_{s+2}), (x_4, y_{s+2}), \ldots, (x_{2h+2}, y_{s+2}), \ldots, (x_{n'-1}, y_{s+2}), \\ \ldots, (x_2, y_m), (x_4, y_m), \ldots, (x_{2h+2}, y_m), \ldots, (x_{n'-1}, y_m) \end{array} \right\}.$$

It is obvious that

$$|V_1| = |V_2| = |V_3| = (2l+2)n.$$

Hence $F_{n'+1}\Box H$ is equitably 3-colorable. By Theorem 2.3, we have

$$\chi^*_{=}(F_{n'+1}\Box H)=3. \qquad \blacksquare$$

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