SOME MEAN INEQUALITIES FOR POSITIVE LINEAR MAPS

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Abstract. If \( m_1^2 \leq A \leq M_1^2 A \) and \( m_2^2 \leq B \leq M_2^2 \) for some positive real numbers \( m_1 \leq M_1 \), \( m_2 \leq M_2 \) and \( \sigma \) is a mean between geometric and arithmetic means, then for positive unital linear map \( \Phi \)

\[
\Phi \left( \frac{M_2m_2}{M_1m_1} A \right) \sigma \Phi(B) \leq \frac{M_2m_2}{2M_1m_1} \Phi(A\sigma B),
\]

\[
\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sigma B) - \Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B) \leq \left( \sqrt{\frac{M_2^2}{m_1}} - \sqrt{\frac{m_2^2}{M_1}} \right)^2,
\]

\[
\left( \frac{1}{M_1^2m_1^2} \Phi(A) \right) \sigma \Phi(A^{-1}) \leq \frac{M_1^2 + m_1^2}{2M_1^2m_1^2} \Phi(A\sigma A^{-1}).
\]

Keywords: \( \sigma \) mean; geometric mean; arithmetic mean; positive linear maps.

1. Introduction

We use the same notation as in [4]. The operator norm is denoted by \( \| \cdot \| \). Let \( M, m \) be scalars and \( I \) be the identity operator. Other capital letters are used to denote the general elements of the \( C^* \)-algebra \( \mathcal{B}(\mathcal{H}) \) of all bounded linear operators acting on a Hilbert space \( (\mathcal{H}, \langle \cdot, \cdot \rangle) \). We write \( A \geq 0 \) to mean that the operator \( A \) is positive. If \( A - B \geq 0 \) \( (A - B \leq 0) \), then we write \( A \geq B \) \( (A \leq B) \).

A linear map \( \Phi \) is positive if \( \Phi(A) \geq 0 \) whenever \( A \geq 0 \). It is said to be unital if \( \Phi(I) = I \).

\( \sigma \) is an arbitrary means between geometric and arithmetic means. A connection \( \sigma \) is a binary operation \( A\sigma B \), which is monotone and continuous from above in each variable and satisfies the transformer inequality, in the sense

\[
C(A\sigma B)C \leq (CAC)\sigmaCBC.
\]
Though Bhagwat and Subramanian [3] introduced power means, for instance, \( \left\{ \frac{1}{p}(A^p + B^p) \right\}^{1/p} \), these means are not monotone in general, nor satisfy the transformer inequality.

A connection \( \sigma \) is a mean if it is normalized, i.e., \( 1\sigma 1 = 1 \) where 1 denotes the identity operator. Such an axiomatic approach is already found in the paper of Nishio and Ando [1], in which parallel addition is given an axiomatic characterization. In addition, there exists an affine order isomorphism between the class of operator means and the class of positive operator monotone functions \( f \) defined on \( (0, \infty) \) with \( f(1) = 1 \) via \( f(t)I = 1\sigma(tI) \) \( (t > 0) \). \( A\sigma B = A^{1/2}f(A^{-1/2}BA^{-1/2})A^{1/2} \) for all strictly positive operators \( A, B \). The operator monotone function \( f \) is called the representing function of \( \sigma \).

For \( A, B > 0 \), the geometric mean \( A\# B \) and arithmetic \( A\nabla B \) mean are defined \( A\# B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2} \) and \( A\nabla B = \frac{A + B}{2} \), respectively.

It is well-known that the arithmetic mean \( \nabla \) is the biggest among symmetric means (see [2]). In this paper, we suppose that \( \# \leq \sigma \leq \nabla \).

Let \( m_1^2 \leq A \leq M_1^2 \) and \( m_2^2 \leq B \leq M_2^2 \) for positive real numbers \( m_1 < M_1, \ m_2 < M_2 \) and \( \sigma \) be a mean between geometric and arithmetic means. Moslehian et al. [5] proved the following inequalities for every positive unital linear map \( \Phi \):

- **Operator Pólya-Szegö inequality:**
  \[
  \Phi(A)\#\Phi(B) \leq \frac{1}{2} \left( \sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}} \right) \Phi(A\#B);
  \]

- **Operator Shisha-Mond inequality:**
  \[
  \Phi^{-\frac{1}{2}}(A\#B)\Phi(B)\Phi^{-\frac{1}{2}}(A\#B) - \Phi^{\frac{1}{2}}(A\#B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\#B) \\
  \leq \left( \sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}} \right)^2;
  \]

- **Operator Kantorovich inequality:**
  \[
  \Phi(A)\#\Phi(A^{-1}) \leq \frac{M_1^2 + m_1^2}{2M_1m_1}.
  \]

In this paper, we will present some inequalities for the matrix mean \( \sigma \) between geometric and arithmetic means which are generalizations of (1.1)~(1.3).

### 2. Main results

Next, we give the main theorem.

**Theorem 1** Let \( A, B \in B(\mathcal{H}) \) be positive invertible operators and \( \Phi : B(\mathcal{H}) \to B(\mathcal{H}) \) be a positive unital linear map. If \( m_1^2 \leq A \leq M_1^2 A \) and \( m_2^2 \leq B \leq M_2^2 \) for some positive real numbers \( m_1 \leq M_1, \ m_2 \leq M_2 \) and \( \sigma \) is a mean between geometric and arithmetic means, then
\[ \Phi\left(\frac{M_2m_2}{M_1m_1} A\right) \sigma \Phi(B) \leq \frac{M_2}{m_1} + \frac{m_2}{M_1} \Phi(A\sigma B), \]

\[ \Phi^{-\frac{1}{2}}(A\sigma B) \Phi(B) \Phi^{-\frac{1}{2}}(A\sigma B) - \Phi^{\frac{1}{2}}(A\sigma B) \Phi^{-\frac{1}{2}}(A) \Phi^{\frac{1}{2}}(A\sigma B) \]

\[ \leq \left( \sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}} \right)^2, \]

\[ \left( \frac{1}{M_1^2m_1^2} \Phi(A) \right) \sigma \Phi(A^{-1}) \leq \frac{M_2^2 + m_2^2}{2M_1^2m_1^2} \Phi(A \sigma A^{-1}). \]

**Proof**: By computation, we have

\[ \left( \frac{M_2}{m_1} - (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} \right) \left( (A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}} - \frac{m_2}{M_1} \right) \geq 0, \]

whence

\[ \frac{M_2m_2}{M_1m_1} A + B \leq \left( \frac{M_2}{m_1} + \frac{m_2}{M_1} \right) A^\circ B. \]

Since \( \Phi \) is a positive linear map, (2.4) becomes the inequality as follows

\[ \frac{M_2m_2}{M_1m_1} \Phi(A) + \Phi(B) \leq \left( \frac{M_2}{m_1} + \frac{m_2}{M_1} \right) \Phi(A^\circ B). \]

By \( \nabla \geq \sigma \geq \# \),

\[ \left( \frac{M_2m_2}{M_1m_1} \Phi(A) \right) \sigma \Phi(B) \leq \left( \frac{M_2m_2}{M_1m_1} \Phi(A) \right) \nabla \Phi(B) \leq \frac{M_2}{m_1} + \frac{m_2}{M_1} \Phi(A^\circ B) \leq \frac{M_2}{m_1} + \frac{m_2}{M_1} \Phi(A \sigma B). \]

By (2.5), we have

\[ \left( \frac{M_2m_2}{M_1m_1} \Phi(A) \right) \sigma \Phi(B) \leq \frac{M_2}{m_1} + \frac{m_2}{M_1} \Phi(A \sigma B). \]

Thus (2.1) holds. It follows from (2.6) that

\[ \Phi^{-\frac{1}{2}}(A\sigma B) \Phi(B) \Phi^{-\frac{1}{2}}(A\sigma B) - \Phi^{\frac{1}{2}}(A\sigma B) \Phi^{-\frac{1}{2}}(A) \Phi^{\frac{1}{2}}(A\sigma B) \]

\[ \leq \left( \frac{M_2}{m_1} + \frac{m_2}{M_1} \right) - \frac{M_2m_2}{M_1m_1} \Phi^{-\frac{1}{2}}(A\sigma B) \Phi(A) \Phi^{-\frac{1}{2}}(A\sigma B) - \Phi^{\frac{1}{2}}(A\sigma B) \Phi^{-\frac{1}{2}}(A) \Phi^{\frac{1}{2}}(A\sigma B) \]

\[ \leq \left( \sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}} \right)^2, \]

\[ \left( \frac{M_2}{m_1} - \frac{m_2}{M_1} \right)^2, \]
which means that
\[
\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sigma B) - \Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B) \leq \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2.
\]
So (2.2) holds.

Suppose \(m_2^2 = \frac{1}{M_1^2} \leq B = A^{-1} \leq \frac{1}{m_1^2} = M_2^2\) in (2.6). Thus, (2.6) becomes
\[
\left(\frac{1}{M_1^2m_1} \Phi(A)\right) \sigma \Phi(A^{-1}) \leq \frac{M_1^2 + m_1^2}{2M_1^2m_1} \Phi(A\sigma A^{-1}),
\]
which is (2.3).

**Remark 2** When \(\sigma = \sharp\), (2.1)–(2.3) are the operator Pólya-Szegö inequality (1.1), the operator Shisha-Mond inequality (1.2) and the operator Kantorovich inequality (1.3), respectively. Thus (2.1)–(2.3) are the generalizations of inequalities (1.1)–(1.3).

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**References**


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