SOME MEAN INEQUALITIES FOR POSITIVE LINEAR MAPS

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Abstract. If $m_1^2 \leq A \leq M_1^2 A$ and $m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$, $m_2 \leq M_2$ and σ is a mean between geometric and arithmetic means, then for positive unital linear map Φ

$$\begin{aligned} \Phi\left(\frac{M_2 m_2}{M_1 m_1} A\right)\sigma\Phi(B) &\leq \frac{\frac{M_2}{m_1}+\frac{m_2}{M_1}}{2}\Phi(A\sigma B), \\ \Phi^{-\frac{1}{2}}(A\sigma B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sigma B)-\Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B) &\leq \left(\sqrt{\frac{M_2}{m_1}}-\sqrt{\frac{m_2}{M_1}}\right)^2, \\ \left(\frac{1}{M_1^2 m_1^2}\Phi(A)\right)\sigma\Phi(A^{-1}) &\leq \frac{M_1^2+m_1^2}{2M_1^2 m_1^2}\Phi(A\sigma A^{-1}). \end{aligned}$$

Keywords: σ mean; geometric mean; arithmetic mean; positive linear maps.

1. Introduction

We use the same notation as in [4]. The operator norm is denoted by $\|\cdot\|$. Let M, m be scalars and I be the identity operator. Other capital letters are used to denote the general elements of the C^* -algebra $\mathcal{B}(\mathcal{H})$ of all bounded linear operators acting on a Hilbert space $(\mathcal{H}, \langle\cdot,\cdot\rangle)$. We write $A \geq 0$ to mean that the operator A is positive. If $A - B \geq 0$ $(A - B \leq 0)$, then we write $A \geq B$ $(A \leq B)$. A linear map Φ is positive if $\Phi(A) \geq 0$ whenever $A \geq 0$. It is said to be unital if $\Phi(I) = I$.

σ is an arbitrary means between geometric and arithmetic means. A connection σ is a binary operation $A\sigma B$, which is monotone and continuous from above in each variable and satisfies the transformer inequality, in the sense

$$C(A\sigma B)C \leqq (CAC)\sigma(CBC).$$

Though Bhagwat and Subramanian [3] introduced power means, for instance, $\{\frac{1}{2}(A^p + B^p)\}^{1/p}$, these means are not monotone in general, nor satisfy the transformer inequality.

A connection σ is a mean if it is normalized, i.e., $1\sigma 1 = 1$ where 1 denotes the identity operator. Such an axiomatic approach is already found in the paper of Nishio and Ando [1], in which parallel addition is given an axiomatic characterization. In addition, there exists an affine order isomorphism between the class of operator means and the class of positive operator monotone functions f defined on $(0, \infty)$ with $f(1) = 1$ via $f(t)I = I\sigma(tI)$ $(t > 0)$. $A\sigma B = A^{\frac{1}{2}} f(A^{-\frac{1}{2}} B A^{-\frac{1}{2}}) A^{\frac{1}{2}}$ for all strictly positive operators A, B. The operator monotone function f is called the representing function of σ.

For $A, B > 0$, the geometric mean $A\sharp B$ and arithmetic $A\nabla B$ mean are defined $A\sharp B = A^{1/2}(A^{-1/2}BA^{-1/2})^{1/2}A^{1/2}$ and $A\nabla B = \dfrac{A+B}{2}$, respectively.

It is well-known that the arithmetic mean ∇ is the biggest among symmetric means (see [2]). In this paper, we suppose that $\sharp \leq \sigma \leq \nabla$.

Let $m_1^2 \leq A \leq M_1^2$ and $m_2^2 \leq B \leq M_2^2$ for positive real numbers $m_1 < M_1$, $m_2 < M_2$ and σ be a mean between geometric and arithmetic means. Moslehian et al. [5] proved the following inequalities for every positive unital linear map Φ:

• Operator Pólya-Szegö inequality:

$$\Phi(A)\sharp\Phi(B) \leq \frac{1}{2}\left(\sqrt{\frac{M_1M_2}{m_1m_2}} + \sqrt{\frac{m_1m_2}{M_1M_2}}\right)\Phi(A\sharp B); \tag{1.1}$$

• Operator Shisha-Mond inequality:

$$\Phi^{-\frac{1}{2}}(A\sharp B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sharp B) - \Phi^{\frac{1}{2}}(A\sharp B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sharp B) \leq \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2; \tag{1.2}$$

• Operator Kantorovich inequality:

$$\Phi(A)\sharp\Phi(A^{-1}) \leq \frac{M_1^2 + m_1^2}{2M_1m_1}. \tag{1.3}$$

In this paper, we will present some inequalities for the matrix mean σ between geometric and arithmetic means which are generalizations of (1.1)~(1.3).

2. Main results

Next, we give the main theorem.

Theorem 1 *Let $A, B \in \mathcal{B}(\mathcal{H})$ be positive invertible operators and $\Phi : \mathcal{B}(\mathcal{H}) \to \mathcal{B}(\mathcal{H})$ be a positive unital linear map. If $m_1^2 \leq A \leq M_1^2 A$ and $m_2^2 \leq B \leq M_2^2$ for some positive real numbers $m_1 \leq M_1$, $m_2 \leq M_2$ and σ is a mean between geometric and arithmetic means, then*

$$\Phi\left(\frac{M_2 m_2}{M_1 m_1}A\right)\sigma\Phi(B) \leq \frac{\frac{M_2}{m_1}+\frac{m_2}{M_1}}{2}\Phi(A\sigma B), \tag{2.1}$$

$$\begin{aligned}&\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sigma B)-\Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B)\\ &\qquad\leq \left(\sqrt{\frac{M_2}{m_1}}-\sqrt{\frac{m_2}{M_1}}\right)^2,\end{aligned} \tag{2.2}$$

$$\left(\frac{1}{M_1^2 m_1^2}\Phi(A)\right)\sigma\Phi(A^{-1}) \leq \frac{M_1^2+m_1^2}{2M_1^2 m_1^2}\Phi(A\sigma A^{-1}). \tag{2.3}$$

Proof : By computation, we have

$$\left(\frac{M_2}{m_1}-(A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}\right)\left((A^{-\frac{1}{2}}BA^{-\frac{1}{2}})^{\frac{1}{2}}-\frac{m_2}{M_1}\right) \geq 0,$$

whence

$$\frac{M_2 m_2}{M_1 m_1}A+B \leq \left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)A\sharp B. \tag{2.4}$$

Since Φ is a positive linear map, (2.4) becomes the inequality as follows

$$\frac{M_2 m_2}{M_1 m_1}\Phi(A)+\Phi(B) \leq \left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)\Phi(A\sharp B).$$

By $\nabla \geq \sigma \geq \sharp$,

$$\begin{aligned}\left(\frac{M_2 m_2}{M_1 m_1}\Phi(A)\right)\sigma\Phi(B) &\leq \left(\frac{M_2 m_2}{M_1 m_1}\Phi(A)\right)\nabla\Phi(B)\\ &\leq \frac{\frac{M_2}{m_1}+\frac{m_2}{M_1}}{2}\Phi(A\sharp B) \leq \frac{\frac{M_2}{m_1}+\frac{m_2}{M_1}}{2}\Phi(A\sigma B).\end{aligned} \tag{2.5}$$

By (2.5), we have

$$\left(\frac{M_2 m_2}{M_1 m_1}\Phi(A)\right)\sigma\Phi(B) \leq \frac{\frac{M_2}{m_1}+\frac{m_2}{M_1}}{2}\Phi(A\sigma B). \tag{2.6}$$

Thus (2.1) holds. It follows from (2.6) that

$$\begin{aligned}&\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sigma B)-\Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B)\\ &\leq \left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)-\frac{M_2 m_2}{M_1 m_1}\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(A)\Phi^{-\frac{1}{2}}(A\sigma B)-\Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B)\\ &\leq \left(\frac{M_2}{m_1}+\frac{m_2}{M_1}\right)-2\sqrt{\frac{M_2 m_2}{M_1 m_1}}\\ &-\left(\sqrt{\frac{M_2 m_2}{M_1 m_1}}(\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(A)\Phi^{-\frac{1}{2}}(A\sigma B))^{\frac{1}{2}}-(\Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B))^{\frac{1}{2}}\right)^2\\ &\leq \left(\sqrt{\frac{M_2}{m_1}}-\sqrt{\frac{m_2}{M_1}}\right)^2,\end{aligned}$$

which means that

$$\Phi^{-\frac{1}{2}}(A\sigma B)\Phi(B)\Phi^{-\frac{1}{2}}(A\sigma B) - \Phi^{\frac{1}{2}}(A\sigma B)\Phi^{-1}(A)\Phi^{\frac{1}{2}}(A\sigma B) \leq \left(\sqrt{\frac{M_2}{m_1}} - \sqrt{\frac{m_2}{M_1}}\right)^2.$$

So (2.2) holds.

Suppose $m_2^2 = \dfrac{1}{M_1^2} \leq B = A^{-1} \leq \dfrac{1}{m_1^2} = M_2^2$ in (2.6). Thus, (2.6) becomes

$$\left(\frac{1}{M_1^2 m_1^2}\Phi(A)\right)\sigma\Phi(A^{-1}) \quad \leq \quad \frac{M_1^2 + m_1^2}{2M_1^2 m_1^2}\Phi(A\sigma A^{-1}),$$

which is (2.3).

Remark 2 When $\sigma = \sharp$, (2.1)–(2.3) are the operator Pólya-Szegö inequality (1.1), the operator Shisha-Mond inequality (1.2) and the operator Kantorovich inequality (1.3), respectively. Thus (2.1)–(2.3) are the generalizations of inequalities (1.1)–(1.3).

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