A COUPLED FIXED POINT THEOREM AND t-NORM OF HADŽÍC TYPE IN FUZZY METRIC SPACE

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Abstract. In this paper, we introduce a coupled fixed point theorem and t-Norm of Hadžíc type in fuzzy metric space and we also give an application of our result.

Keywords: coupled fixed points, fuzzy metric space, occasionally weakly compatible maps.
2010 Mathematics Subject Classification: 47H10, 54H25.

1. Introduction

Zadeh's [10] investigation of the notion of fuzzy set has led to a rich growth of fuzzy Mathematics. Many authors have introduced the concept of fuzzy metric in different ways. There are many view points of the notion of the fuzzy metric spaces in fuzzy topology (see Deng [1], Erceg [2], George and Veeramani [3], Kaleva and Seikkala [5], Kramosil and Michalek [6]). In 2009, Lakshmikantham et.al. [7] introduced coupled fixed point theorems in partially ordered metric spaces. Motivated by them, we introduce a coupled fixed point theorem and t-Norm of Hadžíc type in fuzzy metric space.

2. Preliminaries

Definition 2.1. A t-norm t is a 2-place function $t : [0,1] \times [0,1] \to [0,1]$ which satisfies the following conditions:

(i) $t(0,0) = 0$;
(ii) $t(0,1) = 1$;
(iii) $t(a,b) = t(b,a)$;
(iv) if $a \leq c$ and $b \leq d$, then $t(a,b) \leq t(c,d)$;
(v) $t(t(a,b),c) = t(a,t(b,c))$,

for all $a,b,c,d \in [0,1]$.

Definition 2.2. Let $\sup_{0<t<1} \Delta(t,t) = 1$. A t-norm Δ is said to be of Hadžíc-type (H-type) if the family of functions $\left\{\Delta^m(t)\right\}_{m=1}^{\infty}$ is equi-continuous at $t = 1$, where

$$\Delta^1(t) = t,\ \Delta^{m+1}(t) = \Delta(\Delta^m(t)),\ m = 1,2,3,\ldots,\ t \in [0,1].$$

The t-norm $\Delta_M = \min$ is an example of H-type.

Remark 2.1. Δ is an H-type if and only if, for any $\lambda \in (0,1)$, there exists $\delta(\lambda) \in (0,1)$ such that following implications hold:

$$t > 1 - \delta(\lambda) \text{ implies } \Delta^m(t) > 1 - \lambda.$$

Remark 2.2. Every t-norm Δ_M is of Hadžíc-type but converse need not be true (see [4]).

Definition 2.3. The 3-tuple (X, M, t) is called fuzzy metric space in the sense of Kramosil and Michalek [6] if X is an arbitrary set, t is a continuous t-norm and M is a fuzzy set on $X^2 \times [0,\infty)$ satisfying the following conditions:

(i) $M(x,y,0) = 0$;
(ii) $M(x,y,t) = 1$ if and only if $x = y$;
(iii) $M(x,y,t) = M(y,x,t)$;
(iv) $M(x,z,t+s) \geq t(M(x,y,t), M(y,z,s))$;
(v) $M(x,y,.) : [0,\infty) \to [0,1]$ is left continuous;

for all $x,y,z \in X$ and $s,t > 0$.

Note that the function $M(x,y,t)$ can be thought of as the degree of nearness between x and y with respect to t.

In our theory, we consider the following conditions:

(vi) $\lim_{t\to\infty} M(x,y,t) = 1$ for all x, y in X.

Definition 2.4. A sequence $\{x_n\}$ in a fuzzy metric space (X, M, t) is said to be

(i) convergent to x in X if, for every $\varepsilon > 0$ and $\lambda > 0$, there is an integer $n_0 \in N$ such that $M(x_n, x, \varepsilon) > 1 - \lambda$, for all $n \geq n_0$.

(ii) Cauchy sequence in X if, for every $\varepsilon > 0$ and $\lambda > 0$, there is an integer $n_0 \in N$ such that $M(x_n, x_m, \varepsilon) > 1 - \lambda$, for all $m, n \geq n_0$.

(iii) complete if every Cauchy sequence in X is convergent in X.

Definition 2.5. Define $\Phi = \{\phi : R^+ \to R^+\}$, where $R^+ = [0, +\infty)$ and each $\phi \in \Phi$ satisfying the following conditions:

(ϕ_1) ϕ is non-decreasing.

(ϕ_2) ϕ is upper semi-continuous from the right.

(ϕ_3) $\sum_{n=0}^{\infty} \phi^n(t) < +\infty$ for all $t > 0$, where $\phi^{n+1}(t) = \phi(\phi^n(t)), n \in N$.

Clearly, if $\phi \in \Phi$, then $\phi(t) < t$ for all $t > 0$.

Definition 2.6. An element $x \in X$ is called the common fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $x = f(x, x) = g(x)$.

Definition 2.7. An element $(x, y) \in X \times X$ is called a

(i) coupled fixed point of the mapping $f : X \times X \to X$ if $f(x, y) = x,\ f(y, x) = y$.

(ii) coupled coincidence point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $f(x, y) = g(x),\ f(y, x) = g(y)$.

(iii) common coupled fixed point of the mappings $f : X \times X \to X$ and $g : X \to X$ if $x = f(x, y) = g(x),\ y = f(y, x) = g(y)$.

Definition 2.8. The mappings $f : X \times X \to X$ and $g : X \to X$ are called commutative if $gf(x, y) = f(gx, gy)$ for all $x, y \in X$.

Definition 2.9. The mappings $f : X \times X \to X$ and $g : X \to X$ are called weakly compatible if $f(x, y) = g(x)$, $f(y, x) = g(y)$ implies $gf(x, y) = f(gx, gy)$, $gf(y, x) = f(g(y), g(x))$ for all $x, y \in X$.

Remark 2.3. For convenience, we denote

$$[M(x, y, t)]^n = \underbrace{M(x, y, t) * M(x, y, t) * \cdots * M(x, y, t)}_{n}, \text{ for all } n \in N. \tag{2.1}$$

3. Main result

Theorem 3.1. *Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm of H-type. Let $f : X \times X \to X$ and $g : X \to X$ be two mappings and there exists $\phi \in \Phi$ such that*

$$M(f(x, y), f(u, v), \phi(t)) \geq \psi[M(gx, gu, t) * M(gy, gv, t)], \tag{3.1}$$

for all $x, y, u, v \in X$ and $t > 0$, where $\psi : [0,1] \to [0,1]$ is continuous function such that $\psi(t) \geq t$ for all $t \in [0,1]$.

Suppose that $f(X \times X) \subseteq g(X)$, f and g are weakly compatible, range space of one of the mappings f or g is complete. Then f and g have a coupled coincidence point. Moreover, there exists a unique point x in X such that $x = f(x,x) = g(x)$.

Proof. Let x_0, y_0 be two arbitrary points in X. Since $f(X \times X) \subseteq g(X)$, we can choose x_1, y_1 in X such that $g(x_1) = f(x_0, y_0), g(y_1) = f(y_0, x_0)$.

Continuing in this way, we can construct two sequences $\{x_n\}$ and $\{y_n\}$ in X such that $g(x_{n+1}) = f(x_n, y_n)$ and $g(y_{n+1}) = f(y_n, x_n)$ for all $n \geq 0$.

Step I. We first show that $\{gx_n\}$ and $\{gy_n\}$ are Cauchy sequences.

Since $*$ is a t-norm of H-type, for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{p} \geq (1-\varepsilon), \text{ for all } p \in N. \tag{3.2}$$

Since $\lim_{t\to\infty} M(x,y,t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(gx_0, gx_1, t_0) \geq (1-\delta) \text{ and } M(gy_0, gy_1, t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ_3), we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0). \tag{3.3}$$

Using condition (3.1), we have

$$\begin{aligned} M(gx_1, gx_2, \phi(t_0)) &= M(f(x_0,y_0), f(x_1,y_1), \phi(t_0)) \\ &\geq \psi[M(gx_0, gx_1, t_0) * M(gy_0, gy_1, t_0)] \\ &\geq M(gx_0, gx_1, t_0) * M(gy_0, gy_1, t_0) \end{aligned}$$

$$\begin{aligned} M(gy_1, gy_2, \phi(t_0)) &= M(f(y_0,x_0), f(y_1,x_1), \phi(t_0)) \\ &\geq \psi[M(gy_0, gy_1, t_0) * M(gx_0, gx_1, t_0)] \\ &\geq M(gy_0, gy_1, t_0) * M(gx_0, gx_1, t_0). \end{aligned}$$

Similarly, we can also get

$$\begin{aligned} M(gx_2, gx_2, \phi^2(t_0)) &= M(f(x_1, y_1), f(x_2, y_2), \phi^2(t_0)) \\ &\geq \psi[M(gx_1, gx_2, t_0) * M(gy_1, gy_2, t_0)] \\ &\geq [M(gx_0, gx_1, t_0)]^2 * [M(gy_0, gy_1, t_0]^2 \end{aligned}$$

$$\begin{aligned} M(gy_2, gy_3, \phi^2(t_0)) &= M(f(y_1, x_1), f(y_2, x_2), \phi^2(t_0)) \\ &\geq [M(gy_0, gy_1, t_0)]^2 * [M(gx_0, gx_1, t_0)]^2. \end{aligned}$$

Continuing in this way, we can get

$$\begin{aligned} M(gx_n, gx_{n+1}, \phi^n(t_0)) &\geq [M(gx_0, gx_1, t_0)]^{2^{n-1}} * [M(gy_0, gy_1, t_0)]^{2^{n-1}} \\ M(gy_n, gy_{n+1}, \phi^n(t_0)) &\geq [M(gy_0, gy_1, t_0)]^{2^{n-1}} * [M(gx_0, gx_1, t_0)]^{2^{n-1}}. \end{aligned}$$

So, from (3.2) and (3.3), for $m > n \geq n_0$, we have

$$\begin{aligned} M(gx_n, gx_m, t) &\geq M\left(gx_n, gx_m, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M\left(gx_n, gx_m, \sum_{k=n}^{m-1} \phi^k(t_0)\right) \\ &\geq M\Big(gx_n, gx_{n+1}, \phi^n(t_0)\Big) * M\Big(gx_{n+1}, gx_{n+2}, \phi^{n+1}(t_0)\Big) \\ &\qquad * \cdots * M\Big(gx_{m-1}, gx_m, \phi^{m-1}(t_0)\Big) \\ &\geq \{[M(gx_0, gx_1, t_0)]^{2^{n-1}} * [M(gy_0, gy_1, t_0)]^{2^{n-1}}\} \\ &\quad * \{[M(gx_0, gx_1, t_0)]^{2^n} * [M(gy_0, gy_1, t_0)]^{2^n}\} * \cdots \\ &\quad * \{[M(gx_0, gx_1, t_0)]^{2^{m-2}} * [M(gy_0, gy_1, t_0)]^{2^{m-2}}\} \\ &= [M(gx_0, gx_1, t_0)]^{2^{n-1}(2^{m-n}-1)} * [M(gy_0, gy_1, t_0)]^{2^{n-1}(2^{m-n}-1)} \\ &\geq \underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{2^{n-1}(2^{m-n}-1)} \geq (1-\varepsilon), \end{aligned}$$

which implies that

$$M(gx_n, gx_m, t) \geq (1-\varepsilon) \text{ for all } m, n \in N \text{ with } m > n \geq n_0 \text{ and } t > 0.$$

Therefore, $\{gx_n\}$ is a Cauchy sequence in X.
Similarly, $\{gy_n\}$ is also a Cauchy sequence in X.

Step II. Now, we show that f and g have coupled coincidence point. Without loss of generality, we can assume that $g(X)$ is complete, there exist points x, y in $g(X)$ so that $\lim_{n\to\infty} g(x_{n+1}) = x$ and $\lim_{n\to\infty} g(y_{n+1}) = y$.

For $x, y \in g(X)$ implies the existence of p, q in X such that $g(p) = x,\ g(q) = y$ and hence

$$\lim_{n\to\infty} g(x_{n+1}) = \lim_{n\to\infty} f(x_n, y_n) = g(p) = x,$$
$$\lim_{n\to\infty} g(y_{n+1}) = \lim_{n\to\infty} f(y_n, x_n) = g(q) = y.$$

From (3.1), we have

$$\begin{aligned} M(f(x_n, y_n), f(p,q), \phi(t)) &\geq \psi[M(gx_n, g(p), t) * M(gy_n, g(q), t)] \\ &\geq [M(gx_n, g(p), t) * M(gy_n, g(q), t)]. \end{aligned}$$

Taking limit as $n \to \infty$, we have

$$M(g(p), f(p,q), \phi(t)) = 1, \text{ i.e., } f(p,q) = g(p) = x.$$

Similarly, $f(q,p) = g(q) = y$.
Since, f and g are weakly compatible, so $f(p,q) = g(p) = x$ and $f(q,p) = g(q) = y$ implies $gf(p,q) = f(g(p), g(q))$ and $gf(q,p) = f(g(q), g(p))$, i.e., $g(x) = f(x,y)$ and $g(y) = f(y,x)$.
Thus, f and g have a coupled coincidence point.

Step III. Now we show that $g(x) = x$ and $g(y) = y$.

Since $*$ is a t-norm of H-type, then for any $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{p} \geq (1-\varepsilon), \text{ for all } p \in N.$$

Also, since $\phi \in \Phi$, using condition (ϕ_3), we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.1), we have

$$\begin{aligned} M(gx, x, \phi(t_0)) &= M(f(x,y), f(p,q), \phi(t_0)) \\ &\geq \psi[M(gx, gp, t_0) * M(gy, gq, t_0)] \\ &\geq M(gx, gp, t_0) * M(gy, gq, t_0) \end{aligned}$$

Similarly,

$$M(gy, y, \phi(t_0)) \geq M(gy, gq, t_0) * M(gx, gp, t_0).$$

Continuing in the same way, for all $n \in N$

$$M(gx, x, \phi^n(t_0)) \geq [M(gx, gx, t_0)]^{2^{n-1}} * [M(gy, y, t_0)]^{2^{n-1}}.$$

Thus, we have

$$\begin{aligned} M(gx, x, t) &\geq M\left(gx, x, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M\left(gx, x, \phi^{n_0}(t_0)\right) \\ &\geq \left[M(gx, x, t_0)\right]^{2^{n_0-1}} * \left[M(gy, y, t_0)\right]^{2^{n_0-1}} \\ &\geq \underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{2^{n_0}} \geq (1-\varepsilon), \text{ for all } n_0 \in N. \end{aligned}$$

So, for any $\varepsilon > 0$, we have

$$M(gx, y, t) \geq (1-\varepsilon), \text{ for all } t > 0.$$

This implies that $g(x) = x$. Similarly, $g(y) = y$.

Step IV. Now we show that $x = y$.

Since $*$ is a t-norm of H-type for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{P} \geq (1-\varepsilon), \text{ for all } p \in N.$$

Since $\lim_{t\to\infty} M(x, y, t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(x, y, t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ_3), we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.1), we have

$$\begin{aligned} M(x, y, \phi(t_0)) &= M(f(p, q), f(q, p), \phi(t_0)) \\ &\geq \psi[M(gp, gq, t_0) * M(gq, gp, t_0)] \\ &\geq M(gp, gq, t_0) * M(gq, gp, t_0) \\ &= [M(x, y, t_0)]^2. \end{aligned}$$

Continuing likewise, we have for all $n \in N$, that

$$M(x, y, \phi^n(t_0)) \geq [M(x, y, t_0)]^{2^n}.$$

Thus, we have

$$\begin{aligned} M(x, y, t) &\geq M\left(x, y, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M\left(x, y, \phi^{n_0}(t_0)\right) \\ &\geq \left[M(x, y, t_0)\right]^{2^{n_0}}. \end{aligned}$$

$$\geq \underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{2^{n_0}} \geq (1-\varepsilon), \text{ which implies that } x = y.$$

Thus, f and g have common fixed point x in X.

Step V. Uniqueness:

Let z be any point in X such that $z \neq x$ with $g(z) = z = f(z, z)$.
Since $*$ is a t-norm of H-type for any $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{p} \geq (1-\varepsilon), \text{ for all } p \in N.$$

Since $\lim_{t\to\infty} M(x, y, t) = 1$, for all x, y in X, there exists $t_0 > 0$ such that

$$M(x, y, t_0) \geq (1-\delta).$$

Also, since $\phi \in \Phi$, using condition (ϕ_3), we have

$$\sum_{n=1}^{\infty} \phi^n(t_0) < \infty.$$

Then for any $t > 0$ there exists $n_0 \in N$ such that

$$t > \sum_{k=n_0}^{\infty} \phi^k(t_0).$$

Using condition (3.1), we have

$$\begin{aligned} M(x, z, \phi(t_0)) &= M(f(x, x), f(z, z), \phi(t_0)) \\ &\geq \psi[M(g(x), g(z), t_0) * M(g(x), g(z), t_0)] \\ &\geq M(g(x), g(z), t_0) * M(g(x), g(z), t_0) \\ &= [M(x, z, t_0)]^2. \end{aligned}$$

Thus, we have

$$\begin{aligned} M(x,z,t) &\geq M\left(x, z, \sum_{k=n_0}^{\infty} \phi^k(t_0)\right) \\ &\geq M\left(x, z, \phi^{n_0}(t_0)\right) \\ &\geq \left(\left[M(x,z,t_0)\right]^{2^{n_0-1}}\right)^2 \\ &\geq \left(M(x,z,t_0)\right)^{2^{n_0}} \\ &\geq \underbrace{(1-\delta) * (1-\delta) * \cdots * (1-\delta)}_{2^{n_0}} \geq (1-\varepsilon), \end{aligned}$$

which implies that $x = z$. Thus, f and g have a unique common fixed point in X.

4. An application

Theorem 4.1. *Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm define by $a * b = \min\{a, b\}$ for all a, b in X. Let A, B be weakly compatible self maps on X satisfying the following condition:*

(4.1) $M(X) \subseteq N(X)$,

(4.2) *there exists $\phi \in \Phi$ such that $M(Ax, Ay, \phi(t)) \geq \psi[M(Bx, By, t)]$, for all $x, y, \in X$ and $t > 0$, where $\psi : [0,1] \to [0,1]$ is continuous function such that $\psi(t) \geq t$ for all $t \in [0,1]$.*

If range of any one of the maps A or B is complete, then A and B have a unique common fixed point in X.

Proof. Assuming $f(x,y) = A(x)$ and $g(x) = A(x)$ for all $x, y \in X$ in Theorem 3.1, we get the proof.

Taking $\phi(t) = kt,\ k \in (0,1)$ and $\psi(t) = t$, we have the following:

Corollary 4.1. *Let $(X, M, *)$ be a fuzzy metric space, $*$ being continuous t-norm define by $a * b = \min\{a, b\}$ for all a, b in X. Let A, B be weakly compatible self maps on X satisfying (4.1) and the following condition:*

(4.3) *there exists $k \in (0,1)$ such that $M(Ax, Ay, kt) \geq M(Bx, By, t)$, for all $x, y \in X$ and $t > 0$.*

If range of any one of the maps A or B is complete, then A and B have a unique common fixed point in X.

References

[1] Deng, Z., *Fuzzy pseudo-metric space*, J. Math. Anal. Appl., 86 (1982), 74-95.

[2] Erceg, M.A., *Metric spaces in fuzzy set theory*, J. Math. Anal. Appl., 69 (1979), 205-230.

[3] George, A., Veeramani, P., *On some results in fuzzy metric spaces*, Fuzzy Sets and Systems, 64 (1994), 395-399.

[4] Hadžíc, O., Pap, E., *Fixed point theory in probabilistic metric spaces*, Kluwer Academic Publishers, Dordrecht, 2001.

[5] Kaleva, O., Seikkala, S., *On fuzzy metric spaces*, Fuzzy Sets and Systems, 12 (1984), 215-229.

[6] Kramosil, I., Michalek, J., *Fuzzy metrics and statistical metric spaces*, Kybernetika, 11 (1975), 336-344.

[7] Lakshmikantham, V., Círíc, L., *Coupled fixed point theorems for non-linear contractions in partially ordered metric spaces*, Nonlinear Analysis: Theory, Methods and Applications, 70 (12) (2009), 4341-4349.

[8] Zadeh, L.A., *Fuzzy sets*, Inform. Control, 18 (1965), 338-353.

Accepted: 18.10.2014