THE NONLOCAL BOUNDARY VALUE PROBLEMS
FOR STRONGLY SINGULAR HIGHER-ORDER NONLINEAR
FUNCTIONAL-DIFFERENTIAL EQUATIONS

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Abstract. A priori boundedness principle is proven for the nonlocal boundary value
problems for strongly singular higher-order nonlinear functional-differential equations.
Several sufficient conditions of solvability of the Dirichlet problem under consideration
are derived from the a priori boundedness principle. The proof of the a priori bounded-
ness principle is based on the Agarwal–Kiguradze type theorems, which guarantee the
existence of the Fredholm property for strongly singular higher-order linear differential
equations with argument deviations under the nonlocal boundary conditions.

Key words and phrases: Higher order functional-differential equations, Dirichlet
boundary value problem, strong singularity, Fredholm property, a priori boundedness
principle.

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1. Statement of the main results

1.1. Statement of the problem and a survey of the literature. Consider
the functional differential equation

\[(1.1) \quad u^{(2m+1)}(t) = F(u)(t)\]

with the boundary conditions

\[(1.2) \quad \int_a^b u(s)d\varphi(s) = 0\]

where \(\varphi(b) - \varphi(a) \neq 0\), \(u^{(i)}(a) = 0\), \(u^{(i)}(b) = 0\) \((i = 1, \ldots, m)\).
Here, $-\infty < a < b < +\infty$, $\varphi : [a, b] \to R$ is a function of bounded variation, and the operator $F$ acting from the set of $(m - 1)$-th time continuously differentiable on $]a, b[$ functions, to the set $L_{loc}(]a, b[)$. By $u^{(i)}(a)$ ($u^{(i)}(b)$), we denote the right (the left) limit of the function $u^{(i)}$ at the point $a$ ($b$).

The problem is singular in the sense that for an arbitrary $x$ the right-hand side of equation (1.1) may have nonintegrable singularities at the points $a$ and $b$.

Throughout the paper we use the following notations:

$R^+ = [0, +\infty[$;

$[x]_+$ the positive part of number $x$, that is $[x]_+ = \frac{x + |x|}{2}$;

$L_{loc}(]a, b[)$ ($L_{loc}(]a, b[)$) is the space of functions $y : ]a, b[ \to R$, which are integrable on $[a + \varepsilon, b - \varepsilon]$ for arbitrary small $\varepsilon > 0$;

$L_{\alpha, \beta}(]a, b[)$ ($L_{\alpha, \beta}(]a, b[)$) is the space of integrable (square integrable) with the weight $(t - a)^{\alpha} (b - t)^{\beta}$ functions $y : ]a, b[ \to R$, with the norm

$$||y||_{L_{\alpha, \beta}} = \left( \int_a^b (s - a)^{\alpha} (b - s)^{\beta} |y(s)|^2 ds \right)^{1/2};$$

$L([a, b]) = L_{0,0}(]a, b[)$, $L^2([a, b]) = L^2_{0,0}(]a, b[)$;

$M([a, b])$ is the set of the measurable functions $\tau : ]a, b[ \to ]a, b[)$;

$\tilde{L}^2_{\alpha, \beta}(]a, b[)$ ($\tilde{L}^2_{\alpha}(]a, b[)$) is the Banach space of $y \in L_{loc}(]a, b[)$ ($L_{loc}(]a, b[)$) functions, with the norm

$$||y||_{\tilde{L}^2_{\alpha, \beta}} \equiv \max \left\{ \left[ \int_a^b \left( \int_s^t y(\xi)d\xi \right)^2 ds \right]^{1/2} : a \leq t \leq \frac{a + b}{2} \right\} + \max \left\{ \left[ \int_s^b (b - s)^{\beta} \left( \int_t^s y(\xi)d\xi \right)^2 ds \right]^{1/2} : \frac{a + b}{2} \leq t \leq b \right\} < +\infty;$$

$L_m([a, b])$ is the Banach space of $y \in L_{loc}(]a, b[)$ functions, with the norm

$$||y||_{L_m} = \sup \left\{ \left[ (s - a)(b - t) \right]^{m-1/2} \int_s^t |y(\xi)|d\xi : a < s \leq t < b \right\} < +\infty;$$

$C_{loc}^n([a, b][)$, ($\tilde{C}_{loc}^{n-1}(]a, b[)$) is the space of the functions $y : ]a, b[ \to R$, which are continuous (absolutely continuous) together with $y', y'', ..., y^{(n-1)}$ on $[a + \varepsilon, b - \varepsilon]$ for arbitrarily small $\varepsilon > 0$.

$\tilde{C}^m([a, b][)$ ($m \leq n$) is the space of the functions $y \in \tilde{C}^n_{loc}(]a, b[)$, such that

$$\int_a^b |x^{(m)}(s)|^2 ds < +\infty.$$
$C_2^m([a, b])$ is the Banach space of the functions $y \in C_{loc}^m([a, b])$, such that

$$
\limsup_{t \to b} \frac{|x^{(i)}(t)|}{(b - t)^{m-i+1/2}} < +\infty \quad (i = 1, \ldots, m),
$$

(1.4)

$$
\limsup_{t \to a} \frac{|x^{(i)}(t)|}{(t - a)^{m-i+1/2}} < +\infty \quad (i = 1, \ldots, m),
$$

with the norm:

$$
||x||_{C_2^m} = ||x||_C + \sum_{i=1}^{m} \sup \left\{ \frac{|x^{(i)}(t)|}{\alpha_i(t)} : a < t < b \right\},
$$

where $\alpha_i(t) = (t - a)^{m-i+1/2}(b - t)^{m-i+1/2}$.

$\widetilde{C}_2^m([a, b])$ is the Banach space of the functions $y \in C_{loc}^m([a, b])$, such that conditions $\left( \int_a^b |x^{(m+1)}(s)|^2 ds \right)^{1/2} < +\infty$ and (1.4) hold, with the norm:

$$
||x||_{\tilde{C}_2^m} = ||x||_{C^m} + \left( \int_a^b |x^{(m+1)}(s)|^2 ds \right)^{1/2}.
$$

$D_n([a, b] \times R^+)$ is the set of such functions $\delta : [a, b] \times R^+ \to L_n([a, b])$ that $\delta(t, \cdot) : R^+ \to R^+$ is nondecreasing for every $t \in [a, b]$, and $\delta(\cdot, \rho) \in L_n([a, b])$ for any $\rho \in R^+$.

$D_{2m-2,2m-2}([a, b] \times R^+)$ is the set of such functions $\delta : [a, b] \times R^+ \to \tilde{L}_{2m-2,2m-2}([a, b])$ that $\delta(t, \cdot) : R^+ \to R^+$ is nondecreasing for every $t \in [a, b]$, and $\delta(\cdot, \rho) \in \tilde{L}_{2m-2,2m-2}([a, b])$ for any $\rho \in R^+$.

A solution of problem (1.1), (1.2) is sought in the space $\widetilde{C}_2^{2m,m+1}([a, b])$.

The singular ordinary differential and functional-differential equations, have been studied with sufficient completeness under different boundary conditions, see for example [1], [2], [4] – [14], [17], [25]–[29] and the references cited therein. But the equation (1.1), under the boundary condition (1.2), is not studied even in the case when equation (1.1) has the form

$$
x^{(2m+1)}(t) = \sum_{j=0}^{m} p_j(t)x^{(j)}(\tau_j(t)) + f(x)(t),
$$

(1.5)

where the singularity of the functions $p_j : L_{loc}([a, b])$ be such that the inequalities

$$
\int_a^b (s - a)^{2m-1}(b - s)^{2m-1}(-1)^{n-m}p_1(s)_+ ds < +\infty,
$$

(1.6)

$$
\int_a^b (s - a)^{2m-j}(b - s)^{2m-j}|p_j(s)| ds < +\infty \quad (j = 2, \ldots, m),
$$
are not fulfilled (in this case we said that the linear part of the operator \( F \) is a strongly singular), the operator \( f \) continuously acting from \( C^m_2([a, b]) \) to \( L^2_{2m-2,2m-2}([a, b]) \), and the inclusion

\[(1.7) \quad \sup \{ f(x)(t) : ||x||_{C^m_2} \leq \rho \} \in L^2_{2m-2,2m-2}([a, b]).\]

holds. The first step in studying of the differential equations with strong singularities was made by R.P. Agarwal and I. Kiguradze in the article [3], where the linear ordinary differential equations under conditions (1.2), in the case when the functions \( p_j \) have strong singularities at the points \( a \) and \( b \), are studied. Also the ordinary differential equations with strong singularities under two-point boundary conditions are studied in the articles of I. Kiguradze [15], [16], and N. Partsvania [24]. In the papers [20], [21] these results are generalized for linear differential equation with deviating arguments i.e., are proven the Agarwal-Kiguradze type theorems, which guarantee Fredholm’s property for linear differential equation with deviating arguments.

In this paper, on the basis of articles [3] and [19], we prove a priori boundedness principle for the problem (1.1), (1.2) in the case where equation (1.1) is in form (1.5).

Now, we introduce some results from articles [20], [21], which we need for this work. Consider the equation

\[(1.8) \quad u^{(2m+1)}(t) = \sum_{j=1}^{m} p_j(t)u^{(j-1)}(\tau_j(t)) + q(t) \quad \text{for} \quad a < t < b.\]

By \( h_j : [a, b] \times [a, b] \rightarrow R_+ \) and \( f_j : [a, b] \times M([a, b]) \rightarrow C^{loc}([a, b] \times [a, b]) \) \((j = 1, ..., m)\) we denote the functions and operator, respectively defined by the equalities

\[(1.9) \quad h_1(t, s) = \left| \int_s^t (\xi - a)^{n-2m}((-1)^{n-m}p_1(\xi))_+ d\xi \right|, \]

\[(1.10) \quad f_j(c, \tau_j)(t, s) = \left| \int_s^t (\xi - a)^{n-2m}p_j(\xi) \left| \int_{\tau_j(\xi)}^{\tau_j(c)} (\xi_1 - c)^{2(m-j)}d\xi_1 \right|^{1/2} d\xi \right|, \]

and also we put that

\[f_0(t, s) = \left| \int_s^t p_0(\xi) d\xi \right|.\]
Let $k = 2k_1 + 1$ ($k_1 \in N$), then

$$k!! = \begin{cases} 1 & \text{for } k \leq 0, \\ 1 \cdot 3 \cdot 5 \cdots k & \text{for } k \geq 1. \end{cases}$$

Now, we can introduce the main theorem of the paper [20].

**Theorem 1.1.** Let there exist numbers $t^* \in [a, b]$, $l_{k0} > 0$, $l_{kj} > 0$, $\bar{\gamma}_{kj} \geq 0$, and $\gamma_{kj} > 0$ ($k = 0, 1; j = 1, \ldots, m$) such that along with

$$B_0 \equiv l_{00} \left( \frac{2^{m-1}}{(2m-3)!!} \right)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(t^*-a)^{\gamma_{00}}}{\sqrt{2\gamma_{00}}} \int_a^b \frac{|\varphi(\xi) - \varphi(a)| + |\varphi(\xi) - \varphi(b)|}{|\varphi(b) - \varphi(a)|} d\xi$$

(1.11)

$$+ \sum_{j=1}^m \left( \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(t^*-a)^{\gamma_{0j}}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{0j}}} \right) < \frac{1}{2},$$

$$B_1 \equiv l_{10} \left( \frac{2^{m-1}}{(2m-3)!!} \right)^2 \frac{(b-a)^{m-1/2}}{(2m-1)^{1/2}} \frac{(b-t^*)^{\gamma_{10}}}{\sqrt{2\gamma_{10}}} \int_a^b \frac{|\varphi(\xi) - \varphi(a)| + |\varphi(\xi) - \varphi(b)|}{|\varphi(b) - \varphi(a)|} d\xi$$

(1.12)

$$+ \sum_{j=1}^m \left( \frac{(2m-j)2^{2m-j+1}}{(2m-1)!!(2m-2j+1)!!} + \frac{2^{2m-j-1}(b-t^*)^{\gamma_{1j}}}{(2m-2j-1)!!(2m-3)!!\sqrt{2\gamma_{1j}}} \right) < \frac{1}{2},$$

the conditions

$$\begin{align*}
(t-a)^{m-\gamma_{00}-1/2} f_0(t, s) &\leq l_{00}, & (t-a)^{2m-j} h_j(t, s) &\leq l_{0j}, \\
(t-a)^{m-\gamma_{0j}-1/2} f_j(a, \tau_j)(t, s) &\leq l_{0j}, & \text{for } a < t \leq s \leq t^*,
\end{align*}$$

(1.13)

$$\begin{align*}
(b-t)^{m-\gamma_{10}-1/2} f_0(t, s) &\leq l_{10}, & (b-t)^{2m-j} h_j(t, s) &\leq l_{1j}, \\
(b-t)^{m-\gamma_{1j}-1/2} f_j(b, \tau_j)(t, s) &\leq l_{1j}, & \text{for } t^* \leq s \leq t < b
\end{align*}$$

(1.14)

$j = 1, \ldots, m$ hold. Then for every $q \in \bar{L}_{2m-2, 2m-2}([a, b])$ problem (1.8), (1.2) is uniquely solvable in the space $\bar{C}^{2m, m+1}([a, b])$.

Also, in [21], the following theorem is proven:

**Theorem 1.2.** Let all the conditions of Theorem 1.5 are satisfied. Then the unique solution $u$ of problem (1.8), (1.2) for every $q \in \bar{L}_{2m-2, 2m-2}([a, b])$ admits the estimate

$$||u^{(m+1)}||_{L^2} \leq r ||q||_{\bar{L}^2_{2m-2, 2m-2}},$$

(1.15)

with

$$r = \frac{2^m}{(1 - 2 \max\{B_0, B_1\}(2m - 1))!!},$$

and thus constant $r > 0$ depends only on the numbers $l_{kj}$, $\bar{\gamma}_{kj}$, $\gamma_{kj}$ ($k = 0, 1; j = 0, \ldots, m$), and $a, b, t^*$. 

Remark 1.1. Under conditions of Theorem 1.2, for every \( q \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[) \) the unique solution \( u \) of problem (1.8), (1.2) admits the estimate
\[
||u^{(m+1)}||_{C^2_{2m}} \leq r_{m,\varphi}||q||_{L^2_{2m-2,2m-2}},
\]
with
\[
r_{m,\varphi} = \left(1 + \frac{\int_a^b |\varphi(s) - \varphi(a)| + |\varphi(s) - \varphi(b)|}{|\varphi(b) - \varphi(a)|} ds}{(b-a)^{m-1/2}}(m-1)!/(2m-1)!(2m-1)^{1/2}2m^{-1/2}
+ \sum_{j=1}^m \frac{1}{(m-i)!{(2m-2i+1)^{1/2}(2b-a)^{m-i+1/2}}}(1-2\max\{B_0,B_1\})(2m-1)!!.
\]

1.2. Theorems on a solvability of problem (1.1), (1.2). Define the operator
\[
P : C^m_2(]a,b[) \times C^m_2(]a,b[) \to L_{loc}(]a,b[),
\]
by the equality
\[
P(x,y)(t) = \sum_{j=0}^m p_j(x)(t)y^{(j)}(\tau_j(t)) \quad \text{for } a < t < b
\]
where \( p_j : C^m_2(]a,b[) \to L_{loc}(]a,b[), \) and \( \tau_j \in M(]a,b[). \) Also, for any \( \gamma > 0, \)
define the set \( A_\gamma \) by the relation
\[
A_\gamma = \{ x \in \widetilde{C}^m_2(]a,b[) : ||x||_{\widetilde{C}^m_2} \leq \gamma \}.
\]
To formulate this a priori boundedness principle, we have to introduce

Definition 1.1. Let \( \gamma_0 \) and \( \gamma \) be the positive numbers. We said that the continuous operator \( P : C^m_2(]a,b[) \times C^m_2(]a,b[) \to L_m(]a,b[) \) is \( \gamma_0, \gamma \) consistent with boundary condition (1.2) if:

(i) for any \( x \in A_{\gamma_0} \) and almost all \( t \in ]a,b[ \), the inequality
\[
\sum_{j=0}^m |p_j(x)(t)x^{(j)}(\tau_j(t))| \leq \delta(t, ||x||_{\widetilde{C}^m_2})||x||_{\widetilde{C}^m_2}
\]
holds, where \( \delta \in D_{2m}(]a,b[ \times R^+) \).

(ii) for any \( x \in A_{\gamma_0} \) and \( q \in \widetilde{L}^2_{2m-2,2m-2}(]a,b[) \), the equation
\[
y^{(2m+1)}(t) = \sum_{j=0}^m p_j(x)(t)y^{(j)}(\tau_j(t)) + q(t)
\]
under boundary conditions (1.2), has the unique solution \( y \) in the space \( \widetilde{C}^{2m,m+1}(]a,b[) \) and
\[
||y||_{\widetilde{C}^m_2} \leq \gamma ||q||_{\widetilde{L}^2_{2m-2,2m-2}}.
\]
Definition 1.2. We said that the operator \( P \) is \( \gamma \) consistent with boundary condition (1.2), if the operator \( P \) is \( \gamma_0, \gamma \) consistent with boundary condition (1.2) for any \( \gamma_0 > 0 \).

In the sequel, it will always be assumed that the operator \( F_p \), defined by the equality
\[
F_p(x)(t) = \left| F(x)(t) - \sum_{j=0}^{m} p_j(x)(t)x^{(j)}(\tau_j(t))(t) \right|,
\]
continuously acting from \( C^m_2([a, b]) \) to \( \tilde{L}^{2}_{2m-2,2m-2}([a, b]) \), and
\[
(1.22) \quad \tilde{F}_p(t, \rho) \equiv \sup \{ F_p(x)(t) : \|x\|_{C^m_2} \leq \rho \} \in \tilde{L}^{2}_{2m-2,2m-2}([a, b])
\]
for each \( \rho \in [0, +\infty[. \)

Then, the following theorem is valid.

Theorem 1.3. Let the operator \( P \) be \( \gamma_0, \gamma \) consistent with boundary condition (1.2), and there exist a positive number \( \rho_0 \leq \gamma_0 \), such that
\[
(1.23) \quad \| \tilde{F}_p(\cdot, \min\{2\rho_0, \gamma_0\}) \|_{\tilde{L}^{2}_{2m-2,2m-2}} \leq \frac{\gamma_0}{\gamma}.
\]

Moreover, for any \( \lambda \in ]0, 1[ \), let an arbitrary solution \( x \in A_{\gamma_0} \) of the equation
\[
(1.24) \quad x^{(2m+1)}(t) = (1 - \lambda)P(x, x)(t) + \lambda F(x)(t)
\]
under conditions (1.2), admits the estimate
\[
(1.25) \quad \|x\|_{\tilde{C}^m_2} \leq \rho_0.
\]

Then problem (1.1), (1.2) is solvable in the space \( \tilde{C}^{2m,m+1}([a, b]) \).

From Theorem 1.3 with \( \rho_0 = \gamma_0 \) immediately follows

Corollary 1.1. Let the operator \( P \) be \( \gamma_0, \gamma \) consistent with the boundary condition (1.2), and
\[
(1.26) \quad |F(x)(t) - \sum_{j=1}^{m} p_j(x)(t)x^{(j-1)}(\tau_j(t))(t)| \leq \eta(t, \|x\|_{\tilde{C}^{m-1}_2})
\]
for \( x \in A_{m} \) and almost all \( t \in ]a, b[ \), and
\[
(1.27) \quad \|\eta(\cdot, \gamma_0)\|_{\tilde{L}^{2}_{2m-2,2m-2}} \leq \frac{\gamma_0}{\gamma},
\]
where \( \eta \in D_{2m-2,2m-2}([a, b[ \times R^+) \). Then problem (1.1), (1.2) is solvable in the space \( \tilde{C}^{2m,m+1}([a, b]) \).
Corollary 1.2. Let the operator $P$ be $\gamma$ consistent with the boundary condition (1.2), inequality (1.26) holds for $x \in C_2^m([a, b])$ and almost all $t \in [a, b]$, where $\eta(\cdot, \rho) \in \widetilde{L}_{2m-2}^2([a, b])$ for any $\rho \in R^+$, and

\begin{equation}
\limsup_{\rho \to +\infty} \frac{1}{\rho} ||\eta(\cdot, \rho)||_{\widetilde{L}_{2m-2}^2} < \frac{1}{\gamma}.
\end{equation}

Then, problem (1.1), (1.2) is solvable in the space $\widetilde{C}^{2m, m+1}([a, b])$.

Now define the operators

\begin{align*}
h_j : C_1^{m-1}([a, b]) \times [a, b] & \to R_+ ,
\end{align*}

\begin{align*}
f_j : C_1^{m-1}([a, b]) \times [a, b] \times M([a, b]) & \to R, \quad (j = 1, \ldots, m)
\end{align*}

by the equalities

\begin{align}
h_1(x, t, s) &= \left| \int_s^t (\xi - a)^{n-2m}[-(-1)^{n-m}p_1(x)(\xi)]_+ d\xi \right|,
\end{align}

\begin{align}
h_j(x, t, s) &= \left| \int_s^t (\xi - a)^{n-2m}p_j(x)(\xi) d\xi \right| \quad (j = 2, \ldots, m),
\end{align}

and

\begin{align}
f_j(x, c, \tau_j)(t, s) &= \left| \int_s^t (\xi - a)^{n-2m}p_j(x)(\xi) \left| \int_\xi^{\tau_j(\xi)} (\xi_1 - c)^{2(m-j)} d\xi_1 \right|^{1/2} d\xi \right|.
\end{align}

Theorem 1.4. Let the continuous operator $P : C_1^{m-1}([a, b]) \times C_1^{m-1}([a, b]) \to L_n([a, b])$ admit to condition (1.19), where $\delta \in D_n([a, b] \times R^+)$, $\tau_j \in M([a, b])$ and the numbers $\gamma_0, t^* \in [a, b]$, $l_{kj} > 0$, $\tilde{l}_{kj} > 0$ ($k = 1, 2; j = 1, \ldots, m$), be such that the inequalities

\begin{equation}
(t - a)^{2m-j}h_j(x, t, s) \leq l_{0j}, \quad \limsup_{t \to a} (t - a)^{m-\frac{j}{2} - \gamma_0} f_j(x, a, \tau_j)(t, s) \leq \tilde{l}_{0j}
\end{equation}

for $a < t \leq s \leq t^*$, $||x||_{\widetilde{C}_2^m} \leq \gamma_0$,

\begin{equation}
(b - t)^{2m-j}h_j(x, t, s) \leq l_{1j}, \quad \limsup_{t \to b} (b - t)^{m-\frac{j}{2} - \gamma_0} f_j(x, b, \tau_j)(t, s) \leq \tilde{l}_{1j}
\end{equation}

for $t^* \leq s \leq t < b$, $||x||_{\widetilde{C}_2^m} \leq \gamma_0$, and conditions (1.11), (1.12) hold. Moreover, let the operator $F$ and the function $\eta \in D_{2n-2m-2, 2m-2}([a, b] \times R^+)$ be such that condition (1.26) and the inequality

\begin{equation}
||\eta(\cdot, \gamma_0)||_{\widetilde{L}_{2n-2m-2, 2m-2}} < \frac{\gamma_0}{\tilde{r}_m},
\end{equation}
be fulfilled, where
\[
\tau_n = \left(1 + \sum_{j=1}^{m} \frac{2^{m-j+1/2}}{(m-j)!(2m-2j+1)^{1/2}(b-a)^{m-j+1/2}}\right) \frac{2^m(1+b-a)(2n-2m-1)}{(\tau_n - 2 \max\{B_0, B_1\})(2m-1)!}.
\]

Then, problem (1.1), (1.2) is solvable in the space \( \bar{C}^{m-1,m}|a,b| \).

**Theorem 1.5.** Let the operator \( F \) and function \( \eta \) are such that condition (1.26), (1.28) hold and the continuous operator \( P : C_{1}^{m-1}|a,b| \times C_{1}^{m-1}|a,b| \to L_{n}|a,b| \) admits condition (1.19) where \( \delta \in D_{n}|a,b| \times R^{r} \). Let moreover the measurable functions \( \tau_j \in M|a,b| \) and the numbers \( t^* \in [a,b], l_{kj} > 0, \overline{l}_{kj} > 0, \gamma_{kj} > 0 \), \( (k=1,2; j=1,...,m) \) be such that the inequalities
\[
(t - a)^{2m-j} h_j(x,t,s) \leq l_{0j}, \quad \lim_{t \to a} \sup(t - a)^{m-\frac{1}{2} - \gamma_{0j}} f_j(x,a,\tau_j)(t,s) \leq \overline{l}_{0j}
\]
for \( a < t \leq s \leq t^* \), \( x \in \bar{C}_{1}^{m-1}|a,b| \),
\[
(b - t)^{2m-j} h_j(x,t,s) \leq l_{1j}, \quad \lim_{t \to b} \sup(b - t)^{m-\frac{1}{2} - \gamma_{1j}} f_j(x,b,\tau_j)(t,s) \leq \overline{l}_{1j}
\]
for \( t^* \leq s \leq t < b \), \( x \in \bar{C}_{1}^{m-1}|a,b| \), and conditions (1.11), (1.12) hold. Then, problem (1.1), (1.2) is solvable in the space \( \bar{C}^{m-1,m}|a,b| \).

**Remark 1.2.** Let \( \gamma_0 > 0 \), operators \( \alpha_j(t)p_j(x)(t) \ (j=1,...,m) \) continuously acting from the space \( C_{n}^{m-1}|a,b| \) to the space \( L_{n}|a,b| \), exist the function \( \delta_j \in D_{n}|a,b| \) such that for any \( x \in A_{\gamma_0} \)
\[
|p_j(x)(t)|\alpha_j(t) \leq \delta_j(t,||x||_{\bar{C}_{1}^{m-1}}) \text{ for } a < t < b,
\]
and exists constants \( \kappa > 0, \varepsilon > 0 \) such that
\[
|\tau_j(t) - t| \leq \kappa(t - a) \quad (j = 1, \cdots, m) \text{ for } a < t < a + \varepsilon,
\]
\[
|\tau_j(t) - t| \leq \kappa(b - t) \quad (j = 1, \cdots, m) \text{ for } b - \varepsilon < t < b,
\]
Then, the operator \( P \) defined by equality (1.17), continuously acting from \( A_{\gamma_0} \) to the space \( L_{n}|a,b| \), and there exists the function \( \delta \in D_{n}|a,b| \) such that item (ii) of Definition 1.1 holds.

Now, consider the equation with deviating arguments
\[
u^{(n)}(t) = f(t,u(\tau_1(t)),u'(\tau_2(t)), \cdots, u^{(m-1)}(\tau_m(t))) \text{ for } a < t < b,
\]
where \( -\infty < a < b < +\infty, f : [a,b] \times R^m \to R \) is a function, satisfying the local Carathéodory conditions and \( \tau_j \in M|a,b| \) \( (j = 0,...,n-1) \) are measurable functions.
Corollary 1.3. Let the functions \( \tau_j \in M([a, b[) \) and the numbers \( t^* \in [a, b[, \kappa \geq 0, \varepsilon > 0, l_{kj} > 0, l_{kj} > 0, \gamma_{kj} > 0, (k = 1, 2; j = 1, \ldots, m) \) be such that the conditions (1.11)-(1.14), (1.37) and the inclusions

\[
\alpha_j p_j \in L_n([a, b[) \quad (j = 1, \ldots, m)
\]

are fulfilled. Moreover, let

\[
|f(t, x(\tau_1(t)), x'(\tau_2(t)), \ldots, x^{(m-1)}(\tau_m(t)))| - \sum_{j=1}^{m} p_j(t)x^{(j-1)}(\tau_j(t))|t| \\
\leq \eta(t, ||x||_{C_{1}^{m-1}})
\]

for \( x \in C_{1}^{m-1}([a, b[) \) and almost all \( t \in [a, b[ \), where \( \eta(\cdot, \rho) \in \tilde{L}_{2n-2m-2,2m-2}^{2}([a, b[) \) for any \( \rho \in \mathbf{R}^{+} \), and condition (1.28) holds. Then problem (1.38), (1.2) is solvable in the space \( C_{1}^{m-1, m}([a, b[) \).

Remark 1.3. From conditions (1.39), conditions (1.6) do not follow.

Now, to illustrate our results, consider on \([a, b[\) the second order functional-differential equations

\[
u''(t) = -\frac{\lambda|u(t)|^k}{(t-a)(b-t)^{2+k/2}} u(\tau(t)) + q(x)(t),
\]

(1.41)

\[
u''(t) = -\frac{\lambda|\sin u(t)|^k}{(t-a)(b-t)^2} u(\tau(t)) + q(x)(t),
\]

(1.42)

where \( \lambda, k \in \mathbf{R}^{+} \), the function \( \tau \in M([a, b[) \), the operator \( q : C_{1}^{m-1}([a, b[) \to \tilde{L}_{0,0}^{2}([a, b[) \) is continuous and

\[
\eta(t, \rho) \equiv \sup\{|q(x)(t)| : ||x||_{C_{1}^{m-1}} \leq \rho\} \in \tilde{L}_{0,0}^{2}([a, b[).
\]

Then, from Theorems 1.4 and 1.5, it follows

Corollary 1.4. Let the function \( \tau \in M([a, b[) \), the continuous operator \( q : C_{1}^{m-1}([a, b[) \to \tilde{L}_{0,0}^{2}([a, b[) \), and the numbers \( \gamma_{0} > 0, \lambda \geq 0, k > 0 \) be such that

\[
|\tau(t) - t| \leq \begin{cases} 
(t-a)^{3/2} & \text{for } a < t \leq (a+b)/2 \\
(b-t)^{3/2} & \text{for } (a+b)/2 \leq t < b
\end{cases},
\]

(1.43)

\[
||\eta(t, \gamma_{0})||_{\tilde{L}_{0,0}^{2}} \leq \left(1 + \frac{2}{b-a}\right)^{-1} \frac{(b-a)^2 - 16\gamma_{0}^{k}(1 + [2(b-a)]^{1/4})}{2(b-a)(b-a)^{2}}
\]

(1.44)

\[
\frac{(b-a)^2}{32\gamma_{0}^{k}(1 + [2(b-a)]^{1/4})}
\]

(1.45)

Then, problem (1.41), (1.2) is solvable.
Corollary 1.5. Let the function $\tau \in M([a, b[)$, continuous operator $q : C_t^{m-1}([a, b[) \rightarrow \tilde{L}^2_{0,0}([a, b[)$, and the number $\lambda \geq 0$ by such, that inequalities (1.28) with $n = 2$, (1.43) and

\begin{equation}
\lambda < \frac{(b - a)^2}{32(1 + [2(b - a)]^{1/4})},
\end{equation}

hold. Then, problem (1.42), (1.2) is solvable.

2. Auxiliary propositions

2.1. Lemmas on some properties of the equation $x^{(2m)}(t) = \lambda(t)$. First, we introduce two lemmas without proofs. The first Lemma is proved in [3].

Lemma 2.1. Let $i \in 1, 2, \; x \in \tilde{C}^{m-1}_{t0}([t_0, t_1[)$ and

\begin{equation}
nx^{(j-1)}(t_i) = 0 \; \; (j = 1, \ldots, m), \; \int_{t_0}^{t_1} |x^{(m)}(s)|^2 ds < +\infty.\end{equation}

Then

\begin{equation}
\left| \int_{t_i}^{t} \frac{(x^{(j-1)}(s))^2}{(s - t_i)^{2m-2j+2}} ds \right|^{1/2} \leq \frac{2^{m-j+1}}{(2m - 2j + 1)!!} \int_{t_i}^{t} |x^{(m)}(s)|^2 ds \right|^{1/2}
\end{equation}

for $t_0 \leq t \leq t_1$.

This second lemma is a particular case of Lemma 4.1 in [9]}

Lemma 2.2. If $x \in C_t^{2m-1}([a, a_1])$, then for any $s, t \in ]a, a_1]$, we have the equality

\begin{equation}
(-1)^m \int_{s}^{t} x^{(n)}(\xi)x(\xi) d\xi = w_{2m}(x)(t) - w_{2m}(x)(s) + \int_{s}^{t} |x^{(m)}(\xi)|^2 d\xi
\end{equation}

$w_{2m}(x)(t) = \sum_{j=1}^{m} (-1)^{m+j-1} x^{(2m-j)}(t)x(t)$.

Lemma 2.3. Let the numbers $a_1 \in ]a, b[, \; t_{0k} \in ]a, a_1[, \; \varepsilon_{ik}, \varepsilon_i, \beta_k, \beta \in R^+$, $k \in N, \; i = 1, \ldots, m$ be such that

\begin{equation}
\lim_{k \rightarrow +\infty} t_{0k} = a, \; \lim_{k \rightarrow +\infty} \beta_k = \beta, \; \lim_{k \rightarrow +\infty} \varepsilon_{i,k} = \varepsilon_i.
\end{equation}

Moreover, let

\begin{equation}
\lambda \in \tilde{L}^2_{2m-2,0}([a, a_1]),
\end{equation}
be a nonnegative function, \( x_k \in \tilde{C}^{2m-1,m}([a,b]) \) be a solution of the problem
\[
(2.5) \quad x^{(2m)}(t) = \beta_k \lambda(t),
\]
\[
(2.6) \quad x^{(i-1)}(t_{0,k}) = 0 \quad (i = 1, ..., m), \quad x^{(i-1)}(a_1) = \varepsilon_{i,k} \quad (i = 1, ..., m),
\]
and \( x \in \tilde{C}^{2m-1,m}([a,b]) \) be a solution of the problem
\[
(2.7) \quad x^{(2m)}(t) = \beta \lambda(t),
\]
\[
(2.8) \quad x^{(i-1)}(a) = 0 \quad (i = 1, ..., m), \quad x^{(i-1)}(a_1) = \varepsilon_i \quad (i = 1, ..., m).
\]
Then
\[
(2.9) \quad \lim_{k \to +\infty} x^{(j-1)}_k(t) = x^{(j-1)}(t) \quad (j = 1, ..., 2m) \text{ uniformly in } [a, a_1].
\]

Analogously one can prove

**Lemma 2.4.** Let the numbers \( b_1 \in ]a, b[ , \ t_{0,k} \in ]b_1, b[ , \ \varepsilon_{ik}, \varepsilon_i, \ \beta_k, \beta \in R^+ , \ k \in N , \ i = 1, ..., n - m \) be such that
\[
\lim_{k \to +\infty} t_{0,k} = b, \quad \lim_{k \to +\infty} \beta_k = \beta, \quad \lim_{k \to +\infty} \varepsilon_{i,k} = \varepsilon_i.
\]
Moreover, let \( \lambda \in \tilde{I}^{2}_{0,2m-2}([b_1, b]) \) is a nonnegative function, \( x_k \in \tilde{C}^{n-1,m}([a,b]) \) be a solution of problem (2.5) under the conditions
\[
x^{(i-1)}(b_1) = \varepsilon_{i,k} \quad (i = 1, \ldots, m), \quad x^{(i-1)}(t_{0,k}) = 0 \quad (i = 1, ..., n - m),
\]
and \( x \in \tilde{C}^{n-1,m}([a,b]) \) be a solution of equation (2.7) under the conditions
\[
x^{(i-1)}(b_1) = \varepsilon_i \quad (i = 1, ..., m), \quad x^{(i-1)}(b) = 0 \quad (i = 1, ..., n - m).
\]
Then, equalities (2.9) hold.

**Lemma 2.5.** Let \( a < a_1 < b_1 < b, \ \varepsilon_i \in R^+ \) and
\[
\lambda \in \tilde{I}^{2}_{2m-2m-2,0}([a, a_1]) \quad (\lambda \in \tilde{I}^{2}_{0,2m-2}([b_1, b]))
\]
be nonnegative function. Then for the solution \( x \in \tilde{C}^{n-1,m}([a,b]) \) of problem (2.7), (2.8) (2.7), (2.10)) with \( \beta = 1 \), the estimate
\[
(2.11) \quad \int_a^{a_1} |x^{(m)}(s)|^2 ds \leq \Theta_1(x, a_1, \lambda) \left( \int_{b_1}^b |x^{(m)}(s)|^2 ds \leq \Theta_2(x, b_1, \lambda) \right) \quad (k \in N)
\]
is valid, where
\[
(2.12) \quad \Theta_1(x, a_1, \lambda) = 2|w_n(x)(a_1)| + \gamma_1||\lambda||_{\tilde{I}^{2}_{2m-2m-2,0}([a,a_1])}^2,
\]
\[
\left( \Theta_2(x, b_1, \lambda) = 2|w_n(x)(b_1)| + \gamma_2||\lambda||_{\tilde{I}^{2}_{0,2m-2}([b_1,b])}^2, \right)
\]
and
\[
\gamma_1 = \left( \frac{2^{m-1}(2m+1)}{(2m-1)!!} \right)^2, \quad \gamma_2 = \left( \frac{2^{m-1}(2m+1)(b-a+1)}{(2m-1)!!} \right)^2.
\]
2.2. Lemmas on the Banach space \( C^{m-1}_{1,[a,b]} \). Let the bounded linear operator \( \Gamma : \tilde{C}^{m-1}_{1} \rightarrow \tilde{C}^{m}_{x} \) be defined by the equality

\[
\Gamma(x)(t) = \int_{a}^{b} G(t,s)x(s)ds,
\]

and

\[
G(t,s) = \frac{1}{\varphi(b) - \varphi(a)} \times \begin{cases} 
\varphi(s) - \varphi(b) & \text{for } s \geq t \\
\varphi(s) - \varphi(a) & \text{for } s < t
\end{cases}
\]

is the Green’s function of the problem:

\[
w(t) = 0, \quad \int_{a}^{b} w(s)d\varphi(s) = 0.
\]

where \( \varphi : [a,b] \rightarrow \mathbb{R} \) is a function of bounded variation and \( \varphi(b) - \varphi(a) \neq 0 \).

The problem (2.14) has only the trivial solution, thus \( \Gamma(x) = 0 \) if and only if \( x = 0 \), and we can define the Banach spaces:

\( C^{m-1}_{1,[a,b]} \) is the Banach space of the functions \( x \in C^{m-1}_{loc}([a,b]) \), such that

\[
\lim_{t \to a} \frac{|x^{(i-1)}(t)|}{(t-a)^{m-i+1/2}} < +\infty \quad (i = 1, \ldots, m),
\]

(2.15)

\[
\lim_{t \to b} \frac{|x^{(i-1)}(t)|}{(b-t)^{m-i+1/2}} < +\infty \quad (i = 1, \ldots, m),
\]

with the norm:

\[
\|x\|_{C^{m-1}_{1,[a,b]}} = \|\Gamma(x)\|_{C} + \sum_{i=1}^{m} \sup \left\{ \frac{|x^{(i-1)}(t)|}{\alpha_{i}(t)} : a < t < b \right\},
\]

where \( \alpha_{i}(t) = (t-a)^{m-i+1/2}(b-t)^{m-i+1/2} \).

\( \tilde{C}^{m-1}_{1,[a,b]} \) is the Banach space of the functions \( x \in \tilde{C}^{m-1}_{loc}([a,b]) \), such that conditions (1.3) and (2.15) hold, with the norm:

\[
\|x\|_{\tilde{C}^{m-1}_{1,[a,b]}} = \|\Gamma(x)\|_{C} + \|x\|_{C^{m}_{1}} + \left( \int_{a}^{b} |x^{(m)}(s)|^{2}ds \right)^{1/2}.
\]

\textbf{Definition 2.3.} Let \( \rho \in \mathbb{R}^{+} \) and the function \( \eta \in L_{loc}([a,b]) \) be nonnegative. Then \( S(\rho, \eta) \) is a set of such \( z \in \tilde{C}^{m-1}_{loc}([a,b]) \) that

\[
|z^{(i-1)}\left(\frac{x+b}{2}\right)| \leq \rho \quad (i = 1, \ldots, n),
\]

(2.16)

\[
|z^{(n-1)}(t) - z^{(n-1)}(s)| \leq \int_{s}^{t} \eta(\xi)d\xi \quad \text{for } a < s \leq t < b,
\]

(2.17)

\[
z^{(i-1)}(a) = 0 \quad (i = 1, \ldots, m), \quad z^{(i-1)}(b) = 0 \quad (i = 1, \ldots, m).
\]

(2.18)
Lemma 2.6. For the function \( z \in \tilde{C}^{2m-1,m}([a, b]) \), let conditions (2.18) be satisfied. Then, \( z \in \tilde{C}^{m-1}_{1,\Gamma}([a, b]) \) and the estimates

\[
|z^{(i-1)}(t)| \leq \frac{|t - c_k|^{m-i+1/2}}{(m-i)!(2m-2i+1)^{1/2}} \int_{c_k}^t |z^{(m)}(s)|^2 ds^{1/2} \quad \text{for } a < t < b,
\]

\( i = 1, \ldots, m \), hold for \( k = 1, 2 \), where \( c_1 = a, c_2 = b \).

**Proof.** First, not that in view of inclusion \( z \in \tilde{C}^{2m-1,m}([a, b]) \), the equality

\[
z^{(i-1)}(t) = \sum_{j=i}^l \frac{(t-c)^{j-i}}{(j-i)!} z^{(j-1)}(c) + \frac{1}{(l-i)!} \int_c^t (t-s)^{l-i} z^{(l)}(s) ds
\]

for \( a < t < b, i = 1, \ldots, l, l = 1, \ldots, 2m \), holds, where

1. \( c \in [a, b] \) if \( l \leq m \);
2. \( c \in [a, b] \) if \( l > m \),

and exists \( r > 0 \) such that

\[
\int_a^b |z^{(m)}(s)|^2 ds \leq r.
\]

Equality (2.20), with \( l = m, c = a \) and with \( l = m, c = b \) by conditions (2.18), (2.21) and the Schwartz inequality yields (2.19). But, from (2.19) with \( i = 1 \), we have that \( ||z||_C < +\infty \), and then, by the inequality

\[
|\Gamma(z)(t)| \leq \int_a^b \frac{|\varphi(s) - \varphi(a)| + |\varphi(s) - \varphi(b)|}{|\varphi(b) - \varphi(a)|} ds ||z||_C,
\]

we have

\[
||\Gamma(z)||_C < +\infty.
\]

From (2.19), (2.21) and (2.23) it is clear that \( z \in \tilde{C}^{m-1}_{1,\Gamma}([a, b]) \).

Lemma 2.7. Let \( \rho \in \mathbb{R}^+ \), and \( \eta \in \tilde{L}^{2}_{2m-2,2m-2}([a, b]) \) is a nonnegative function. Then \( S(\rho, \eta) \) is a compact subset of the space \( \tilde{C}^{m-1}_{1,\Gamma}([a, b]) \).

**Proof.** Condition (2.17) yields the inequality \( |z^{(2m)}(t)| \leq \eta(t) \). Thus there exists such a function \( \eta_1 \in \tilde{L}^{2}_{2m-2,2m-2}([a, b]) \) that

\[
z^{(2m)}(t) = \eta_1(t), \quad \text{for } a < t < b
\]
(2.25) \[ |\eta_i(t)| \leq \eta(t) \quad \text{for} \quad a < t < b \]

From Theorem 1.1, it follows that problem (2.24), (2.18) has a unique solution \( z \in C^{2m-1,m}([a, b]) \), i.e., there exists \( r > 0 \) such that the inequality (2.21) holds.

For any \( z \in S(\rho, \eta) \), from equality (2.20) with \( l = 2m \), by (2.16), (2.24) and (2.25) we get

(2.26) \[ |z^{(i-1)}(t)| \leq \gamma_i(t) \quad \text{for} \quad a < t < b, \quad (i = 1, \ldots, 2m), \]

where \( \gamma_i(t) = \rho_i + \frac{1}{(2m - i)!} \int_a^t (t - s)^{n-i} \eta(s)ds \) \( \quad (i = 1, \ldots, 2m). \)

Now, let \( z_k \in S(\rho, \eta) \) \( (k \in N) \). By virtue of the Arzelà-Ascoli lemma and conditions (2.17), (2.26), the sequence \( \{z_k\}_{k=1}^{+\infty} \) contains a subsequence \( \{z_{k_l}\}_{l=1}^{+\infty} \) such that \( \{z_{k_l}^{(i-1)}\}_{l=1}^{+\infty} \) are uniformly convergent on \( [a, b] \). Thus, without loss of generality, we can assume that \( \{z_k^{(i-1)}\}_{k=1}^{+\infty} \) \( (i = 1, \ldots, 2m-1) \) are uniformly convergent on \( [a, b] \). Let \( \lim_{k \to +\infty} z_k(t) = z_0(t) \), then \( z_0 \in C^{2m-1}_{loc}([a, b]) \) and

(2.27) \[ \lim_{k \to +\infty} z_k^{(i-1)}(t) = z_0^{(i-1)}(t) \quad (i = 1, \ldots, 2m) \quad \text{uniformly on} \quad [a, b]. \]

From (2.27), in view of the inclusions \( z_k \in S(\rho, \eta) \), immediately it follows that

(2.28) \[ \left| z_0^{(i-1)} \left( \frac{a + b}{2} \right) \right| \leq \rho \quad (i = 1, \ldots, 2m), \]

(2.29) \[ z_0^{(i-1)}(a) = 0 \quad (j = 1, \cdots, m), \quad z_0^{(i-1)}(b) = 0 \quad (j = 1, \ldots, m). \]

(2.30) \[ |z_0^{(2m-1)}(t) - z_0^{(2m-1)}(s)| \leq \int_s^t \eta(s)ds \quad \text{for} \quad a < s \leq t < b. \]

From (2.28)-(2.30), it is clear that \( z_0 \in S(\rho, \eta) \). To finish the proof, we must show that

(2.31) \[ \lim_{k \to +\infty} ||z_k(t) - z_0(t)||_{C_{1, \Gamma}^{2m-1}} = 0, \]

(2.32) \[ S(\rho, \eta) \subset C_{1, \Gamma}^{2m-1}([a, b]). \]

First, note that, from (2.27), by the conditions \( z_k \in S(\rho, \eta) \) and (2.29), we have

(2.33) \[ \lim_{k \to +\infty} ||z_k - z_0||_C = 0, \]

from which, by (2.22), we get

(2.34) \[ \lim_{k \to +\infty} ||\Gamma(z_k - z_0)||_C = 0. \]

Let, \( x_k = z_0 - z_k \), and \( a_1 \in ]a, b[ \), \( b_1 \in ]a_1, b[ \). Then, it is clear that \( x_k \in S(\rho', \eta') \), where \( \rho' = 2\rho, \eta' = 2\eta \). Thus, for any \( x_k \), there exists \( \eta_k \in \tilde{L}^2_{2m-2,2m-2}([a, b]) \) such that

(2.35) \[ x_k^{(2m)}(t) = \eta_k(t), \]

(2.35) \[ x_k^{(i-1)}(a) = 0 \quad (i = 1, \ldots, m), \quad x_k^{(i-1)}(b) = 0 \quad (i = 1, \ldots, m). \]
where

\[ |\eta_k(t)| \leq 2\eta(t) \quad \text{for } a < t < b \quad (k \in N). \tag{2.36} \]

On the other hand, from (2.19) with \( z = x_k \), in view of (2.35), we get

\[ |x_k^{(i-1)}(t)| \leq \left( \int_a^t |x_k^{(m)}(s)|^2 \, ds \right)^{1/2} (t - a)^{m-i+1/2} \quad \text{for } a < t < a_1, \tag{2.37} \]

\[ |x_k^{(i-1)}(t)| \leq \left( \int_t^b |x_k^{(m)}(s)|^2 \, ds \right)^{1/2} (b - t)^{m-i+1/2} \quad \text{for } b_1 < t < b, \]

for \( i = 1, \ldots, m \).

Now, let \( w_{2m} \) be the operator defined in Lemma 2.2 and \( \Theta_1, \Theta_2 \) are functions defined by (2.12) with \( \lambda = \eta_k \). Then, conditions (2.27) yields

\[ \lim_{k \to +\infty} w_{2m}(x_k)(a_1) = 0, \quad \lim_{k \to +\infty} w_{2m}(x_k)(b_1) = 0 \quad (k \in N), \tag{2.38} \]

and, from the definition of the norm \( || \cdot ||_{L^2_{a,b}} \), (2.36) and (2.38), it follows that, for any \( \varepsilon > 0 \), we can choose \( a_1 \in]a, \min\{a + 1, b\}[, \ b_1 \in]\max\{b - 1, b\}[, \text{ and } k_0 \in N, \text{ such that} \]

\[ \Theta_1(x_k, a_1, 2\eta) \leq \frac{\varepsilon}{6} (b - b_1)^{m-1/2} \quad (k \geq k_0), \tag{2.39} \]

\[ \Theta_2(x_k, b_1, 2\eta) \leq \frac{\varepsilon}{6} (a_1 - a)^{m-1/2} \quad (k \geq k_0). \]

By using Lemma 2.5 for \( x_k \), in view of (2.37) and (2.39), we get

\[ \int_a^{a_1} |x_k^{(m)}(s)|^2 \, ds \leq \frac{\varepsilon}{6} \quad \int_{b_1}^b |x_k^{(m)}(s)|^2 \, ds \leq \frac{\varepsilon}{6} \quad (k \geq k_0), \tag{2.40} \]

\[ \frac{|x_k^{(i-1)}(t)|}{\alpha_i(t)} \leq \frac{\varepsilon}{2m} \quad \text{for } t \in]a, a_1] \cup ]b_1, b[. \quad (1 \leq i \leq m, \ k \geq k_0). \tag{2.41} \]

Also, in view of (2.27) without loss of generality we can assume that

\[ \frac{|x_k^{(i-1)}(t)|}{\alpha_i(t)} \leq \frac{\varepsilon}{2m} \quad \text{for } a_1 \leq t \leq b_1, \quad (1 \leq i \leq n - 1, \ k \geq k_0), \tag{2.42} \]

\[ \int_{a_1}^{b_1} |x_k^{(m)}(s)|^2 \, ds \leq \frac{\varepsilon}{6} \quad (k \geq k_0). \tag{2.43} \]

From (2.33), (2.40)-(2.43), equality (2.31) immediately follows.

Let, now \( z \in S(\rho, \eta) \) and \( z_k = \delta_k z \), where \( \lim_{k \to +\infty} \delta_k = 0 \). Then, by (2.27), it is clear that \( z_0 \equiv 0 \) and then, from (2.31) it follows \( z \in \tilde{C}^m_{1,1}([a, b][), \text{ i.e., inclusion (2.32) holds.} \)
Lemma 2.8. Let $\tau_j \in M([a, b])$, $\alpha \geq 0$, $\beta \geq 0$ and exists $\delta \in ]0, b - a[\text{ such that }$

(2.44) \quad |\tau_j(t) - t| \leq k_1(t - a)^\beta \quad \text{for } a < t \leq a + \delta.

Then

$$\left| \int_t^{\tau(t)} (s - a)^\alpha ds \right| \leq \begin{cases} k_1[1 + k_1\delta^{\beta - 1}]^\alpha(t - a)^{\alpha + \beta} & \text{for } \beta \geq 1 \\ k_1[\delta^{1 - \beta} + k_1]^\alpha(t - a)^{\alpha + \beta} & \text{for } 0 \leq \beta < 1 \end{cases}$$

for $a < t \leq a + \delta$.

Proof. First note that

$$\left| \int_t^{\tau(t)} (s - a)^\alpha ds \right| \leq (\max\{\tau(t), t\} - a)^\alpha |\tau(t) - t| \quad \text{for } a \leq t \leq a + \delta,$n

and $\max\{\tau(t), t\} \leq t + |\tau(t) - t| \quad \text{for } a \leq t \leq a + \delta$. Then, in view of condition (2.44), we get

$$\left| \int_t^{\tau(t)} (s - a)^\alpha ds \right| \leq k_1[(t - a) + k_1(t - a)^\beta]^{\alpha}(t - a)^\beta \quad \text{for } a \leq t \leq a + \delta.$$

The last inequality yields the validity of our lemma.

Analogously, one can prove

Lemma 2.9. Let $\tau_j \in M([a, b])$, $\alpha \geq 0$, $\beta \geq 0$ and exists $\delta \in ]0, b - a[\text{ such that }$

(2.45) \quad |\tau_j(t) - t| \leq k_1(b - t)^\beta \quad \text{for } b - \delta \leq t < b.

Then

$$\left| \int_t^{\tau(t)} (b - t)^\alpha ds \right| \leq \begin{cases} k_1[1 + k_1\delta^{\beta - 1}]^\alpha(b - t)^{\alpha + \beta} & \text{for } \beta \geq 1 \\ k_1[\delta^{1 - \beta} + k_1]^\alpha(b - t)^{\alpha + \beta} & \text{for } 0 \leq \beta < 1 \end{cases}$$

for $b - \delta \leq t < b$.

2.3. Lemmas on the solutions of auxiliary problems.

Lemma 2.10. For any function $x \in C_1^{m-1}([a, b])$ ($x \in \tilde{C}_1^{m-1}([a, b])$) the equality

$$||\Gamma(x)||_{C_2^m} = ||x||_{C_1^{m-1}}^{\prime}, \quad (||\Gamma(x)||_{\tilde{C}_2^m} = ||x||_{\tilde{C}_1^{m-1}}^{\prime})$$

holds.
Proof. From the definition of the norms $|| \cdot ||_{\tilde{C}^m_{1, \Gamma}}$ and $|| \cdot ||_{\tilde{C}^{m-1}_{1, \Gamma}}$ we have

$$||\Gamma(x)||_{\tilde{C}^m_{1, \Gamma}} = ||\Gamma(x)||_{C} + \sum_{i=1}^{m} \sup_{0 < t < b} \left\{ \frac{\Gamma^{(i)}(x)(t)}{\alpha_i(t)} : a < t < b \right\}$$

$$+ \left( \int_{a}^{b} |\Gamma^{(m+1)}(x)(s)|^2 ds \right)^{1/2}$$

$$= ||\Gamma(x)||_{C} + \sum_{i=1}^{m} \sup_{0 < t < b} \left\{ \frac{|x^{(i-1)}(t)|}{\alpha_i(t)} : a < t < b \right\}$$

$$+ \left( \int_{a}^{b} |x^{(m)}(s)|^2 ds \right)^{1/2} = ||x||_{\tilde{C}^{m-1}_{1, \Gamma}}.$$

Now, define the continuous operators $P_\Gamma : C^{m-1}_{1, \Gamma}([a, b]) \times C^{m-1}_{1, \Gamma}([a, b]) \to L_n([a, b])$ as

$$P_\Gamma(x, y)(t) = P(\Gamma(x), \Gamma(y))(t) \quad \text{for} \quad a < t < b,$$

i.e.,

$$(2.46) \quad P_\Gamma(x, y)(t) = \sum_{j=1}^{m} p_{j, \Gamma}(x)(t) \Gamma^{(j)}(y)(\tau_j(t)) + p_{0, \Gamma}(x)(t) \int_{a}^{b} G(\tau_0(t), s) \Gamma(y)(s) ds$$

where the operators $p_{j, \Gamma} : C^{m-1}_{1, \Gamma}([a, b]) \to L_n([a, b])$ ($j = 0, ..., m$) are defined by the equalities

$$p_{j, \Gamma}(x)(t) = p_{j}(\Gamma(x))(t),$$

and $F_{p, \Gamma} : C^{m-1}_{1, \Gamma}([a, b]) \to \tilde{L}^2_{2m-2, 2m-2}([a, b])$ as

$$(2.47) \quad F_{p, \Gamma}(x)(t) = F_{p, \Gamma}(\Gamma(x))(t) = |F(\Gamma(x))(t) - P_\Gamma(x, x)(t)|.$$

Lemma 2.11. If the operator $P : C^{n}_{2}([a, b]) \times C^{n}_{2}([a, b]) \to L_n([a, b])$ is consistent with the numbers $\gamma_0, \gamma$, and the set $A_{\gamma_0, \Gamma} \subset \tilde{C}^{m-1}_{1, \Gamma}([a, b])$ is defined by the equality

$$A_{\gamma_0, \Gamma} = \{ x \in \tilde{C}^{m-1}_{1, \Gamma}([a, b]) : ||x||_{\tilde{C}^{m-1}_{1, \Gamma}} \leq \gamma_0 \},$$

then:

(i) for any $x_0 \in A_{\gamma_0, \Gamma}$ and almost all $t \in [a, b]$, the inequality

$$(2.48) \quad \sum_{j=1}^{m} |p_{j, \Gamma}(x_0)(t)x_0^{(j-1)}(\tau_j(t))| + |p_{0, \Gamma}(x_0)(t)\int_{a}^{b} G(\tau_0(s), s)x_0(s)ds|$$

$$\leq \eta(t, ||x_0||_{\tilde{C}^{m-1}_{1, \Gamma}})||x_0||_{\tilde{C}^{m-1}_{1, \Gamma}},$$

holds, where $\delta \in D_n([a, b] \times R^+)$.
Lemma 2.12.

and

Proof. In view of Lemma 2.10, for any \( x_0 \in A_{\gamma_0, r} \) we have \( y = \Gamma(x_0) \in A_{\gamma_0} \), where \( A_{\gamma_0} \) is defined by (1.18), and then, from inequality (1.19), we get

\[
\sum_{j=1}^{m} |p_j, \Gamma(x_0)(t)x_0^{(j-1)}(\tau_j(t))| + |p_0, \Gamma(x_0)(t)G(\tau_0(t), s)x_0(s)ds| = \sum_{j=1}^{m} |p_j(\Gamma(x_0)))(t)\Gamma^{(j)}(x_0)(\tau_j(t))| + |p_0(\Gamma(x_0)))(t)\Gamma(x_0)(t)|
\]

\[
= \sum_{j=0}^{m} |p_j(y)(t)y^{(j)}(\tau_j(t))| \leq \eta(t, ||y||_{\mathcal{C}_2^m})||y||_{\mathcal{C}_2^m} = \eta(t, ||x_0||_{\mathcal{C}_2^{m+1}})||x_0||_{\mathcal{C}_2^{m}}.
\]

Now, let \( x_0 \in A_{\gamma_0, r} \) and \( q \in \hat{L}_{2m-2,2m-2}([a, b]) \), the function \( y \in \hat{C}^{2m,m+1}([a, b]) \) is a solution of problem (1.20), (1.2), for \( x = \Gamma(x_0) \). Let also \( z = y' \), then \( z \in \hat{C}^{2m-1,m}([a, b]) \) and in view of (1.2) we have the representation \( y = \Gamma(z) \), and from (1.20) and (1.21) it follows

\[
z^{(2m)}(t) = \sum_{j=1}^{m} p_j(\Gamma(x_0))(t)\Gamma^{(j)}(z)(\tau_j(t)) + q(t)
\]

\[
= \sum_{j=1}^{m} p_j, \Gamma(x_0)(t)z^{(j-1)}(\tau_j(t)) + p_0, \Gamma(x_0)(t)G(\tau_0(t), s)z(s)ds + q(t)
\]

and

\[
||z||_{\mathcal{C}_2^{m-1}} = ||\Gamma(z)||_{\mathcal{C}_2^m} = ||y||_{\mathcal{C}_2^m} \leq \gamma ||q||_{\hat{L}_{2m-2,2m-2}}.
\]

Thus, relations (2.48), (2.48), and (2.48) are valid.

Lemma 2.12. Let (1.22) and all the conditions of Theorem 1.3 hold. Then

\[
\hat{F}_{p, \Gamma}(t, \rho) = \sup \{F_{p, \Gamma}(x)(t) : ||x||_{\mathcal{C}_2^{m-1}} \leq \rho \} \in \hat{L}_{2m-2,2m-2}([a, b]),
\]

(2.51)

\[
||\hat{F}_{p, \Gamma}(\cdot, \min \{2\rho_0, \gamma_0\})||_{\hat{L}_{2m-2,2m-2}} \leq \frac{\gamma_0}{\gamma},
\]

(2.52)
and, for any \( \lambda \in [0, 1] \), an arbitrary solution \( z \in A_{\gamma, \Gamma} \) of the equation

\[
z^{(2m)}(t) = (1 - \lambda)P_1(z, z)(t) + \lambda F(\Gamma(z))(t)
\]

under condition (2.18), admits the estimate

\[
||z||_{C_1, \Gamma}^{m-1} \leq \rho_0,
\]

**Proof.** In view of Lemma 2.10, and (2.47), by the notation \( y \equiv \Gamma(x) \in C_2^m[\alpha, \beta] \) and (1.22), we get

\[
(2.53)
\]

Thus (2.51) holds.

From the last relation with \( \rho = \min\{2\rho_0, \gamma_0\} \), in view of (1.23), we get (2.52).

Let now \( z \in A_{\gamma, \Gamma} \) is a solution of problem (2.53), (2.18), and \( x \equiv \Gamma(z) \). Then

\[
x^{(2m+1)}(t) = \Gamma^{(2m+1)}(z)(t) = z^{(2m)}(t) = (1 - \lambda)P_1(z, z)(t) + \lambda F(\Gamma(z))(t)
\]

i.e., \( x \) is a solution of problem (1.24), (1.2) and then the estimate (1.25) holds, from which by Lemma 2.10 we get (2.54).

Now, let the operator \( q : C_1^{m-1}(\alpha, \beta) \rightarrow \tilde{L}_2^{2m-2, 2m-2}([\alpha, \beta]) \) be continuous and for any \( x \in \tilde{C}_1^{m-1}(\alpha, \beta) \subset C_1^{m-1}(\alpha, \beta) \) consider the nonhomogeneous equation

\[
z^{(2m)}(t) = \sum_{j=1}^{m} p_{j, \Gamma}(x_0)(t)z^{(j-1)}(\tau_j(t)) + p_{0, \Gamma}(x_0)(t) \int_a^b G(\tau_0(t), s)z(s)ds + q(x)(t)
\]

and the corresponding homogeneous equation

\[
z^{(2m)}(t) = \sum_{j=1}^{m} p_{j, \Gamma}(x_0)(t)z^{(j-1)}(\tau_j(t)) + p_{0, \Gamma}(x_0)(t) \int_a^b G(\tau_0(t), s)z(s)ds,
\]

and let \( E^n \) be a set of the solutions of problem (2.55), (2.18). Assume that the operator \( P : C_2^m([\alpha, \beta]) \times C_2^m([\alpha, \beta]) \rightarrow L_n([\alpha, \beta]) \) be \( \gamma_0, \gamma \)-consistent with the boundary condition (1.2), and operator \( q : C_2^m([\alpha, \beta]) \rightarrow \tilde{L}_2^{2m-2, 2m-2}([\alpha, \beta]) \), be continuous. Then, by Lemma 2.11, problem (2.55), (2.18) has the unique solution \( z \in \tilde{C}_1^{2m-1, m}([\alpha, \beta]) \). But, in view of Lemma 2.6, it is clear that \( z \in \tilde{C}_1^{m-1}([\alpha, \beta]) \). Thus, \( E^n \cap \tilde{C}_1^{m-1}([\alpha, \beta]) \neq \emptyset \), and there exists the operator \( U : \tilde{C}_1^{m-1}([\alpha, \beta]) \rightarrow E^n \cap \tilde{C}_1^{m-1}([\alpha, \beta]) \) defined by the equality

\[
U(x)(t) = z(t).
\]
Lemma 2.13. \( U : \tilde{C}^{m-1}_{1,\Gamma}(a, b) \rightarrow E^n \cap \tilde{C}^{m-1}_{1,\Gamma}(a, b) \) is a continuous operator.

**Proof.** Let \( x_k \in \tilde{C}^{m-1}_{1,\Gamma}(a, b) \) and \( z_k(t) = U(x_k)(t) \) \((k = 1, 2)\), \( y = y_2 - y_1 \), and the operator \( P \) is defined by (1.17). Then

\[
z^{(2m)}(t) = P_1(x_2, z)(t) + q_0(x_1, x_2)(t)
\]

where \( q_0(x_1, x_2)(t) = P_1(x_2, y_1)(t) - P_1(x_1, y_1)(t) + q(x_2)(t) - q(x_1)(t). \) Hence, by item \( \textit{ii.} \) of Lemma 2.11 we have

\[
||U(x_2) - U(x_1)||_{\tilde{C}^{m-1}_{1,\Gamma}} \leq \gamma ||q_0, r(x_1, x_2)||_{L^{2m-2, 2m-2}}.
\]

Since the operators \( P_1 \) and \( q \) are continuous, this estimate implies the continuity of the operator \( U \).

3. Proofs

**Proof of Remark 1.1.** Let \( z \) be a solution of problem (2.49), (2.18) with \( p_j, \Gamma(x_0) = p_j \) \((j = 0, \ldots, m)\), then \( u = \Gamma(z) \) is a solution of problem (1.8), (1.2), and from inequalities (2.19) it follows the estimate

\[
|z^{(i-1)}(t)| \leq \frac{|(b - t)(t - a)|^{m-i+1/2}}{(m - i)!(2m - 2i + 1)^{1/2}} (\frac{2}{b - a})^{m-i+1/2} ||z^{(m)}||_{L^2},
\]

for \( a \leq t \leq b \), and

\[
||z||_C \leq \frac{(b - a)^{m-1/2}}{(m - 1)!(2m - 1)^{1/2}2^{m-1/2}} ||z^{(m)}||_{L^2}.
\]

From the last estimates, by the definition of the norm \( || \cdot ||_{\tilde{C}^{m-1}_{1,\Gamma}} \) and (2.22), we have

\[
||z||_{\tilde{C}^{m-1}_{1,\Gamma}} \leq \int_a^b \frac{(|\varphi(s) - \varphi(a)| + |\varphi(s) - \varphi(b)|)}{|\varphi(b) - \varphi(a)|} ds \frac{(b - a)^{m-1/2}}{(m - 1)!(2m - 1)^{1/2}2^{m-1/2}} ||z^{(m)}||_{L^2}
\]

\[
+ \left(1 + \sum_{j=1}^m \frac{1}{(m - i)!(2m - 2i + 1)^{1/2}} (\frac{2}{b - a})^{m-i+1/2} \right) ||z^{(m)}||_{L^2},
\]

from which, by Lemma 2.10, estimate (1.15) and the fact that \( u' = z, (1.16) \) immediately follows.

**Proof of Theorem 1.3.** Let \( \delta \) and \( \lambda \) are the functions and numbers appearing in Definition 1.1, and the operator \( \tilde{F}_{p, \Gamma}(t, \rho) \) be defined by (2.51). We set

\[
(3.57) \quad \eta(t) = \delta(t, \gamma_0)\gamma_0 + \tilde{F}_{p, \Gamma}(t, \min\{2\rho_0, \gamma_0\}),
\]
χ(s) = \begin{cases} 
1 & \text{for } 0 \leq s \leq \rho_0 \\
2 - s/\rho_0 & \text{for } \rho_0 < s < 2\rho_0 \\
0 & \text{for } s \geq 2\rho_0 
\end{cases}

(3.58)

q(x)(t) = \chi(||x||_{C_{\Gamma^{-1}}^{1}})F_{p,\Gamma}(x)(t).

(3.59)

From (1.22) it is clear that the nonnegative functions \(F_{p,\Gamma}, \eta\), admits the inclusion

\[
\tilde{F}_{p,\Gamma}(\cdot, \min\{2\rho_0, \gamma_0\}), \eta \in \tilde{L}_{2m-2,2m-2}^{2}(\mathbb{R})
\]

and, for every \(x \in A_{\gamma_0,\Gamma} \subset \tilde{C}_{1,\Gamma}^{m-1}(a, b)\) and almost all \(t \in [a, b]\), the inequality

\[
|q(x)(t)| \leq \tilde{F}_{p,\Gamma}(t, \min\{2\rho_0, \gamma_0\}) \quad \text{for } a < t < b
\]

holds.

Let \(U : A_{\gamma_0,\Gamma} \to E^{n} \cap \tilde{C}_{1,\Gamma}^{m-1}(a, b)\) is a operator appeared in Lemma 2.13, from which it follows that \(U\) is a continuous operator. From (i) and (ii) of Definition 1.1, i.e., by Lemma 2.11, from (2.48) and (2.50) it is clear that, for each \(x \in A_{\gamma_0,\Gamma}\), the conditions

\[
||z||_{\tilde{C}_{1,\Gamma}^{m-1}} \leq \gamma_0, \quad |z^{(n-1)}(t) - z^{(n-1)}(s)| \leq \int_{s}^{t} \eta(\xi)d\xi \quad \text{for } a < t < b
\]

hold. Thus, in view of Definition 2.3, the operator \(U\) maps the ball \(A_{\gamma_0,\Gamma}\) into its own subset \(S(\rho_1, \eta)\). From Lemma 2.2, it follows that \(S(\rho_1, \eta)\) is the compact subset of the ball \(A_{\gamma_0,\Gamma} \subset \tilde{C}_{1,\Gamma}^{m-1}(a, b)\), i.e., the operator \(u\) maps the ball \(A_{\gamma_0,\Gamma}\) into its own compact subset. Therefore, owing to Schauder’s principle, there exists \(z \in S(\rho_1, \eta) \subset A_{\gamma_0,\Gamma}\), such that

\[
z(t) = U(z)(t) \quad \text{for } a < t < b.
\]

Thus, by (2.55) and notation (3.59), the function \(z(z \in A_{\gamma_0})\) is a solution of problem (2.53), (2.18), where

\[
\lambda = \chi(||z||_{\tilde{C}_{1,\Gamma}^{m-1}}).
\]

(3.62)

If \(\gamma_0 = \rho_0\), then, in view of condition \(z \in A_{\gamma_0,\Gamma}\), by (3.58) we have that \(\lambda = 1\), and then, in view of (2.53) and (3.59), the function \(z\) is a solution of the equation

\[
z^{(2m)}(t) = F(\Gamma(z))(t)
\]

(3.63)

under boundary conditions (2.18). Thus the function \(u = \Gamma(z)\) (where \(u' = z\)) is a solution of problem (1.1), (1.2).

Let us show now, that \(z\) admits estimate (2.50) in the case when \(\rho_0 < \gamma_0\). Assume the contrary. Then either

\[
\rho_0 < ||z||_{\tilde{C}_{1,\Gamma}^{m-1}} < 2\rho_0,
\]

(3.64)
or

\(|z|_{C^m_{1-1}} \geq 2\rho_0.\)

If condition (3.64) holds, then by virtue of (3.58) and (3.62), \(z\) is a solution of equation (2.53) with \(\lambda \in [0, 1]\), and then (2.54) holds. But this contradicts (3.64).

Assume now that (3.65) is fulfilled. Then, by virtue of (3.58) and (3.62), we have that \(\lambda = 0\). Therefore, \(x \in A_{\gamma_0, \Gamma}\) is a solution of problem (2.56), (2.18). Thus, from (ii) of Lemma 2.11, it is obvious that \(z \equiv 0\), because problem (2.49), (2.18) has only a trivial solution. But this contradicts condition (3.65), i.e., estimate (2.54) is valid. From estimate (2.54) and (3.58), we have that \(\lambda = 1\), and then, in view of (2.55) and (3.59), the function \(z\) is a solution of problem (3.63), (2.18), which admits to the estimate (2.50). Thus, the function \(u = \Gamma(z)\) (where \(u' = z\)) is a solution of problem (1.1), (1.2).

**Proof of Corollary 1.2.** First, note that, in view of condition (1.28), there exists such a \(\gamma_0 > 2\rho_0\) that condition (1.23) holds, and, in view of Definition 1.2, the operator \(P\) is \(\gamma_0, \gamma\) consistent.

On the other hand, from (1.28), it follows the existence of the number \(\rho_0\), such that

\(|\eta(\cdot, \rho)||I_{2n-2m-2,2m-2} < \rho\) for \(\rho > \rho_0.\)

Let \(x\) be a solution of problem (1.24), (1.2) for some \(\lambda \in [0, 1]\). Then \(y = x\) is also a solution of problem (1.20), (1.2), where \(q(t) = \lambda(F(x)(t) - P(x, x)(t))\).

Now, let \(\rho = ||x|_{C^m_{1-1}}\) and assume that

\(|\eta(\cdot, \rho)||I_{2n-2m-2,2m-2} < \rho\) for \(\rho > \rho_0.\)

holds. Then, in view of the \(\gamma\)-consistency of the operator \(p\) with boundary conditions (1.2), inequality (1.21) holds and thus, by condition (1.26), we have

\(\rho = ||x|_{C^m_{1-1}} \leq \gamma||q(x)||I_{2n-2m-2,2m-2} \leq \gamma||\eta(\cdot, \rho)||I_{2n-2m-2,2m-2}.\)

But the last inequality contradicts (3.66). Thus, assumption (3.67) is not valid and \(\rho \leq \rho_0\). Therefore, for any \(\lambda \in [0, 1]\), an arbitrary solution of problem (1.24), (1.2) admits the estimate (1.25). Therefore, all the conditions of Theorem 1.3 are fulfilled, from which the solvability of problem (1.1), (1.2) follows.

**Proof of Theorem 1.4.** Let \(r_n\) be the constant defined in Remark 1.1. First, let us prove that the operator \(P\) is \(\gamma_0, r_n\) consistent with boundary conditions (1.2). From the conditions of our theorem, it is obvious that the item (i) of Definition 1.1 is satisfied. Let now \(x\) be an arbitrary fixed function from the set \(A_{\gamma_0}\) and let \(p_j(t) = \gamma_j(x)(t)\). Thus in view of (1.31), (1.32) all the assumptions of Theorem 1.1 are satisfied, and then for any \(q \in L^2_{2n-2m-2,2m-2}([a, b])\) the problem (1.20), (1.2) has unique solution \(y\). Also in view of Remark 1.1 there exists the constant \(r_n > 0\),
it is clear that the item (i) of Definition 1.1 is satisfied. Let now \(\gamma\) be the conditions of our theorem it is obvious that the item (i) of Definition 1.1 follows.

**Proof of Theorem 1.5.** Let \(r_n\) be the constant defined in Remark 1.1. First prove that operator \(P\) is \(r_n\) consistent with boundary conditions (1.2). From the conditions of our theorem it is obvious that the item (i) of Definition 1.1 satisfies. Let now \(\gamma_0\) be an arbitrary nonnegative number, \(x\) be arbitrary fixed function from the space \(A_{\gamma_0}\) and let \(p_j(t) \equiv p_j(x)(t)\). Then, in view of (1.34), (1.35), all the assumptions of Theorem 1.1 are satisfied and then, for any \(q \in \bar{L}_{2n-2m-2m-2}^2([a, b]),\) problem (1.20), (1.2) has a unique solution \(y\). Also, in view of Remark 1.1, there exists the constant \(r_n > 0\), (which depends only on the numbers \(l_{kj}, \bar{l}_{kj}, \gamma_{kj} (k = 1,2; j = 1,\ldots, m),\) and \(a, b, t^*, n,\)) such that estimate (1.21) holds with \(\gamma = r_n\). I.e., the operator \(P\) is \(\gamma_0, r_n\) consistent with boundary conditions (1.2). Therefore, all the assumptions of Corollary 1.1 are fulfilled, from which the solvability of problem (1.1), (1.2) follows.

**Proof of Remark 1.2.** By Schwartz’s inequality, the definition of the norm \(||y||_{C^{m-1}}\) and inequalities (1.36), (2.2) for any \(x, y \in A_{\gamma_0}\) and \(z = y - x\), we have

\[
|p_j(y)(t)z^{(j-1)}(\tau_j(t))| = |p_j(y)(t)z^{(j-1)}(t)| + |p_j(y)(t)| \int_{t}^{\tau_j(t)} z^{(j)}(\psi)d\psi | \\
\leq ||z||_{C^{m-1}} |p_j(y)(t)| \alpha_j(t) \left(1 + \frac{1}{\alpha_j(t)} \left( \int_{t}^{\tau_j(t)} (\psi - a)2m-2j d\psi \right)^{1/2} \right)
\]

(3.68)

for \(a < t < b\). On the other hand, from conditions (1.37), by Lemmas 2.8 and 2.9, it is clear that

\[
\alpha_j^{-1}(t) \left( \int_{s}^{\tau_j(s)} (\xi - a)2m-2j d\xi \right)^{1/2} \leq \frac{\sqrt{\kappa (1 + \kappa)}}{\varepsilon^{m-j+1/2}} \quad \text{for } s \in [a, a + \varepsilon] \cup [b - \varepsilon, b],
\]

\[
\alpha_j^{-1}(t) \left( \int_{s}^{\tau_j(s)} (\xi - a)2m-2j d\xi \right)^{1/2} \leq \varepsilon^{-2m+2j-1} \left( \int_{a}^{b} (\xi - a)2m-2j d\xi \right)^{1/2} = \frac{(b - a)^{m-j+1/2}}{\sqrt{2m - 2j + \frac{m-2j}{\varepsilon^{2m-2j+1}}}} \quad \text{for } s \in ]a + \varepsilon, b - \varepsilon[.
\]

Then, if we put

\[
\kappa_0 = \min \left\{ \frac{\sqrt{\kappa (1 + \kappa)}}{\varepsilon^{m-j+1/2}}, \frac{(b - a)^{m-j+1/2}}{\sqrt{2m - 2j + \frac{m-2j}{\varepsilon^{2m-2j+1}}}} \right\},
\]

(3.69)
from (3.68), by the last estimates, we get the inequality
\[
|p_j(y(t)z^{(j-1)}(\tau_j(t)))| \leq ||z||c_{\delta}^{-1}(1 + \kappa_0)|p_j(y(t))|\alpha_j(t) \\
\leq ||z||c_{\delta}^{-1}(1 + \kappa_0)\delta_j(t) \leq ||z||c_{\delta}^{-1},
\]
for \(a < t < b\). Analogously, we get that
\[
|p_j(y(t) - p_j(x(t)))z^{(j-1)}(\tau_j(t))| \leq ||z||c_{\delta}^{-1}(1 + \kappa_0)|p_j(y(t)) - p_j(x(t))|\alpha_j(t)
\]
for \(a < t < b\). From (3.70) and the last inequality, it is obvious that the operator
\(P\) defined by equality (1.17) continuously acting from \(A_{\gamma_0}\) to the space \(L_n([a, b])\), and the item (ii) of Definition 1.1 holds, with \(\delta(t, \rho) = (1 + \kappa_0) \sum_{j=1}^{m} \delta_j(t, \rho)\).

**Proof of Corollary 1.3.** From conditions (1.40) and (1.37), by Remark 1.2, we obtain that the operator \(P\) defined by equality (1.17) with \(p_j(x)(t) = p_j(t)\), continuously acting from \(A_{\gamma_0}\) to the space \(L_n([a, b])\), for any \(\gamma_0 > 0\), i.e., continuously acting from \(c_{\delta}^{-1}([a, b])\) to the space \(L_n([a, b])\). Therefore, it is clear that all the conditions of Theorem 1.5 would be satisfied with
\[
F(x)(t) = f(t, x(\tau_1(t)), x'(\tau_2(t)), \ldots, x^{(m-1)}(\tau_m(t))), \quad \delta(t, \rho) = (1 + \kappa_0) \sum_{j=1}^{m} |p_j(t)|,
\]
where the constant \(\kappa_0\) is defined by equality (3.69). Thus problem (1.38), (1.2) is solvable.

**Proof of Corollary 1.4.** Let the operators \(F, p_1 : C^{m-1}([a, b]) \to L_{loc}([a, b])\), and the function \(\eta : [a, b] \times R^+ \to R^+\) be defined by equalities
\[
F(x)(t) = -\frac{\lambda|x(t)|^k}{|(t-a)(b-t)|^{2+k/2}}x(\tau(t)) + q(x)(t), \quad p_1(x)(t) = -\frac{\lambda|x(t)|^k}{|(t-a)(b-t)|^{2+k/2}}.
\]
Then, it is easy to verify that in view of (1.43)-(1.45), conditions (1.11), (1.12), (1.26), (1.31)-(1.40) are satisfied with
\[
\delta(t, \rho) = \frac{\rho^k\lambda}{|(t-a)(b-t)|^2}, \quad l_0 = \frac{4\gamma_0^k\lambda}{(b-a)^2}, \quad \tilde{l}_0 = \frac{16\gamma_0^k\lambda}{(b-a)^2},
\]
\[
(3.71) \quad r_2 = \left(1 + \sqrt{\frac{2}{b-a}}\right) \frac{2(1 + b - a)(b-a)^2}{(b-a)^2 - 16\gamma_0^k(1 + [2(b-a)]^{1/4})},
\]
\[
B_0 = B_1 = \frac{16\gamma_0^k}{(b-a)^2}(1 + [2(b-a)]^{1/4}), \quad t^* = (a + b)/2, \quad \gamma_0 = \gamma_{11} = \frac{1}{4}.
\]
Thus all the condition of Theorem 1.4 are satisfied, from which follows solvability of problem (1.41), (1.2).

**Proof of Corollary 1.5.** Let the operators \(F, p_1 : C^{m-1}([a, b]) \to L_{loc}([a, b])\), and the function \(\eta : [a, b] \times R^+ \to R^+\) be defined by equalities
\[
F(x)(t) = -\frac{\lambda|\sin x(t)|}{|(t-a)(b-t)|^2}x(\tau(t)) + q(x)(t), \quad p_1(x)(t) = -\frac{\lambda|\sin x(t)|}{|(t-a)(b-t)|^2}.
\]
Then it is easy to verify that in view of (1.28), (1.43), and (1.46), all the conditions of Theorem 1.5 follow, where \( \delta, l_{01}, l'_{01}, r_2, B_0, B_1, t^*, \gamma_{01}, \gamma_{11}, \) are defined by (3.71) with \( \rho = 1, \gamma_0 = 1, \) from which the solvability of problem (1.41), (1.2) follows.

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References


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